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GRADED PSEUDO-VALUATION RINGS

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Abstract. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a commutative ring graded by an arbitrary torsionless monoid Γ. A homogeneous prime ideal P of R is said to be strongly homogeneous prime if aP and bR are comparable for any homogeneous elements a, b of R. We will say that R is a graded pseudo-valuation ring (gr-PVR for short) if every homogeneous prime ideal of R is strongly homogeneous prime. In this paper, we introduce and study the graded version of the pseudo-valuation rings which is a generalization of the gr-pseudo-valuation domains in the context of arbitrary Γ-graded rings (with zero-divisors). We then study the possible transfer of this property to the graded trivial ring extension and the graded amalgamation. Our goal is to provide examples of new classes of Γ-graded rings that satisfy the above mentioned property.

1. Introduction

Throughout this paper it is assumed that all rings are with identity and all modules are nonzero unitary and Γ denotes a torsionless grading monoid (i.e., a commutative, cancellative monoid, and the quotient group of Γ, $\langle\Gamma\rangle = \{a-b \mid a, b \in \Gamma\}$ is a torsion-free Abelian group). Our goal is to generalize the study of pseudo-valuation rings (PVRs for short) to the context of Γ-graded rings. The notion of PVRs was introduced by D. F. Anderson, Badawi and Dobbs [9] (this article is our motivation) and has been studied extensively thereafter in [7, 10, 20].

Let A be a ring and E be an A-module. Then $A \propto E$, the *trivial* (*ring*) *extension of A by E*, is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is defined by $(a, e)(b, f) := (ab, af + be)$ for all $a, b \in A$ and all $e, f \in E$. (This construction is also known by other terminology and other notation, such as the *idealization* $A(+)E$.) The basic properties of trivial ring extensions are summarized in the books [14, 19]. Trivial ring extensions have been generalized and studied extensively in graded ring theory, often because of their usefulness in constructing new classes of

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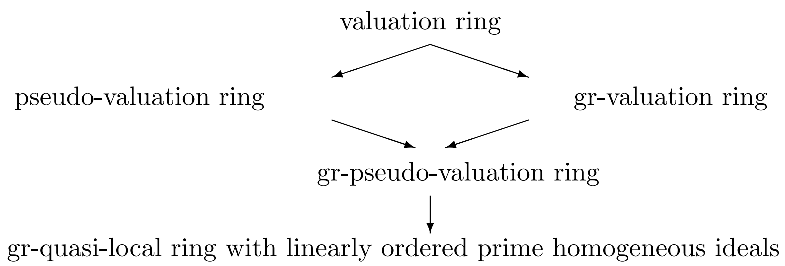
examples of graded rings satisfying various properties (cf. [3, 6, 15]). Let Γ be a commutative monoid. Let $A = \bigoplus_{\alpha\in\Gamma} A_\alpha$ be a Γ-graded ring and $E = \bigoplus_{\alpha\in\Gamma} E_\alpha$ be a Γ-graded A-module. Then $A \propto E$ is a Γ-graded ring with $(A \propto E)_\alpha = A_\alpha \oplus E_\alpha$ for every $\alpha \in \Gamma$ (cf. [6, Proposition 2]). Consequently, $h(A \propto E) = \bigcup_{\alpha\in\Gamma}(A \propto E)_\alpha$. The next two paragraphs summarize most of the background on the underlying domain-theoretic properties and their ring-theoretic generalizations that will be relevant in this paper. A reader may choose to skim the next long paragraph first, and return to it later as needed.

In [18], Hedstrom and Houston introduced the class of pseudo-valuation domains (PVDs for short), which is closely related to the class of valuation domains. A domain R with quotient field K is called a PVD if, whenever P is a prime ideal of R and $xy \in P$ with $x \in K$ and $y \in K$, either $x \in P$ or $y \in P$. For more information on PVDs, see the interesting survey article [8]. In [1], M. T. Ahmed *et al.* generalized the study of pseudo-valuation domains (PVDs) to the context of Γ-graded integral domains. Among other things they proved that every gr-PVD is gr-local [1, Corollary 2.3]; and every gr-valuation domain is a gr-PVD [1, Corollary 2.4], the converse fails in general: A counterexample is obtained by taking R to be the group ring $D[X; \langle\Gamma\rangle]$ of $\langle\Gamma\rangle$ over a non-valuation domain PVD D (see [1, Example 2.11]). They further characterized the gr-PVD in [2]. In [26, 27], P. Sahandi also characterized gr-PVDs of special types. In [9], D. F. Anderson, Badawi and Dobbs generalized the study of PVDs to the context of arbitrary rings (possibly with nontrivial zero-divisors) as follows. A prime ideal P of a ring R is said to be a *strongly prime ideal* (*of* R) if aP and bR are comparable (under inclusion) for all $a, b \in R$; a ring R is called a *pseudo-valuation ring* (PVR for short) if every prime ideal of R is a strongly prime ideal of R. Obviously, every valuation ring is a PVR; the converse fails [9, Examples 10]. As explained in [9, p. 58], a domain is a PVR if and only if it is a PVD.

We can now specify the main objectives of this paper. In Section 2, we review the definitions and preliminary results needed in this paper. In Section 3, we introduce the notion of graded pseudo-valuation rings (gr-PVRs), which can be seen as an extension of two concepts: PVDs to the setting of arbitrary Γ-graded rings (with zero-divisors) and the graded version of PVRs. Among other things, and for a useful kind of condition, we show that a nontrivially graded ring $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is never a PVR (Proposition 3.2). In Theorem 3.7, several characterizations of gr-PVRs are given; for example, a graded ring $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is a gr-PVR if and only if R has a unique maximal homogeneous ideal M and M is strongly homogeneous prime. As a consequence, we consider the stability of the class of gr-PVRs with respect to the graded homomorphic images (see Corollary 3.8), and every gr-valuation ring is a gr-PVR (Corollary 3.9); Proposition 4.7 and Example 4.15 show that the converse fails. In Section 4, we study the possible transfer of this generalized property for the graded trivial ring extension $(A \propto E)$ and the graded amalgamation $(A \bowtie^f J)$. For the main transfer result in this paper, see Theorems 4.4 and 4.9.

As we continue to study the above classes of graded rings, the reader may find it helpful to keep in mind the implications noted in the following figure.



Let $A = \bigoplus_{\alpha\in\Gamma} A_\alpha$ and $B = \bigoplus_{\alpha\in\Gamma} B_\alpha$ be two commutative rings graded by an arbitrary commutative monoid Γ, J be a homogeneous ideal of B and $f : A \longrightarrow B$ be a graded ring homomorphism. Then $R := A \bowtie^f J$, the amalgamation of A with B along J with respect to f (introduced and studied by D'Anna, Finocchiaro, and Fontana in [11, 12]), is a graded ring $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ with the set $h(R) = \bigcup_{\alpha\in\Gamma} R_\alpha$ of all homogeneous elements of R, where for each $\alpha \in \Gamma$

$$R_\alpha = \left(A \bowtie^f J\right)_\alpha := \{(a_\alpha, f(a_\alpha) + j_\alpha) \mid a_\alpha \in A_\alpha, j_\alpha \in J_\alpha\}.$$

This construction, which was introduced and studied in [17], enables us to meet the challenges posed by providing ample examples that enrich the current literature of graded ring-theoretic. One of the key tools for studying $A \bowtie^f J$ is based on the fact that the graded amalgamation can be studied in the frame of pullback constructions [16, Section 3]. Other classical constructions such as the graded amalgamated duplication of a ring along an ideal denoted by $A \bowtie I$ and the graded trivial ring extension of A by E [6] ($A \ltimes E$), can be interpreted as particular cases of the general graded amalgamation construction [17, Example 3.3 & 3.4].

2. Preliminaries

This section introduces some basic properties of graded rings and modules used in the rest of the paper. Let Γ be a torsionless grading monoid (written additively), with an identity element denoted by 0, and let the quotient group of Γ be $\langle\Gamma\rangle = \{a - b \mid a, b \in \Gamma\}$, a torsion-free Abelian group. It is well known that a cancellative monoid is torsionless if and only if it can be given a total order compatible with the monoid operation [23, p. 123].

Recall that a (not necessarily unital) ring R is called a Γ-graded ring, or simply a graded ring, if $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$, each R_γ is an additive subgroup of R, and $R_\gamma R_\delta \subseteq R_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. The set of homogeneous elements of R is denoted by $h(R) = \bigcup_{\gamma\in\gamma} R_\gamma$. The nonzero elements of R_γ are called homogeneous of degree γ, and we write $\deg(r) = \gamma$ if $r \in R_\gamma \backslash \{0\}$. We call the set

$$\Gamma_R = \{\gamma \in \Gamma \mid R_\gamma \neq 0\}$$

the support of R. We say that R has a trivial grading, or R is concentrated in degree zero if the support of R is the trivial group, i.e., $R_0 = R$ and $R_\gamma = 0$ for every $\gamma \in \Gamma \backslash \{0\}$. Clearly R_0 is a subring of R (intuitively $1 \in R_0$) and each R_α is an R_0-module.

