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DIFFERENT CHARACTERIZATIONS OF CURVATURE IN THE CONTEXT OF LIE ALGEBROIDS

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Abstract. We consider a vector bundle map $F\colon E_1 \longrightarrow E_2$ between Lie algebroids E_1 and E_2 over arbitrary bases M_1 and M_2. We associate to it different notions of curvature which we call A-curvature, Q-curvature, P-curvature, and S-curvature using the different characterizations of Lie algebroid structure, namely Lie algebroid, Q-manifold, Poisson and Schouten structures. We will see that these curvatures generalize the ordinary notion of curvature defined for a vector bundle, and we will prove that these curvatures are equivalent, in the sense that F is a morphism of Lie algebroids if and only if one (and hence all) of these curvatures is null. In particular we get as a corollary that F is a morphism of Lie algebroids if and only if the corresponding map is a morphism of Poisson manifolds (resp. Schouten supermanifolds).

1. Introduction

The definition of morphism of Lie algebroids over arbitrary bases was introduced by Higgins and Mackenzie [4] in 1990, and the justification for the definition was considerably involved and elaborate (see as well [6, Section 4.3]). Let us consider a vector bundle map $(F, f)\colon (E_1, M_1, \pi_1, a_1, [,]_1) \to (E_2, M_2, \pi_2, a_2, [,]_2)$ between Lie algebroids E_1 and E_2 over arbitrary bases M_1 and M_2 and with anchors a_1 and a_2, respectively. The difficulty of finding the right definition of morphism of Lie algebroids is due to the fact that the map f does not necessarily induce a map of sections from E_1 to E_2, unlike in the case when both Lie algebroids are over a same manifold M and f is the identity map (more details can be found in [6]). In this paper we review the definition of morphism of Lie algebroids over arbitrary bases: We motivate the definition of morphism of Lie algebroids by considering first a diffeomorphism f between the base manifolds M_1 and M_2, and this will induce a map on the sections. After that we can weaken the condition of f being a diffeomorphism

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and this will enable us to extend the definition naturally to the case when f is an arbitrary map.

It is known that a Lie algebroid structure on a vector bundle E is equivalent to one of the following equivalent formulations (see for example [7, 10]):

- A Q-manifold structure on ΠE, where $\Pi E = (M, \Gamma\Lambda(E^*))$ with $\Lambda(E^*)$ being the exterior bundle of E^*.
- A Poisson bracket on $C^\infty(E^*)$, where E^* is the dual vector bundle of E.
- A Schouten bracket on $C^\infty(\Pi E^*)$, where $\Pi E^* = (M, \Gamma\Lambda(E))$ with $\Lambda(E)$ being the exterior bundle of E.

Given a vector bundle morphism $F : E_1 \to E_2$, where E_1 and E_2 are Lie algebroids over bases M_1 and M_2, then we get the corresponding maps $F^\Pi \colon \Pi E_1 \longrightarrow \Pi E_2$, $H \colon E_2^* \longrightarrow E_1^*$ and $T \colon \Pi E_2^* \longrightarrow \Pi E_1^*$ between Q-manifolds, Poisson manifolds and Schouten supermanifolds, respectively. A theorem of Vaintrob [7] states that F is a morphism of Lie algebroids if and only if F^Π is a morphism of Q-manifolds. We get corresponding results considering the Poisson and Schouten structures under the stronger condition of $f : M_1 \to M_2$ being a diffeomorphism. This means that the map F is a morphism of Lie algebroids if and only if H is a morphism of Poisson manifolds (likewise, if and only if T is a morphism of Schouten supermanifolds). And as far as we know this has not been discussed in the literature.

To this end and using the different characterizations of Lie algebroid structure, we introduce different notions of curvature which we call A-curvature, Q-curvature, P-curvature and S-curvature. Any of these curvatures is defined to be the failure of the corresponding map to be a morphism of the corresponding structure, for example P-curvature will represent the failure of the map $H \colon E_2^* \longrightarrow E_1^*$ to be a morphism of Poisson manifolds. Hence any of these curvatures is null if and only if the corresponding map is a morphism with respect to the corresponding structure. We will prove that these curvatures are equivalent, in the sense that the vector bundle map F is a morphism of Lie algebroids if and only if one (and hence all) of these curvatures is null.