Let I be an ideal of R. Then I is said to be a homogeneous ideal of R if one of the following equivalent conditions holds: (i) $I = \bigoplus_{\alpha \in \Gamma} I_\alpha$, where $I_\alpha = I \cap R_\alpha$ for all $\alpha \in \Gamma$ and (ii) $a = a_{\alpha_1} + a_{\alpha_2} + \cdots + a_{\alpha_n} \in I$ implies that $a_{\alpha_i} \in I$, where $a_{\alpha_i} \in R_{\alpha_i}$. A homogeneous ideal M of R is called a maximal homogeneous ideal (gr-maximal for short) if it is maximal among proper homogeneous ideals; equivalently, if every nonzero homogeneous element of R/M is invertible. A graded ring is said to be graded local (gr-local) if it has a unique gr-maximal ideal, and a graded ring R is called a graded-field (gr-field for short) if every nonzero homogeneous element of R is invertible. Obviously, every field is a gr-field, but the converse is not generally true, see [22, p. 44].

We will use the following definition (which is the classical one if R is a graded integral domain). Let R be a graded ring. A *regular homogeneous element* a is indeed a homogeneous element a such that $(0 : a) = 0$. Denoted by H the multiplicative set of regular homogeneous elements of R. Then, by extending some definitions to the case where rings have zero-divisors, R_H, called the homogeneous total ring of quotients of R, is a ring graded by $\langle \Gamma \rangle$, where $R_H = \bigoplus_{\alpha \in \langle \Gamma \rangle} (R_H)_\alpha$ with

$$(R_H)_\alpha = \left\{ \frac{r}{s} \mid r \in R_\beta, s \text{ regular homogeneous in } R_\gamma \text{ and } \beta - \gamma = \alpha \right\}.$$

If R is a graded integral domain (an integral domain graded by Γ), then R_H is called the homogeneous quotient field of R. Obviously, every nonzero homogeneous element of R_H is invertible and $(R_H)_0$ is a field. A graded R-module E is *gr-divisible* if for every homogeneous element $e \in E$ and every regular homogeneous element a of R there exists $f \in E$ such that $e = af$.

Finally, as usual, for any Γ-graded ring R, h-Spec(R) denotes the set of prime homogeneous ideals of R, h-Max(R) denotes the set of maximal homogeneous ideals of R, and h-Jac(R) denotes the graded Jacobson radical of R, i.e., the intersection of all maximal homogeneous ideals of R.

Let I be a proper homogeneous ideal of a graded ring R. Then the graded radical of I is denoted by h-rad$(I) = \{x = \sum_{\alpha \in \Gamma} x_\alpha \in R \mid$ for every $\alpha \in \Gamma$, there exists $n \in \mathbb{N}$ such that $x_\alpha^n \in I\}$. It is easy to see that h-rad(I) is always a homogeneous ideal of R. Note that, if x is a homogeneous element, then $x \in$ h-rad(I) if and only if $x^n \in I$ for some positive integer n (see [24]). In the case where the grading monoid Γ is a group, [24, Proposition 2.5] shows that h-rad(I) is the intersection of all prime homogeneous ideals of R containing I. The following proposition gives another proof for any grading monoid Γ.

Proposition 2.1. *Let R be a graded ring and I be a homogeneous ideal of R. Then* h-rad(I) *is the intersection of all prime homogeneous ideals of R containing I.*

Proof. It is clear that $\text{h-rad}(I) \subseteq \bigcap_{I \subseteq P, P \in \text{h-Spec}(R)} P$. For the reverse containment, let $x = \sum_{\alpha \in \Gamma} x_\alpha \in R \setminus \text{h-rad}(I)$. Then there exists $\alpha \in \Gamma$ such that $x_\alpha^n \notin I$ for each $n > 0$. Now, it is clear that the set

$$\Omega := \{L, \text{ a homogeneous ideal of } R \mid I \subseteq L \text{ and } x_\alpha^n \notin L \text{ for all integers } n > 0\}$$

is not empty (it contains I) and inductive. Thus, by Zorn's lemma, Ω contains a maximal element P. We claim that P is a prime homogeneous ideal of R. Otherwise, there exist $a, b \in h(R) \setminus P$ such that $ab \in P$. By the maximality of P we have $P + aR \notin \Omega$ and $P + bR \notin \Omega$. Since $I \subseteq P$, we get $I \subseteq P + aR$ and $I \subseteq P + bR$. Hence there exist integers $n, m > 0$ such that $x_\alpha^n \in P + aR$ and $x_\alpha^m \in P + bR$. Thus $x_\alpha^{n+m} \in P$, a contradiction. So P is a prime homogeneous ideal of R and $x_\alpha \notin P$, and so $x \notin \bigcap_{I \subseteq P, P \in \text{h-Spec}(R)} P$. Hence $\text{h-rad}(I) = \bigcap_{I \subseteq P, P \in \text{h-Spec}(R)} P$. □

For future reference, we note the following results (cf. [5, Lemma 3.2, Corollary 3.7, Proposition 3.8 and Theorem 3.10, p. 366–369]).

Theorem 2.2. *The following assertions are equivalent for graded rings $R \subseteq T$:*

(1) $\text{h-Spec}(R) = \text{h-Spec}(T)$;
(2) $\text{h-Max}(R) = \text{h-Max}(T)$;
(3) $\text{h-Max}(R) \subset \text{h-Max}(T)$;
(4) $\text{h-Max}(T) \subset \text{h-Max}(R)$;
(5) *R and T have the same graded radical ideals.*

Furthermore, if (*any of the above conditions hold*) *and $R \neq T$, then R is gr-local.*

In fact, the following two lemmas are needed to prove Theorem 2.2.

Lemma 2.3. *If R is a proper graded subring of a graded ring T, then R and T have at most one maximal homogeneous ideal in common.*

Proof. Suppose, on the contrary, that M and N are distinct maximal homogeneous ideals common to both R and T. Since M and N are comaximal homogeneous ideals of R, we have $M + N = R$. Similarly, since M and N are comaximal homogeneous ideals of T, we have $M + N = T$ and hence $R = T$, as required. □

Lemma 2.4. *Let $R \subseteq T$ be graded rings such that R is gr-local and its maximal homogeneous ideal M is also a homogeneous ideal of T. Then $M \subseteq$ h-Jac(T).*

Proof. It suffices to show that $M \subseteq N$ for any maximal homogeneous ideal N of T. If this condition fails, then $N \subsetneq M + N$ for some maximal homogeneous ideal N of T. By maximality of N we get $M + N = T$, and so $m + n = 1$ for some $m \in M$ and $n \in N$ (not necessarily homogeneous). Set $n = \sum_{i=0}^{k} n_{\alpha_i}$ with $\deg(n_{\alpha_i}) = \alpha_i$ for each $i \in \{0, \dots, k\}$. Now if we assume that all $\alpha_i \neq 0$ for all $i \in \{0, \dots, k\}$, then decomposing m into its homogeneous components gives: $m = n-1 = n_{\alpha_0} + \cdots + n_{\alpha_k} - 1 \in M$, and thus $1 \in M$ by the homogeneity of M, which is a contradiction. Therefore, without loss of generality, we can assume that $\alpha_0 = 0$. Then we have $n - 1 \in M$ and

$$n - 1 = \sum_{i=0}^{k} n_{\alpha_i} - 1 = (n_{\alpha_0} - 1) + \sum_{i=1}^{k} n_{\alpha_i} \in M.$$

Thus $n_{\alpha_0} - 1 \in M$. So n_{α_0} is a unit of R, since n_{α_0} is nonzero homogeneous element and, a fortiori, a unit of T, which contradicts that $n \in N$. $\square$

Lemma 2.5. *Let $R \subseteq T$ be graded rings such that R is gr-local with maximal homogeneous ideal M. Then* h-Spec(R) = h-Spec(T) *if and only if* $M \in$ h-Max(T).

Proof. The "only if" half is trivial. Conversely, suppose that $M \in$ h-Max(T). By Lemma 2.4, $M \subseteq$ h-Jac(T), and so M is the only maximal homogeneous ideal of T. Hence, h-Spec$(T) \subseteq$ h-Spec(R). For the reverse inclusion, let $P \in$ h-Spec(R). First, we show that P is a homogeneous ideal of T. To do this, it is enough to show that for every $t = \sum_{i=0}^{k} t_{\alpha_i} \in T$ and $x = \sum_{j=0}^{n} x_{\alpha_j} \in P$ (not necessarily homogeneous elements), we get $tx \in P$. As $tx = \sum_{i=0}^{k} \sum_{j=0}^{n} t_{\alpha_i} x_{\alpha_j}$, it is only necessary to show that $t_{\alpha_i} x_{\alpha_j} \in P$ for all $0 \leq i \leq k$ and $0 \leq j \leq n$. Note that $x_{\alpha_j} \in P \subseteq M$. So $t_{\alpha_i}^2 x_{\alpha_j} \in M \subseteq R$, since M is an ideal of T. Then we obtain

$$(t_{\alpha_i} x_{\alpha_j})^2 = x_{\alpha_j} t_{\alpha_i}^2 x_{\alpha_j} \in MTP = MP \subseteq P.$$

Since $t_{\alpha_i} x_{\alpha_j}$ is a homogeneous element of R and $P \in$ h-Spec(R), we have that $t_{\alpha_i} x_{\alpha_j} \in P$. Therefore, P is a homogeneous ideal of T. Finally, to see that P is prime homogeneous in T, suppose $x_{\alpha_i} y_{\alpha_j} \in P$ for homogeneous elements x_{α_i} and y_{α_j} of T; our task is to show that at least one of x_{α_i} and y_{α_j} is in P. This is obvious in the case that both x_{α_i} and y_{α_j} are in M, since P is prime homogeneous in R. For the remaining possibility, suppose without loss of generality that $x_{\alpha_i} \in T \backslash M$. Then $x_{\alpha_i}^{-1} \in T$ and $y_{\alpha_j} = x_{\alpha_i}^{-1}(x_{\alpha_i} y_{\alpha_j}) \in TP \subset P$, as needed. $\square$

Now we can give the demonstration of Theorem 2.2.

Proof of Theorem 2.2. We will prove this theorem in the following order: (1) $\Rightarrow$ (5) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and (2) $\Leftrightarrow$ (4). Since every graded radical ideal of a ring is just the intersection of prime homogeneous ideals of the graded ring, (1) $\Rightarrow$ (5) is immediate. (1) $\Rightarrow$ (5) is immediate. Moreover, (5) $\Rightarrow$ (2) holds, since the maximal homogeneous ideals of a graded ring are just the maximal elements (with respect to inclusion) of the set of graded radical ideals of the ring. The implication (2) $\Rightarrow$ (3) is trivial. To show that (3) $\Rightarrow$ (1), we can assume $R \neq T$. By Lemma 2.3 and (3), R is gr-local (thus disposing of the last statement of the theorem), and Lemma 2.5 can be applied to obtain (1).

Since (2) $\Rightarrow$ (4) is trivial, it remains to show (4) $\Rightarrow$ (2). Assume that h-Max(T) $\subset$ h-Max(R). Then all maximal homogeneous ideals of T are also maximal homogeneous ideals of R. Thus, T is a graded-local ring according to Lemma 2.3. Let M denote the unique maximal homogeneous ideal of T. It suffices to prove that R cannot have a maximal homogeneous ideal $N \neq M$. Given such N, choose $x \in M \backslash N$ and $y \in N \backslash M$, x (resp. y) can be assumed to be homogeneous. Since $y \in T \backslash M$, it follows that y is a unit of T, and so $xy^{-1} \in MT = M$, from which $x = (xy^{-1})\, y \in MN \subset N$, the desired contradiction. $\square$

3. Graded pseudo-valuation rings

In this section, we introduce the notion of graded pseudo-valuation rings. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded ring. Then a homogeneous prime ideal P of R is said to be strongly homogeneous prime if aP and bR are comparable for every homogeneous elements a, b of R. We will say that a graded ring R is a graded pseudo-valuation ring (gr-PVR for short) if every homogeneous prime ideal of R is strongly homogeneous prime. Of course, the notions of "gr-PVRs" and "PVRs" coincide if the ring is trivially graded. Note that a nontrivially graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is never a PVD (see [1, Proposition 2.1]). Next, we explain why this result does not hold beyond the context of graded integral domains.

Example 3.1. Let k be a field, and let X and Y be indeterminates. Then $R = k[X, Y]/\left(X^2, XY, Y^2\right)$ is a PVR by [9, Example 10], which is not trivially graded (R is a $\mathbb{Z}$-graded ring with $R_0 = k$, $R_1 = kX + kY$, and $R_n = 0$ for $n \in \mathbb{Z} \backslash \{0, 1\}$).

However, in the case where the Γ-graded ring R has a homogeneous regular element of nonzero degree that is not a unit, we show that R is never a PVR.

Proposition 3.2. *Let R be a Γ-graded ring which has a homogeneous nonunit regular element x with nonzero degree. Then R is never a PVR.*

Proof. Let x be a homogeneous nonunit regular element of R with nonzero degree. First we claim that $(1 - x^2)R$ is a proper ideal of R. Suppose, for contradiction, that $(1 - x^2)R = R$. This implies that $1 - x^2$ is a unit, i.e., there exist $n \in \mathbb{N}$ and $a = a_{\alpha_0} + \cdots + a_{\alpha_n}$ with $a_{\alpha_i} \in R_{\alpha_i} \setminus \{0\}$ such that

$$1 = a(1 - x^2) = a_{\alpha_0} + \cdots + a_{\alpha_n} - x^2 a_{\alpha_0} - \cdots - x^2 a_{\alpha_n}.$$

On the other hand, since Γ is torsionless, we get

$$\{\alpha_0, \ldots, \alpha_n\} \neq \{2deg(x) + \alpha_0, \ldots, 2deg(x) + \alpha_n\}.$$

Let $\alpha_i, \alpha_j \in \Gamma$ such that

$$\alpha_i \in \{\alpha_0, \ldots, \alpha_n\} \setminus \{2deg(x) + \alpha_0, \ldots, 2deg(x) + \alpha_n\}$$

and

$$(2deg(x)+\alpha_j)\in\{2deg(x)+\alpha_0,\dots,2deg(x)+\alpha_n\}\setminus\{\alpha_0,\dots,\alpha_n\}.$$

Because of the grading, the grade α_i part of 1 (resp., $2deg(x)+\alpha_j$) must be $a_{\alpha_i}\neq 0$ (resp., $-x^2a_{\alpha_j}\neq 0$ as x is regular), which contradicts the fact that 1 is homogeneous and $\alpha_i\neq 2deg(x)+\alpha_j$. Now we claim that neither $x^2R\subseteq(1+x^2)R$ nor $(1+x^2)R(1-x^2)R\subseteq x^2R$, in which case R is not a PVR by [9, Theorem 5 (3)], and we are done. Indeed, if $x^2R\subseteq(1+x^2)R$, then $1=(1+x^2)-x^2\in(1+x^2)R$, and so $1+x^2$ is a unit, which is a contradiction, as we mentioned above. If $(1+x^2)R(1-x^2)R\subseteq x^2R$, then $1=(1+x^2)(1-x^2)+x^4\in x^2R$, and so x^2 is a unit, which is a contradiction. Hence R is not a PVR. □

After considering Proposition 3.2, we have observed in Example 3.1 that R has a graduation of type $\mathbb{Z}$ (clearly a torsionless grading monoid). However, R is a PVR with $M=Z(R)$, which shows the necessity of the condition that R must have a homogeneous nonunit regular element x with nonzero degree in Proposition 3.2.

Remark 3.3. (cf. [1, Proposition 2.2]) If $R=\bigoplus_{\alpha\in\Gamma}R_\alpha$ is a graded integral domain and P is a homogeneous prime ideal of R, then aP and bR are comparable for any homogeneous elements a,b of R if and only if whenever $xy\in P$ with x,y homogeneous elements of R_H, then either $x\in P$ or $y\in P$.

Example 3.4. Armed with Remark 3.3, we already have many examples of the gr-PVR:

(1) Graded valuation domains; see [4].
(2) Graded pseudo-valuation domains, in particular; the group ring $D[X;\langle\Gamma\rangle]$ of $\langle\Gamma\rangle$ over a non-valuation domain PVD D (see [1, Example 2.11]).
(3) Any graded homomorphic image of a graded-valuation domain (Corollary 3.8).

Proposition 3.5. *Let I be a homogeneous ideal of a graded ring R and P be a strongly homogeneous prime ideal of R. Then I and P are comparable.*

Proof. Suppose I is not contained in P. Then there exists $b\in I\setminus P$. So we can assume that b is homogeneous, and for $a=1$ (with $\deg(a)=0$), bR is not contained in $P=aP$, and so $P\subseteq bR\subseteq I$. □

The following corollary is an immediate consequence of Proposition 3.5 (cf. [9, Lemma 1]).