A Koszul connection $\nabla \colon \mathfrak{X}(M) \times \Gamma E \longrightarrow \Gamma E$ on a vector bundle E can be reinterpreted as a vector bundle map $\widehat{\nabla} \colon TM \longrightarrow \mathfrak{D}(E)$ between the Lie algebroids TM and $\mathfrak{D}(E)$, where $\mathfrak{D}(E)$ is the Lie algebroid of derivations on E (see for instance [6, Section 5.2]). The details will be discussed in Example 4.5. On the other hand, if E is a Lie algebroid with anchor a, then a *Lie algebroid connection* is defined to be a vector bundle map $s \colon TM \longrightarrow E$ such that $a \circ s = \mathrm{id}_{TM}$, which can be seen to extend the concept of Koszul connection on a vector bundle (see for example [6]); and moreover a Lie algebroid connection is flat (i.e. the curvature is null) if and only if it is a morphism of Lie algebroids. In this paper instead of considering a Lie algebroid connection $s \colon TM \longrightarrow E$, we consider a vector bundle map $F \colon E_1 \longrightarrow E_2$ between Lie algebroids E_1 and E_2 over arbitrary bases M_1 and M_2, and thus the curvatures mentioned earlier (i.e. A-curvature, Q-curvature, P-curvature and S-curvature) associated with F will extend the ordinary notion of curvature defined for a vector bundle.

The paper is organized as follows: In Section 2, we review the definition of morphism of Lie algebroids and give a natural motivation of the definition in the case when we have Lie algebroids over arbitrary bases. In Section 3, we consider a vector bundle map $F : E_1 \to E_2$, where E_1 and E_2 are Lie algebroids over bases M_1 and M_2 and treat the different curvatures A-curvature, Q-curvature, P-curvature and S-curvature, and show that the vector bundle map F is a morphism Lie algebroids if and only if one of these curvatures is null. In particular we get that F is a morphism of Lie algebroids if and only if the corresponding map is a morphism of Poisson manifolds (resp. Schouten supermanifolds). In the appendix we give some background material on the theory of supermanifolds necessary for understanding the paper.

2. Morphism of Lie algebroids

Definition. A *Lie algebroid* on a manifold M is a vector bundle (E, M, π) with a vector bundle map $a\colon E \longrightarrow TM$ over M, and a Lie bracket $[\ ,\ ] : \Gamma E \times \Gamma E \longrightarrow \Gamma E$ such that

(1) $[s, ft] = f\,[s,t] + a\,(s)\,(f)\,t$,
(2) $a\,[s,t] = [a\,(s)\,,a\,(t)]$ for all $s, t \in \Gamma E$ and $f \in C^\infty(M)$.

The map a is called the *anchor* of the Lie algebroid E. The Lie algebroid will be denoted by $(E, M, \pi, a, [,])$. We use as well the notation $[s, h] = a(s)(h)$, for $s \in \Gamma E$ and $h \in C^\infty(M)$.

In what follows we consider two Lie algebroids $(E_1, M_1, \pi_1, a_1, [,]_1)$ and $(E_2, M_2, \pi_2, a_2, [,]_2)$, and let

$$(F, f)\colon\ (E_1, M_1, \pi_1, a_1, [,]_1) \longrightarrow (E_2, M_2, \pi_2, a_2, [,]_2)$$

be a vector bundle morphism.