Corollary 3.6. *Let $R=\bigoplus_{\alpha\in\Gamma}R_\alpha$ be a gr-PVR. Then* h-Spec(R) *is totally ordered under inclusion. In particular, R is gr-local.*

Next, we give several equivalent "comparability" conditions for a (not necessarily gr-local) graded ring R to be a gr-PVR (cf. [1, Theorem 2.8] and [9, Theorem 5]).

Theorem 3.7. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded ring. Then the following conditions are equivalent.*

(1) *R is a gr-PVR.*
(2) *(R, M) is a gr-local ring and M is strongly homogeneous prime.*
(3) *For all homogeneous elements α, β of R, either $\alpha|\beta$ or $\beta|\alpha\gamma$ for every nonunit homogeneous element γ of R.*
(4) *For all homogeneous ideals I, J of R, either $I \subseteq J$ or $JL \subseteq I$ for every proper homogeneous ideal L of R.*
(5) *For all homogeneous elements α, β of R, either $\alpha|\beta$ or $\alpha N \subseteq \beta N$, where N is the set of all nonunit homogeneous elements of R.*

Proof. We will prove this theorem in the following order: (1) $\Rightarrow$ (2)$\Rightarrow$ (3)$\Rightarrow$ (4) $\Rightarrow$ (5)$\Rightarrow$ (2) and (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) This follows directly from Corollary 3.6 and the fact that a maximal homogeneous ideal is prime.

(2) $\Rightarrow$ (3) This is straightforward.

(3) $\Rightarrow$ (4) Let I, J be homogeneous ideals of R. Suppose $I \not\subseteq J$. Let $x \in I \setminus J$. To show that $JL \subseteq I$, where L is a proper homogeneous ideal of R, let $y \in J$. Since I (resp. J) is homogeneous, x (resp. y) can be assumed to be homogeneous elements, and by (3) we get $x|ry$ for every homogeneous element $r \in L$. So $ry \in I$ for all homogeneous elements $r \in L$ and $y \in J$. So $JL \subseteq I$.

(4) $\Rightarrow$ (5) Let α, β be two homogeneous elements of R. Set $I := \beta R$, $J := \alpha R$, in the case where $I \subseteq J$, we get $\alpha|\beta$. In the remaining case, we want to show that $\alpha N \subseteq \beta N$. Let $n \in N$. Then by (4) we have $JL \subseteq I$ for $L = nR$. Consequently, there exists a homogeneous element t in R such that $\alpha n = \beta t$. Note that $t \in N$ since $I \not\subseteq J$. Hence $\alpha N \subseteq \beta N$.

(5) $\Rightarrow$ (2) First, we observe that R is gr-local. If not, R has distinct maximal homogeneous ideals P and Q. Pick two homogeneous elements $a \in P \setminus Q$ and $b \in Q \setminus P$. If $a|b$, then $b \in P$, a contradiction. If $aN \subset bN$, then $a^2 \in bN \subseteq Q$, and hence $a \in Q$, which is also a contradiction. So by (5), R is gr-local with maximal homogeneous ideal M, so $\langle N\rangle = M$. Let α, β be two homogeneous elements in R. Suppose $\alpha|\beta$. Then $\beta = \alpha r$ for some homogeneous element $r \in R$. Then $\beta R \subseteq \alpha M$ if $r \in M$; otherwise, $\alpha = r^{-1}\beta$ and $\alpha M \subseteq \beta R$. If $\alpha \nmid \beta$, then by (5), $\alpha M \subseteq \beta M \subseteq \beta R$. So M is strongly homogeneous prime.

(2) $\Rightarrow$ (1) Suppose R is gr-local and M is strongly homogeneous prime. We need to show that every non-maximal homogeneous prime ideal P of R is strongly homogeneous prime. Let a and b be two homogeneous elements in R. We show that aP and bR are comparable. Since M is strongly homogeneous prime, aM and bR are comparable. If $aM \subseteq bR$, then $aP \subseteq aM \subseteq bR$. Thus we can assume that bR is properly contained in aM, and hence $b = am$ for some $m \in M$ (m can be assumed to be homogeneous). If $m \in P$, then $b = am \in aP$, and hence $bR \subset aP$. So we can assume that $m \notin P$. We show that $P \subset mM$. Let x be a homogeneous element in P. Then xR and mM are comparable. If $mM \subseteq xR \subseteq P$, then either $m \in P$ or $M \subseteq P$, a contradiction.

So $xR \subset mM$ for every homogeneous element x in P, and hence $P \subset mM$. Thus $aP \subseteq amM = bM \subseteq bR$. □

Corollary 3.8. *Every graded homomorphic image of a gr-PVR is a gr-PVR.*

It is well known that a ring R is a valuation ring if its ideals are linearly ordered by inclusion (equivalently, its principal ideals are linearly ordered by inclusion). Analogously, we define this notion in the setting of Γ-graded rings as follows. A graded ring $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is called a gr-valuation ring if its homogeneous ideals are linearly ordered by inclusion (equivalently, its principal homogeneous ideals are linearly ordered by inclusion). Indeed, if R is a valuation ring, then it is a gr-valuation ring; while the converse is not true in general; the easiest example is given by any gr-valuation ring $R = \bigoplus_{\alpha \in G} R_\alpha$ with $2 \neq 0$ which has a homogeneous regular element x of R with $deg(x) \neq 0$ and G is a torsion-free Abelian group (for example, $R = A[X, X^{-1}]$, where A is a valuation ring such that $2 \neq 0$). Then we can observe that $(1+x^4)R$ is not comparable with $(1+x^2)R$. If $(1+x^4)R \subseteq (1+x^2)R$, then $2 = 1+x^4+(1-x^2)(1+x^2) \in (1+x^2)R$ and hence $2 = (1+x^2)a$ is a homogeneous element, a contradiction since x is regular and G is a torsion-free Abelian group (analogous to the proof of Proposition 3.2). Also if $(1+x^2)R \subseteq (1+x^4)R$, then $2 = 1+x^4+(1-x^2)(1+x^2) \in (1+x^4)R$ and thus, as in above, $2 = (1+x^4)a$ is a homogeneous element, which is a contradiction.

Corollary 3.9. *Every gr-valuation ring R is a gr-PVR.*

Proof. Let R be a gr-valuation ring with maximal homogeneous ideal M and $a, b \in h(R)$. Since R is a gr-valuation ring, the homogeneous ideals aM and bR are comparable. Thus M is strongly homogeneous prime, and so R is a gr-PVR by Theorem 3.7(2). □

In the light of Theorem 2.2, if R is a proper graded subring of a graded ring T, then h-Spec(R) = h-Spec(T) if and only if h-Max(R) is comparable to h-Max(T), and in this case R (and hence T) is gr-local. In the context of this study, we have the following result.

Theorem 3.10. *Let T be a gr-local ring with maximal homogeneous ideal M and R be a graded subring of T with maximal homogeneous ideal M (so* h-Spec(R) = h-Spec(T)*). Then R is a gr-PVR if and only if T is a gr-PVR.*

Proof. First, suppose that R is a gr-PVR. Let $a, b \in h(T)$. We can assume that $a, b \in M$. Then aM and bR are comparable since R is a gr-PVR. Thus $aM \subset bR$ implies $aM \subseteq bR \subseteq bT$; so we can assume that $bR \subseteq aM$. But then $bT \subseteq aMT = aM$, and hence T is a gr-PVR by Theorem 3.7(2).

Conversely, suppose that T is a gr-PVR. Let $a, b \in h(R)$. Again, we can assume that $a, b \in M$. Then aM and bT are comparable since T is a gr-PVR. If $bT \subseteq aM$, then $bR \subseteq aM$; so we can assume that $aM \subseteq bT$. If aM is not contained in bR, then $am = bt$ for some homogeneous elements $m \in M$ and

$t \in T \setminus R$. Thus t is a unit of T; so $b = a\left(mt^{-1}\right) \in aM$, and thus $bR \subseteq aM$. Hence R is a gr-PVR by Theorem 3.7. □

Let R be a Γ-graded ring and H be the multiplicative subset of regular homogeneous elements of R. An overring of a ring R is a subring of the total quotient ring of R containing R. An overring T of R is said to be a homogeneous overring of R if $R \subseteq T \subseteq R_H$ and $T = \bigoplus_{\alpha \in \langle \Gamma \rangle}(T \cap (R_H)_\alpha)$. So T is a $\langle \Gamma \rangle$-graded ring. Obviously, for any homogeneous ideal I of R, the subset $(I : I) = \{x \in R_H \mid xI \subseteq I\}$ is a homogeneous overring. If R is a gr-local ring with maximal homogeneous ideal M, we denote $(M : M)$ as the largest homogeneous overring of R in R_H in which M is a homogeneous ideal. Armed with the previous theorem, we have the next result.

Proposition 3.11. *Let R be a gr-local ring with maximal homogeneous ideal M. If $V = (M : M)$ is a gr-valuation ring with maximal homogeneous ideal M, then R is a gr-PVR.*

Proof. By Corollary 3.9, V is a gr-PVR. Hence R is a gr-PVR by Theorem 3.10. □

Let R be a gr-local ring (e.g. a gr-PVR) with maximal homogeneous ideal M. If every homogeneous element of M is a homogeneous zero-divisor of R, then $R_H = R$ and in particular $(M : M) = R$. If M contains a regular homogeneous element, then this observation can be strengthened to give the converse of Proposition 3.11.

Theorem 3.12. *Let R be a gr-PVR whose maximal homogeneous ideal M contains a regular homogeneous element. Then $V = (M : M)$ is a gr-valuation ring with maximal homogeneous ideal M.*

Proof. Let a/s, b/t be two homogeneous elements of V, where $a, b \in h(R)$ and $s, t \in R$ are regular homogeneous elements. Then at, bs are homogeneous in R. We show that atV and bsV are comparable, and hence $(a/s)V$ and $(b/t)V$ are comparable. So we only need to show that aV and bV are comparable for a, b homogeneous in R. In fact, we can assume that $a, b \in M$. First, aM and bR are comparable since R is a gr-PVR. If $bR \subseteq aM$, then $b = an$ for some $n \in M$, and hence $bV \subseteq aV$. Thus we can assume that aM is properly contained in bR. Let $s \in h(M)$ be regular. Then $as = br$ for some $r \in R$, where r can be assumed to be homogeneous. Therefore rM and sR are comparable since R is a gr-PVR. If $sR \subseteq rM$, then $s = rm$ for some $m \in M$ (m can be assumed to be homogeneous). Then r and m are regular since s is regular. Thus, since $as = br$, we have $arm = br$, and $am = b$ since r is regular. Thus $bV \subseteq aV$. So we can assume that rM is properly contained in sR. Hence $rM \subseteq sM$, so $(r/s)M \subseteq M$, and hence $r/s \in (M : M) = V$. Since $as = br$, it follows that $a = b(r/s)$ with $r/s \in V$. Hence $aV \subset bV$, and V is a gr-valuation ring.

Finally, we show that M is the maximal homogeneous ideal of V, i.e., every homogeneous element $x \in V \setminus M$ is a unit of V. Let $x = r/s \in V \setminus M$, where

r is a homogeneous element of R and s is a regular homogeneous element of R. If $s \in R \setminus M$, then s is a unit of R. Thus $x = rs^{-1} \in R \setminus M$, and so x is a unit in R, and thus a unit in V. Hence we can assume that $s \in M$. Thus sM and rR are comparable since R is a gr-PVR. If $rR \subseteq sM$, then $r = sm$ for some $m \in M$, and hence $x = r/s = sm/s = m \in M$, a contradiction. Thus sM is properly contained in rR. Hence $s^2 = rt$ for some $t \in R$, where t can be assumed to be homogeneous. Thus both r and t are regular since s is regular. Hence $x^{-1} = s/r \in R_H$. Since sM is properly contained in rR, it follows that $sM \subset rM$, and hence $(s/r)M \subset M$. Thus $x^{-1} = s/r \in (M : M) = V$. So M is the unique maximal ideal of V. □

4. Graded pseudo-valuation rings in some graded ring constructions

Before stating the first main result of this section which investigates the possible transfer of the gr-PVR between a graded ring A and a graded trivial ring extension $A \propto E$, we make the following useful remark.

Remark 4.1. Let E be a graded R-module. Then E is *gr-divisible* if and only if $ax = e$, with $e \in E$ and a regular element $a \in h(R)$, has a solution in E.

Proof. If $ax = e$, where a is a regular homogeneous element of R and $e \in E$ has a solution in E, then E is gr-divisible. Conversely, suppose E is gr-divisible and let a be a regular homogeneous element of R and $e = \sum_{g\in\Gamma} e_g \in E$. Then for every $g \in \Gamma$ there exists $f_g \in E$ such that $af_g = e_g$ and hence $a\sum_{g\in\Gamma} f_g = e$, as desired. □

Proposition 4.2. *Let $A = \bigoplus_{\alpha\in\Gamma} A_\alpha$ be a graded ring, $E = \bigoplus_{\alpha\in\Gamma} E_\alpha$ be a nonzero graded A-module, and $R := A \propto E$ be the graded trivial ring extension. If R is a gr-PVR, then A is a gr-PVR and E is a gr-divisible A-module.*

Proof. This follows easily from Corollary 3.8 and the observation that A is a graded homomorphic image of R. It remains to show that E is a gr-divisible A-module. We show that if r is a regular homogeneous element of A and e is a nonzero homogeneous element of E, then there exists $f \in E$ such that $rf = e$. Without loss of generality, r is a nonunit of R, and so $r \in P$ for some $P \in$ h-Spec(A). Put $Q := P \propto E \in$ h-Spec(R), and consider the homogeneous elements $a := (r, 0)$ and $b := (0, e)$ of R. Simple calculations show that $Qa = Pr \propto rE$ and $Rb = 0 \propto Ae$. If $Qa \subseteq Rb$, then $Pr = 0$, from which $P = 0$ (since r is regular) and thus $r = 0$ by the choice of P, which contradicts the condition that r is regular. Hence, since R is a gr-PVR, $Rb \subseteq Qa$. In particular, $Ae \subseteq rE$, and so a suitable f can be found. □

In this context, it is very important to note that Badawi and Dobbs stated in their paper [10] that: If A is a PVD and E is a divisible A-module, then $R := A \propto E$ is a PVR (cf. [10, Theorem 3.1(b)]). However, a counterexample to this theorem was recently given by Riffi in [25]. This allows us to state that the converse of Proposition 4.2 is false.

Example 4.3. Let $A := \mathbb{Q}+X\mathbb{R}[X], E := \mathbb{C}\left[X, X^{-1}\right]$, and $R = A \propto E$, where X an indeterminate over $\mathbb{C}$. Then A is $\mathbb{Z}$-graded via $A_0 = \mathbb{Q}, A_n = \mathbb{R}X^n$ for $n \geq 1$, and $A_n = 0$ otherwise; and E is $\mathbb{Z}$-graded via $E_n = \mathbb{C}X^n$ for $n \in \mathbb{Z}$. Now, A is a gr-PVD by [2, Example 3.16] and E is a gr-divisible A-module. But R is not a gr-PVR. Indeed, for $u := (0, X)$, $v := (0, iX), P = X\mathbb{R}[X]$, and $Q := P \propto E$, we have neither $u \in Qv$ nor $Qv \subset Ru$.

Theorem 4.4. *Let $A = \bigoplus_{\alpha\in\Gamma} A_\alpha$ be a graded ring, $E = \bigoplus_{\alpha\in\Gamma} E_\alpha$ be a nonzero graded A-module, and $R := A \propto E$ be the graded trivial ring extension. Then:*

(1) *If R is a gr-PVR, then so is A.*
(2) *Suppose* $\mathrm{Ann}(E) \neq 0$. *Then R is a gr-PVR if and only if A is a gr-local ring with maximal homogeneous ideal M such that $M^2 = 0$ and $ME = 0$.*
(3) *Suppose R is a gr-PVR. Then A is a gr-local ring, say with maximal homogeneous ideal M; E is a gr-divisible A-module; either A is a gr-*PVD *or $M^2 = 0$; and for every homogeneous elements $e, f \in E$, either $Ae \subseteq Mf$ or $Mf \subseteq Ae$.*
(4) *If E is a nonzero finitely generated graded A-module and R is a gr-PVR, then A is gr-local and its maximal homogeneous ideal, say M, satisfies $M^2 = 0$.*

Proof. (1) If R is a gr-PVR, then by Proposition 4.2 A is too.

(2) Since the assignment $P \mapsto P \propto E$ determines an order-isomorphism h-Spec$(A) \to$ h-Spec(R), we can assume, without loss of generality, that A is gr-local, with unique maximal homogeneous ideal M. Then R is gr-local, with unique maximal homogeneous ideal $N := M \propto E$ (cf. [6, Theorem 2]). By hypothesis, we can pick a nonzero homogeneous element $a \in \mathrm{Ann}(E)$.