Suppose that r is a section of E_1 and that there are sections v_α of E_2, and functions $r^\alpha \in C^\infty(M_1)$ such that $F \circ r = \sum_\alpha r^\alpha f^*(v_\alpha)$. Then the decomposition $\sum_\alpha r^\alpha f^*(v_\alpha)$ is called an *F-decomposition* for r. An F-decomposition always exists and is not necessarily unique (see for example [4, Section 1] and [1]). However it is easy to see that a local F-decomposition for a section $r \in \Gamma E_1$ exists, that is there exist open neighbourhoods U_1 and U_2 of M_1 and M_2 respectively and local sections $v_\alpha \in \Gamma E_2|_{U_2}$ and functions $r^\alpha \in C^\infty(U_1)$ such that $F \circ r|_{U_1} = \sum_\alpha r^\alpha f^*(v_\alpha)$. To see this suppose that (e_i) is a local basis of ΓE_1 on an open neighbourhood U_1 and (v_α) a local basis of ΓE_2 on an open neighbourhood U_2 such that $f(U_1) \subseteq U_2$ and that $r|_{U_1} = \sum_i s^i e_i$. Then

$$F \circ r|_{U_1} = \sum_{i,\alpha} s^i F_i^\alpha f^*(v_\alpha), \tag{1}$$

for some functions $F_i^\alpha \in C^\infty(U_1)$. Since a Lie bracket is local, that is, its value at a point x depends only on an open neighbourhood of x, it is sufficient to consider local F-decompositions.

We suppose that the map $f\colon M_1 \longrightarrow M_2$ is a diffeomorphism with inverse g. We define the map $\bar{F}\colon C^\infty(M_1) \cup \Gamma E_1 \longrightarrow C^\infty(M_2) \cup \Gamma E_2$ by

$$\bar{F}(h) = g^*(h) \qquad \text{and} \qquad \bar{F}(r) = F \circ r \circ g$$

for $h \in C^\infty(M_1)$, and $r \in \Gamma E_1$. Therefore the map $\bar{F}$ gives us a one-to-one correspondence between $C^\infty(M_1)$ and $C^\infty(M_2)$ and between ΓE_1 and ΓE_2.

We define the map $R_1\colon \Gamma E_1 \times C^\infty(M_1) \longrightarrow C^\infty(M_1)$ by

$$R_1(r, h) = f^* \bar{F}[r, h]_1 - f^*[\bar{F}(r), \bar{F}(h)]_2, \tag{2}$$

and therefore if $F \circ r = \sum_\alpha r^\alpha f^*(v_\alpha)$, then we have

$$R_1(r, h) = [r, h]_1 - \sum_i r^\alpha f^*[v_\alpha, g^* h]_2. \tag{3}$$

We define as well the map $\Delta\colon \Gamma E_1 \times C^\infty(M_2) \longrightarrow C^\infty(M_1)$ by

$$\Delta(r, h) = [r, f^* h]_1 - \sum_\alpha r^\alpha f^*[v_\alpha, h]_2. \tag{4}$$

As can be seen the definition of Δ is valid even when the map f is not a diffeomorphism, and that when f is a diffeomorphism we have $\Delta(r, h) = R_1(r, f^* h)$. When both Δ and R_1 are defined we have $\Delta = 0$ if and only if $R_1 = 0$, and any of the equations $\Delta = 0$ and $R_1 = 0$ is equivalent to the *anchor preservation* condition $a_2 \circ F = \mathrm{d}f \circ a_1$, which can be verified without difficulty. We could have defined only the map Δ, but for the convenience of presentation we introduced the map R_1 as well.

For $r, z \in \Gamma E_1$, suppose that $F \circ r = \sum_\alpha r^\alpha f^*(v_\alpha)$, and $F \circ z = \sum_\beta z^\beta f^*(w_\beta)$, for some functions r^α, z^β in $C^\infty(M_1)$, and some sections v_α, w_β in ΓE_2. Since $\bar{F}(r) = \sum_\alpha g^*(r^\alpha) v_\alpha$ and $\bar{F}(z) = \sum_\beta g^*(z^\beta) w_\beta$, we have

$$\begin{aligned}
[\bar{F}(r), \bar{F}(z)]_2 = \sum_{\alpha,\beta} \big(& g^*(r^\alpha) g^*(z^\beta)[v_\alpha, w_\beta]_2 + g^*(r^\alpha)[v_\alpha, g^*(z^\beta)]_2 w_\beta \\
& - g^*(z^\beta)[w_\beta, g^*(r^\alpha)]_2 v_\alpha \big),
\end{aligned} \tag{5}$$

and this gives

$$\begin{aligned}
f^* \left[\bar{F}(r), \bar{F}(z)\right]_2 = \sum_{\alpha,\beta} \big(& r^\alpha z^\beta f^*[v_\alpha, w_\beta]_2 + f^*[g^*(r^\alpha) v_\alpha, g^*(z^\beta)]_2 f^*(w_\beta) \\
& - f^*[g^*(z^\beta) w_\beta, g^*(r^\alpha)]_2 f^*(v_\alpha) \big).
\end{aligned} \tag{6}$$