Suppose first that R is a gr-PVR. To show that $ME = 0$, it suffices to prove that if $\mu \in h(M)$ and $e \in h(E)$, then $\mu e = 0$. In fact, since R is a gr-PVR, either $(a, 0)R \subseteq (0, e)N$ or $(0, e)N \subseteq (a, 0)R$. The first possibility cannot hold since $(0, e)N \subseteq 0 \propto E$ and $a \neq 0$, and so $(0, e)N \subseteq (a, 0)R$. Thus $(0, e)(\mu, 0) = (a, 0)(b, t)$ for some $(b, t) \in R$, whence $(0, \mu e) = (ab, at) = (ab, 0)$, and so $\mu e = 0$, which proves that $ME = 0$.

Next, to show that $M^2 = 0$, it suffices to prove that $mn = 0$ for all homogeneous elements $m, n \in M$. By hypothesis, we can choose a nonzero homogeneous element $g \in E$. Since R is a gr-PVR, either $(0, g)R \subseteq (m, 0)N$ or $(m, 0)N \subseteq (0, g)R$. The first possibility cannot hold since $(m, 0)N \subseteq mM \propto mE \subseteq M^2 \propto 0$ and $g \neq 0$, and so $(m, 0)N \subseteq (0, g)R$. Thus $(m, 0)(n, 0) = (0, g)(c, s)$ for some $(c, s) \in R$, from which $mn = 0$, which proves that $M^2 = 0$.

Conversely, suppose that $M^2 = 0$ and $ME = 0$. To show that R is a gr-PVR, Theorem 3.7(2) reduces our task to showing that if $\xi = (d, k), \eta = (h, f)$ are homogeneous elements in R, then either $\xi R \subseteq \eta N$ or $\eta N \subseteq \xi R$. We will examine three cases. If $d \notin M$, then ξ is a unit of R (cf. [6, Proposition 5]), so that $\eta N \subseteq R = \xi R$. Next, if $d \in M$ and $h \notin M$, then η is a unit of R, so that

$\xi R \subseteq dA \propto E \subseteq M \propto E = N = \eta N$. Finally, if both d and h are in M, then, since $M^2 = 0$ and $ME = 0$, we have $\eta N \subseteq hM \propto (hE + Mf) \subseteq M^2 \propto ME = \{(0,0)\} \subseteq \xi R$.

(3) As in the proof of (2), A is a gr-local ring, say with maximal homogeneous ideal M, so that R is a gr-local ring with maximal homogeneous ideal $N = M \propto E$. Let e, f be two homogeneous elements of E. Since R is a gr-PVR, either $(0,e)R \subseteq (0,f)N$ or $(0,f)N \subseteq (0,e)R$. Since $(0,e)R = 0 \propto Ae$ and $(0,f)N = 0 \propto Mf$, it follows that either $Ae \subseteq Mf$ or $Mf \subseteq Ae$.

Next, by Proposition 4.2, E is a gr-divisible A-module. Also, by Proposition 4.2 (or (1)), A is a gr-PVR, and so if A is a graded integral domain, then A is a gr-PVD. So all that remains is to prove that either A is a graded integral domain or $M^2 = 0$.

By (2) we can assume, without loss of generality, that $\mathrm{Ann}(E) = 0$. Suppose first that $A = A_H$. We will prove that in this case $M^2 = 0$. It suffices to show that $am = 0$ for all homogeneous elements $a, m \in M \setminus \{0\}$. Since A is a homogeneous total quotient ring, every regular homogeneous element of A is a unit. Thus there exists a nonzero homogeneous element $b \in M$ such that $ba = 0$. Since $b \notin \mathrm{Ann}(E)$, there exists $g = \sum_{\alpha\in\Gamma} g_\alpha \in E$ such that $bg = \sum_{\alpha\in\Gamma} bg_\alpha \neq 0$. Then $bg_\alpha \neq 0$ for some $\alpha \in \Gamma$. Since R is a gr-PVR, either $(0, g_\alpha)R \subseteq (a,0)N$ or $(a,0)N \subseteq (0, g_\alpha)R$. The first possibility cannot hold, because if $(0, g_\alpha) = (a,0)(c,d)$ for some $(c,d) \in N$, then $g_\alpha = ad$, which gives $bg_\alpha = (ba)d = 0 \cdot d = 0$, a contradiction. So $(a,0)N \subseteq (0, g_\alpha)R$. In particular, $(a,0)(m,0) \in (0, g_\alpha)R \subseteq 0 \propto E$, hence $am = 0$. This completes the proof that $M^2 = 0$ when $A = A_H$.

In the remaining case, $A \neq A_H$, we will give an indirect proof that A is a graded integral domain. In other words, we assume that A is not a graded integral domain and look for a contradiction. Pick two homogeneous elements $a, b \in A \setminus \{0\}$ such that $ab = 0$. Since A is not the homogeneous total quotient ring, M must contain some homogeneous regular element d of A. Since R is a gr-PVR, we get that for every nonzero homogeneous element $e \in E$, either $(a,0)N \subseteq (0,e)R$ or $(0,e) \in (a,0)N$. The first possibility cannot hold, because $(0,e)R \subseteq 0 \propto E$ and $a \neq 0$. Therefore, $(0,e) = (a,0)(c,f)$ for some $(c,f) \in N$. It follows that $e = af$, and so $be = baf = (ab)f = 0 \cdot f = 0$; i.e., $b \in \mathrm{Ann}(E)$. However, since R is a gr-PVR, we also have that either $(b,0)N \subseteq (0,e)R$ or $(0,e) \in (b,0)N$. We see as above that the first condition cannot hold (since $b \neq 0$). So for every $0 \neq e \in h(E)$, there exists $(d,h) \in N$ such that $(0,e) = (b,0)(d,h)$. Then $e = bh \in bE = 0$, the desired contradiction.

(4) As above, A is a gr-local ring, say with maximal homogeneous ideal M, and R is gr-local with maximal homogeneous ideal $N = M \propto E$. Suppose the claim fails. Then, by (2) and the proof of (3), A is a gr-PVD but not a gr-field. Pick nonzero homogeneous elements $d \in M$ and $e \in h(E)$. Since R is a gr-PVR, either $(d,0)N \subseteq (0,e)R$ or $(0,e) \in (d,0)N$. As above, the first condition cannot hold (since $d \neq 0$). Hence $(0,e) = (d,0)(c,f)$ for some $(c,f) \in N$.

Therefore, $e = df \in ME$. Consequently $E = ME$. Then Nakayama's graded analog Lemma (cf. [21, Lemma 4.2]) gives $E = 0$, the desired contradiction. □

Remark 4.5. Theorem 4.4 can be viewed as a graded analog of [13, Theorem 2.8]. However, it is important to note that the proof of [13, Theorem 2.8] depends on pre-existing results, among which [10, Theorem 3.1(b)] has been proven **incorrect**. To circumvent these issues, we opted not to rely on external work in our proof. Instead, all necessary results have been carefully examined and detailed in the preceding sections. This approach ensures the reliability and accuracy of our results without dependence on potentially problematic sources.

Corollary 4.6. *Let A be a graded integral domain. Then:*

(1) *Let E be a nonzero finitely generated graded A-module. Then $R := A \propto E$ is a gr-PVR if and only if A is a gr-field.*
(2) *$A \propto A$ is a gr-PVR if and only if A is a gr-field.*

Proof. Since (2) is a special case of (1), we only need to prove (1). If R is a gr-PVR, then Theorem 4.4(4) gives that the graded integral domain A is gr-local and its maximal homogeneous ideal M satisfies $M^2 = 0$, hence $M = 0$ and A is a gr-field. Conversely, if A is a gr-field (regardless of whether E is finitely generated as a graded A-module), then R is gr-local ring with a maximal homogenous ideal $M := 0 \propto E$. Since (R, M) is a gr-local ring with $M^2 = 0$, by applying Theorem 3.7(4) we can conclude that R is a gr-PVR. □

Recall from [6] that a graded ring R is said to be a graded finite-conductor ring if $(0 : a)$ and $Rb \cap Rc$ are finitely generated for every homogeneous elements a, b, c of R.

Proposition 4.7. *Let (A, M) be a gr-local ring, but not a gr-field, such that $M^2 = 0$. Let E be a non-zero graded A-module such that $ME = 0$ and $R := A \propto E$ be the graded trivial ring extension. Then:*

(1) *R is a gr-PVR.*
(2) *Suppose M is a finitely generated homogeneous ideal of A and E is a finitely generated graded A-module. Then R is a graded finite-conductor ring.*
(3) *Suppose that M is the only homogeneous prime ideal of A. Then $R_H = R$ is not a gr-valuation ring.*

Proof. The hypotheses ensure that $\mathrm{Ann}(E) \neq 0$. Thus, (1) follows from Theorem 4.4 (2).

(2) Since (A, M) is a gr-local ring, so is (R, N), where $N := M \propto E$. Moreover, N is a finitely generated homogeneous ideal of R, for if $M = \sum Am_i$ and $E = \sum Ae_j$, then $N = \sum_i R(m_i, 0) + \sum_j R(0, e_j)$. Since $M^2 = 0$ and $ME = 0$, it also follows that $N^2 = 0$. Hence, by [6, Proposition 9], R is a graded finite-conductor ring.

(3) Since $M \propto E$ is the only homogeneous prime ideal of R, it immediately follows that $R_H = R$. Now, suppose the statement fails. In other words, suppose R is a gr-valuation ring. Since A is not a gr-field, we conclude that $M \neq 0$. Pick a nonzero homogeneous element $m \in M$ and let v be a homogeneous element of E. Then either $(m,0)|(0,v)$ or $(0,v)|(0,m)$. As above, the first condition cannot hold (since $m \neq 0$). Therefore $(0,v) = (0,m)(c,f)$ for some $(c,f) \in R$. Thus $v = mc = 0$ and so $E = 0$, the desired contradiction. □

Next, we study the transfer of the gr-PVR to the graded amalgamation of rings. We start with the following lemma.

Lemma 4.8. *Let A and B be a pair of graded rings and $f : A \longrightarrow B$ be a graded ring homomorphism. Let J be a nonzero proper homogeneous ideal of B. If $A \bowtie^f J$ is a gr-PVR, then so are A and $f(A) + J$ and $J \subseteq$ h-Jac(B).*

Proof. If $A \bowtie^f J$ is a gr-PVR, then it follows from [17, Theorem 3.5(4)] that $\frac{A\bowtie^f J}{\{0\}\times J} \cong A$ and $\frac{A\bowtie^f J}{f^{-1}(J)\times\{0\}} \cong f(A)+J$. Thus, by Corollary 3.8, A and $f(A)+J$ are gr-PVRs. Moreover, since $A \bowtie^f J$ is gr-local, then $J \subseteq$ h-Jac(B) by [16, Theorem 4.1(4)]. □

Our next theorem studies the transfer of the gr-PVR to graded amalgamation of rings.

Theorem 4.9. *Let A and B be a pair of graded rings, $f : A \longrightarrow B$ be a graded ring homomorphism, and J be a nonzero homogeneous ideal of B such that $f^{-1}(J) \neq 0$. Then $R := A \bowtie^f J$ is a gr-PVR if and only if A is a gr-local ring with maximal homogeneous ideal M such that $J^2 = 0$, $f(M)J = 0$ and $M^2 = 0$.*

Proof. Suppose R is a gr-PVR. Then R is gr-local with maximal homogeneous ideal $M'^f_g = M \bowtie^f J$ such that M is the maximal homogeneous ideal of A. Let $0 \neq i \in f^{-1}(J)$ and $0 \neq j \in J$. Since $f^{-1}(J)$ (resp. J) is homogeneous, i (resp. j) can be assumed to be homogeneous, $(i,0)$ (resp. $(0,j)$) is a homogeneous element of $A \bowtie^f J$. Then $(i,0) \in (0,j)M'^f_g$ or $(0,j)M'^f_g \subseteq (i,0)R$. If $(i,0) = (0,j)(m, f(m)+k)$ for some $(m, f(m)+k) \in M'^f_g$, then $i = 0$, which is a contradiction. So for any homogeneous element $m \in M$ we have $(0,j)(m,f(m)) = (i,0)(a, f(a)+k)$ for some $(a, f(a)+k) \in R$. Also, for any homogeneous element $t \in J$ we have $(0,j)(0,t) = (i,0)(b, f(b)+r)$ for some $(b, f(b)+r) \in R$. So $f(m)j = 0$ and $jt = 0$. Hence $f(M)J = 0$ and $J^2 = 0$. To show that $M^2 = 0$, it suffices to prove that $mn = 0$ for all homogeneous elements $m, n \in M$, together with the hypothesis that R is a gr-PVR. Let $0 \neq j \in J \cap h(B)$ and $m \in h(M)$. We conclude that $(0,j) \in (m,f(m))M'^f_g$ or $(m,f(m))M'^f_g \subseteq (0,j)R$. If $(0,j) = (m, f(m)(n, f(n)+ k)$ for some $(n, f(n)+k) \in M'^f_g$, then $mn = 0$ and $j = f(mn) + f(m)k = 0$, which is a contradiction. Hence for any homogeneous element $n \in M$ we have

$(m, f(m))(n, f(n)) = (0, j)(a, f(a) + k)$ for some $(a, f(a) + k) \in R$. Hence $mn = 0$, which implies that $M^2 = 0$.

Conversely, suppose that A is a gr-local ring with maximal homogeneous ideal M such that $J^2 = 0$, $f(M)J = 0$, and $M^2 = 0$. To show that R is a gr-PVR, Theorem 3.7(2) reduces our task to showing that R is a gr-local ring with maximal homogeneous ideal M'^f_g and M'^f_g is strongly homogeneous prime. By [16, Theorem 4.1(4)], R is a gr-local ring with maximal homogeneous ideal M'^f_g. Let $(a, f(a) + i)$ and $(b, f(b) + j)$ be homogeneous elements in R. We may assume, without loss of generality, that $(a, f(a)+i)$ and $(b, f(b)+j)$ are in M'^f_g. So a and b are in $h(M)$. Then $(b, f(b)+j)M'^f_g = ((0, 0)) \subseteq (a, f(a)+i)R$. Therefore R is a gr-PVR. □

In light of Theorem 4.9, we have the following corollary.

Corollary 4.10. *Let A be a graded ring and I be a nonzero proper homogeneous ideal of A. Then $A \bowtie I$ is a gr-PVR if and only if A is a gr-local ring with maximal homogeneous ideal M such that $M^2 = 0$.*

Proof. If $B := A$ and $I = \bigoplus_{\alpha\in\Gamma} I_\alpha$ is a homogeneous ideal of A. Consider the identity map $f = id_A$. Then we have $f^{-1}(I) = I \neq 0$. Therefore, by Theorem 4.9, if $A \bowtie I = A \bowtie^f I$ is a gr-PVR, then A is a gr-local ring with maximal homogeneous ideal M and $M^2 = 0$. Conversely, suppose A is a gr-local ring with maximal homogeneous ideal M and $M^2 = 0$. Then it is easy to see that $I^2 = 0$ and $MI = 0$ since $I \subseteq M$. So $A \bowtie I$ is a gr-PVR. □

If A is a graded integral domain and J is a nonzero proper homogeneous ideal of B, then we have the following result.

Proposition 4.11. *Let A and B be graded rings, $f : A \longrightarrow B$ be a graded ring homomorphism, and J be a nonzero proper homogeneous ideal of B. Suppose A is a graded integral domain. Then $R = A \bowtie^f J$ is a gr-PVR if and only if the following conditions hold:*

(a) *$f^{-1}(J) = 0$, $J \subseteq$ h-Jac(B), and A is a gr-PVD with maximal homogeneous ideal M.*
(b) *$J = (f(a) + j)J$ for every homogeneous element $0 \neq a \in A$, $j \in J$ such that* $\deg(a) = \deg(j)$.
(c) *For any two homogeneous elements $i, j \in J, j(f(A)+J)$ and $i(f(M)+J)$ are comparable.*

Proof. Assume that R is a gr-PVR. Then Lemma 4.8 gives that $J \subseteq$ h-Jac(B), and A is a gr-PVD with maximal homogeneous ideal M. Hence, $M'^f_g := M \bowtie^f J$ is the maximal homogeneous ideal of R. Let $a \in f^{-1}(J)$ and $0 \neq j \in J$. Then a (resp. j) can be assumed to be homogeneous. Since R is a gr-PVR, we get $(a, 0)M'^f_g \subseteq (0, j)R$ or $(0, j) \subseteq (a, 0)M'^f_g$. If $(0, j) \subseteq (a, 0)M'^f_g$, then $j = 0$, which is impossible. Hence $(a, 0)(a, 0) = (0, j)(r, f(r) + k)$ for some $(r, f(r) + k) \in R$. This implies $a^2 = 0$ and thus $a = 0$, since A is a graded integral domain. Therefore, $f^{-1}(J) = 0$. Now let $0 \neq a \in A$, $j \in J$ be homogeneous elements such that $\deg(a) = \deg(j)$. Then $(a, f(a) + j)$ is a homogeneous element of R. Let t be a nonzero homogeneous element in J. Without loss of generality, we may assume that a is a nonunit homogeneous element. Hence $(a, f(a) + j)M_g'^f \subseteq (0, t)R$ or $(0, t)R \subseteq (a, f(a) + j)M_g'^f$. If $(a, f(a)+j)M_g'^f \subseteq (0, t)R$, then $a = 0$, which gives a contradiction. Hence there exists $(r, f(r) + k) \in R$ such that $(0, t) = (a, f(a) + j)(r, f(r) + k)$. Therefore $r = 0$ and $t = (f(a) + j)k$ and hence $J = (f(a) + j)J$. On the other hand, for any two homogeneous elements $i, j \in J$ we have $(0, i)M_g'^f \subseteq (0, j)R$ or $(0, j)R \subseteq (0, i)M_g'^f$. Therefore $i(f(M) + J) \subseteq j(f(A) + J)$ or $j(f(A) + J) \subseteq i(f(M) + J)$.