On the other hand we have

$$\begin{aligned}
\sum_\alpha f^* \left[g^*(r^\alpha) v_\alpha, g^*(z^\beta)\right]_2 &= f^*[g^*(F \circ r), g^*(z^\beta)]_2 \\
&= [r, z^\beta]_1 - R_1(r, z^\beta),
\end{aligned}$$

and therefore (6) becomes

$$
\begin{aligned}
&f^*[\bar{F}(r), \bar{F}(z)]_2 \\
&= \sum_{\alpha,\beta} r^\alpha z^\beta f^*[v_\alpha, w_\beta]_2 + \sum_\beta [r, z^\beta]_1 f^*(w_\beta) - \sum_\alpha [z, r^\alpha]_1 f^*(v_\alpha) \\
&\quad + \sum_\alpha R_1(z, r^\alpha) f^*(v_\alpha) - \sum_\beta R_1(r, z^\beta) f^*(w_\beta).
\end{aligned} \tag{7}
$$

Then we will say that (F, f) is a morphism of Lie algebroids if $\bar{F}[r, h]_1 = [\bar{F}(r), \bar{F}(h)]_2$ and $\bar{F}[r, z]_1 = [\bar{F}(r), \bar{F}(z)]_2$, for all $r, z \in \Gamma E_1$ and all $h \in C^\infty(M_1)$. This is equivalent to requiring that

(1) $\Delta = 0$ and
(2) $F \circ [r, z]_1 = \sum_{\alpha,\beta} r^\alpha z^\beta f^*[v_\alpha, w_\beta]_2 + \sum_j [r, z^\beta]_1 f^*(w_\beta) - \sum_\alpha [z, r^\alpha]_1 f^*(v_\alpha)$,

which represent the anchor preservation and the *bracket preservation* conditions respectively.

As can be seen the first and second conditions do not involve g, and therefore they can be extended as a definition of morphism of Lie algebroids in the case when f is arbitrary, i.e. not necessarily a diffeomorphism. However in this case we need to make sure that this definition does not depend on the F-decompositions of the sections r and z. And now we are ready to give the definition of morphism of Lie algebroids in the general case, i.e. when the map f is arbitrary.

Definition. Let $(E_1, M_1, \pi_1, a_1, [,]_1)$ and $(E_2, M_2, \pi_2, a_2, [,]_2)$ be Lie algebroids and $(F, f)\colon (E_1, M_1, \pi_1, a_1, [,]_1) \to (E_2, M_2, \pi_2, a_2, [,]_2)$ a vector bundle morphism. Then we say that (F, f) is a *Lie algebroid morphism* if

(1) $a_2 \circ F = \mathrm{d} f \circ a_1$, and
(2) $F \circ [r, z]_1 = \sum_{\alpha,\beta} r^\alpha z^\beta f^*[v_i, w_\beta]_2 + \sum_\beta [r, z^\beta]_1 f^*(w_\beta) - \sum_\alpha [z, r^\alpha]_1 f^*(v_\alpha)$,

whenever $F \circ r = \sum_i r^\alpha f^*(v_\alpha)$, and $F \circ z = \sum_\beta z^\beta f^*(w_\beta)$.

Here we need to check that the second condition does not depend on the F-decompositions of the sections r and z. For $r \in \Gamma E_1$, let

$$
\begin{aligned}
\mathcal{K}(z) = \{&(z^\beta, w_\beta)_\beta \colon \beta = 1, 2, \ldots, n \text{ for some } n,\ z^\beta \in C^\infty(M_1), \\
&w_\beta \in \Gamma E_2,\ F \circ z = \sum_\beta z^\beta f^*(w_\beta)\}.
\end{aligned} \tag{8}
$$

Hence $\mathcal{K}(z)$ represents the set of all F-decompositions of the section $z \in \Gamma E_1$.