Conversely, since A is gr-local with maximal homogeneous ideal M and $J \subseteq$ h-Jac(B), it follows by [16, Theorem 4.1(4)] that $M_g'^f := M \bowtie^f J$ is the maximal homogeneous ideal of R. Let $(a, f(a) + i)$ and $(b, f(b) + j)$ be two homogeneous elements of R. Consider first that $aA \nsubseteq bM$. Since A is a gr-PVD, we have $bM \subseteq aA$. Hence if $m \in M$, there is $r \in A$ such that $bm = ar$. Since $a \neq 0$ and $\deg(a) = \deg(i)$, we have $J = (f(a) + i)J$. For $t \in J$, there is $k \in J$ such that $f(b)t + j(f(m) + t) = (f(a) + i)k + f(r)i$. Thus we have that $(b, f(b) + j)(m, f(m) + t) = (a, f(a) + i)(r, f(r) + k) \in (a, f(a) + i)R$. Now, suppose that $a = bm$ for some $m \in M$. If $b = 0$, then $a = 0$, and so it follows by (c) that $(0, j) \in (0, i)M_g'^f$ or $(0, i)M_g'^f \subseteq (0, j)R$. We can assume that $b \neq 0$. Then there exists $k \in J$ such that $i = jf(m) + (f(b) + j)k$. So we have that $(a, f(a) + i) = (b, f(b) + j)(m, f(m) + k) \in (b, f(b) + j)M_g'^f$. Thus R is a gr-PVR. $\square$

Remark 4.12. (1) It is easy to see that if $J = 0$, then $A \bowtie^f J \cong A$, and so $A \bowtie^f J$ is a gr-PVR if and only if so is A.

(2) If $J \neq 0$ and $A \bowtie^f J$ is a gr-PVR, then either A is a homogeneous total quotient ring or A is a graded integral domain. In fact, assume that $A \bowtie^f J$ is a gr-PVR. By Lemma 4.8, A is a gr-PVR with maximal homogeneous ideal M. If f is not injective, then $f^{-1}(J) \neq 0$. Hence, Theorem 4.9 implies that $M^2 = 0$ and so $A = A_H$.

Now, assume that f is injective such that A is not a homogeneous total quotient ring. Let a and b be nonzero homogeneous elements of A such that $ab = 0$ and a nonzero homogeneous element $j \in J$. Since $A \bowtie^f J$ is a gr-PVR, we get that $A \bowtie^f J$ is a gr-local ring with maximal homogeneous ideal $M \bowtie^f J$. Furthermore, we have $(0, j) \in (a, f(a))M \bowtie^f J$ or $(a, f(a))M \bowtie^f J \subseteq (0, j)A \bowtie^f J$. If $(a, f(a))M \bowtie^f J \subseteq (0, j)A \bowtie^f J$, then $(a, f(a))(d, f(d)) = (0, j)(t, f(t) + k)$ for some regular element $d \in M$ and $(t, f(t) + k) \in A \bowtie^f J$. Hence $ad = 0$ implies that $a = 0$, which gives a contradiction. Therefore $(0, j) = (a, f(a))(t, f(t) + k)$ for some $(t, f(t) + k) \in M \bowtie^f J$. Hence $at = 0$ and $j = f(a)(f(t) + k)$. Then $f(b)j = 0$ for every homogeneous element $j \in J$. By similar reasoning as above, we have $(0, j) \in (b, f(b))M \bowtie^f J$. Hence

$j = f(b)k$ for some $k \in J$ and thus $j = 0$, which is a contradiction. Therefore A is a graded integral domain.

Next, we give another result for the gr-PVR in graded amalgamation.

Theorem 4.13. *Let A and B be a pair of graded rings and $f : A \longrightarrow B$ be a graded ring homomorphism. Let J be a nonzero homogeneous ideal of B with no nontrivial homogeneous nilpotent. Let $R := A \bowtie^f J$. Then R is a gr-PVR if and only if f is injective, $f(A) \cap J = 0$, and $f(A) + J$ is a gr-PVR.*

Proof. Suppose R is a gr-PVR. Then R is gr-local with maximal homogeneous ideal $M_g'^f := M \bowtie^f J$ such that M is a maximal homogeneous ideal A. To prove that f is injective, it suffices to show that the restriction of f to A_α is injective for any $\alpha \in \Gamma$. Let i_α be nonzero homogeneous in $\operatorname{Ker}(f)$ and $0 \neq j_\beta \in h(J)$. Then $(i_\alpha, 0) \in (0, j_\beta)M_g'^f$ or $(0, j_\beta)M_g'^f \subseteq (i_\alpha, 0)R$. If $(i_\alpha, 0) \in (0, j_\beta)(r, f(r) + k)$ for some $(r, f(r) + k) \in M_g'^f$, then $i = 0$, which is a contradiction. If $(0, j_\beta)M_g'^f \subseteq (i_\alpha, 0)R$, then $(0, j_\beta)(0, j_\beta) = (i_\alpha, 0)(a, f(a)+t)$ for some $(a, f(a)+t) \in A \bowtie^f J$. Hence $j^2 = 0$ and so $j = 0$, a desired contradiction. Therefore $\operatorname{Ker}(f) = 0$ and so f injective. Now, let $0 \neq f(a) \in f(A) \cap J$ such that $a = \sum_{\alpha\in\Gamma} a_\alpha$. Then it is easy to see that $(a_\alpha, 0), (0, f(a_\alpha)) \in h(A \bowtie^f J)$ for each $\alpha \in \Gamma$. Hence $(a_\alpha, 0) \in (0, f(a_\alpha))M_g'^f$ or $(0, f(a_\alpha))M_g'^f \subseteq (a_\alpha, 0)R$. If $(a_\alpha, 0) = (0, f(a_\alpha))(r, f(r)+k)$ for some $(r, f(r)+k) \in M_g'^f$, then $a_\alpha = 0$ and so $f(a_\alpha) = 0$ for every $\alpha \in \Gamma$, which implies $f(a) = 0$. If $(0, f(a_\alpha))M_g'^f \subseteq (a_\alpha, 0)R$, then $(0, f(a_\alpha))(0, f(a_\alpha)) = (a_\alpha, 0)(b, f(b)+t)$ for some $(b, f(b)+t) \in R$. Hence $f(a_\alpha)^2 = 0$ and so $f(a) = 0$. In any case we have $f(a) = 0$, which is a contradiction. So $f(A) \cap J = 0$. By Lemma 4.8, $f(A) + J$ is a gr-PVR.

Conversely, since f is injective and $f(A) \cap J = 0$, we have $R \cong f(A) + J$. So R is a gr-PVR, since $f(A) + J$ is a gr-PVR. □

The following corollary is a direct consequence of Theorem 4.13, which examines the case of graded amalgamated duplication.

Corollary 4.14. *Let A be a graded integral domain and I be a proper homogeneous ideal of A. Then $A \bowtie I$ is a gr-PVR if and only if A is a gr-PVR and $I = 0$.*

Proof. Note that $A \bowtie I = A \bowtie^f I$, where $f = id : A \longrightarrow A$. Suppose that I is a nonzero homogeneous ideal of A. By Theorem 4.13 , if $A \bowtie I$ is a gr-PVR, then $f(A) \cap I = A \cap I = 0$. Hence $I = 0$, a contradiction. Therefore, Theorem 4.13 implies that $A \bowtie I$ is a gr-PVR if and only if A is a gr-PVR and $I = 0$. □

Example 4.15. Let K be a gr-field and E be a graded K-vector space. Set $A := K \propto E$. Then A is a gr-PVR by the proof of Corollary 4.6, with a maximal homogeneous ideal $M := 0 \propto E$ satisfying $M^2 = 0$. Now if we consider the graded amalgamation duplication $R = A \bowtie M$. So R is a gr-PVR according to Corollary 4.10. However R is not a gr-valuation ring. To address this, let a be a non-zero homogeneous element in M. As $R(0,a) = 0 \propto Aa$ and $R(a,0) = aA \propto aM = aA \propto 0$. We get $(0,a) \notin R(a,0)$ and $(a,0) \notin R(0,a)$.

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