Let $(r^\alpha, v_\alpha)_\alpha \in \mathcal{K}(z)$, that is, we fix some F-decomposition of the section r, and consider the map $N\colon \mathcal{K}(z) \longrightarrow \Gamma f^*E_2$ given by

$$\begin{aligned} N\left((z^\beta, w_\beta)_\beta\right) = & \sum_{\alpha,\beta} r^\alpha z^\beta f^*[v_\alpha, w_\beta]_2 + \sum_\beta [r, z^\beta]_1 f^*(w_\beta) \\ & - \sum_\alpha [z, r^\alpha]_1 f^*(v_\alpha). \end{aligned} \tag{9}$$

By a direct computation we get

$$N((z^\beta, h^\beta w_\beta)_\beta) = N((z^\beta f^*(h^\beta), w_\beta)_\beta) - \sum_\beta z^\beta \Delta(r, h^\beta) f^*(w_\beta). \tag{10}$$

Suppose that $F \circ z = \sum_\beta z^\beta f^*(w_\beta) = \sum_\beta \bar{z}^\beta f^*(\bar{w}_\beta)$, and let (ε_γ) be a local basis of ΓE_2. Suppose that $w_\beta = \sum_\gamma m_{\beta\gamma}\varepsilon_\gamma$ and $\bar{w}_\beta = \sum_\gamma \bar{m}_{\beta\gamma}\varepsilon_\gamma$. Therefore $F \circ z = \sum_{\beta,\gamma} z^\beta f^*(m_{\beta\gamma}) f^*(\varepsilon_\gamma) = \sum_{\beta,\gamma} \bar{z}^\beta f^*(\bar{m}_{\beta\gamma}) f^*(\varepsilon_\gamma)$.

Given that $\Delta = 0$ since $a_2 \circ F = \mathrm{d} f \circ a_1$, therefore we get

$$\begin{aligned} N((z^\beta, w_\beta)_\beta) &= N\left((z^\beta, \sum_\gamma m_{\beta\gamma}\varepsilon_\gamma)_\beta\right) \\ &= \sum_\gamma N\left((z^\beta, m_{\beta\gamma}\varepsilon_\gamma)_\beta\right) \\ &= N\left(\left(\sum_\gamma z^\beta f^*(m_{\beta\gamma}), \varepsilon_\gamma\right)_\beta\right) - \sum_{\beta,\gamma} z^\beta \Delta(r, m_{\beta\gamma}) f^*(\varepsilon_\gamma) \\ &= N((\bar{z}^\beta, \bar{w}_\beta)_\beta). \end{aligned}$$

We conclude that $N((\bar{z}^\beta, \bar{w}_\beta)_\beta) = N((z^\beta, w_\beta)_\beta)$, that is the right-hand side of the second condition does not depend on the F-decompositions of the section z, and since the expression $\sum_{\alpha,\beta} r^\alpha z^\beta f^*[v_\alpha, w_\beta]_2 + \sum_\beta [r, z^\beta]_1 f^*(w_\beta) - \sum_i [z, r^\alpha]_1 f^*(v_\alpha)$ is antisymmetric, likewise it does not depend on the F-decompositions of the section r, and therefore the definition is good.

3. Derivations along a map of supermanifolds

We state below two propositions that will be needed in what follows. Some necessary background material about supermanifolds is given in the appendix.

Definition (see [2, Definition 1.2])**.** If $f\colon \mathcal{M} \longrightarrow \mathcal{N}$ is a map between supermanifolds, then a *homogeneous derivation along* f (or simply a *homogeneous* f*-derivation*) is a homogeneous $\mathbb{R}$-linear map $D\colon C^\infty(\mathcal{N}) \longrightarrow C^\infty(\mathcal{M})$ such that

$$D(gh) = D(g) f^*(h) + (-1)^{\widetilde{D}\widetilde{g}} f^*(g) D(h),$$

for all homogeneous g, $h \in C^\infty(\mathcal{N})$. If $\widetilde{D} = 0$ then D is said to be an *even* f*-derivation*, and if $\widetilde{D} = 1$ then D is said to be an *odd* f*-derivation.* An (arbitrary) f*-derivation* is a sum of an even f-derivation and an odd one. As can be noticed when f is the identity map then D is just an ordinary derivation (or a vector field). As in the case of derivations the following proposition gives us a general representation of an f-derivation in a local coordinate system.

Proposition 3.1 (see [2, Proposition 2.1])**.** *Suppose that* $f\colon \mathcal{M} \longrightarrow \mathcal{N}$ *is a map between supermanifolds, and* $D\colon C^\infty(\mathcal{N}) \longrightarrow C^\infty(\mathcal{M})$ *an* f-derivation. *Then locally*

$$D = \sum_A D^A f^* \circ \partial_{y^A}$$

for some functions $D^A \in C^\infty(\mathcal{M})$*, where* (y^A) *are local coordinates on* $\mathcal{N}$*.*

Proof. The proposition was given in [2] without proof, we shall provide one in what follows.

It is sufficient to prove that if $D(y^A) = 0$ for all A where (y^A) are local coordinates on $\mathcal{N}$, then D is identically 0. To see this suppose that the previous supposition is valid and consider the map $X = D - \sum_A D(y^A) f^* \circ \partial_{y^A}$ which is an f-derivation and satisfies $X(y^A) = 0$ for all A, and therefore $X = 0$ by our assumption, that is $D = \sum_A D(y^A) f^* \circ \partial_{y^A}$.

Before proceeding to the rest of the proof we need the following lemma.

Lemma 3.2 (**The generalized Hadamard lemma**)**.** *Suppose that* $g\colon V \longrightarrow \mathbb{R}$ *is a smooth function, where* V *is an open convex subset of* $\mathbb{R}^n$ *containing* b*. Then there are real numbers* a_μ *and smooth functions* $g_\mu\colon V \longrightarrow \mathbb{R}$ *such that*

$$g(y) = \sum_{|\mu| \le k} a_\mu (y-b)^\mu + \sum_{|\mu|=k+1} g_\mu(y)(y-b)^\mu,$$

where $y = (y^1, y^2, \ldots, y^n)$*,* $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$*,* $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_n$*,* $y^\mu = (y^1)^{\mu_1}(y^2)^{\mu_2}\cdots(y^n)^{\mu_n}, \mu_\alpha \in \{0, 1, \ldots, k\}$*.*

In the case when $k = 0$ the lemma is just the (classical) Hadamard lemma, i.e. for a smooth function $g\colon V \longrightarrow \mathbb{R}$ where V is an open subset of $\mathbb{R}^n$, there are smooth functions $g_\alpha\colon V \longrightarrow \mathbb{R}$ such that

$$g(y) = g(b) + \sum_{\alpha=1}^{n} g_\alpha(y)(y^\alpha - b^\alpha).$$

The generalized Hadamard lemma can be obtained by repeatedly applying the Hadamard lemma to the functions g_μ.

To finish the proof of the proposition, suppose that D is a homogeneous f-derivation, and hence

$$D(gh) = D(g) f^*(h) + (-1)^{\widetilde{D}\widetilde{g}} f^*(g) D(h).$$

Let U be an open neighbourhood U of $\mathcal{M}$ with coordinates (x^i, θ^j) and V an open neighbourhood of $\mathcal{N}$ with coordinates (y^α, η^β) such that $f(U) \subseteq V$, with $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, k$, $\alpha = 1, 2, \ldots, n$ and $\beta = 1, 2, \ldots, p$. Without loss of generality we can assume that U is an open neighbourhood of $\mathbb{R}^m$ and V a convex open neighbourhood of $\mathbb{R}^n$. Let $a \in U$ and $b = f(a)$.

We claim that if $D\left(y^{\alpha}\right)=0$ and $D\left(\eta^{\beta}\right)=0$ for all α, β, then $D=0$. First let $g\in C^{\infty}(V)$ and suppose that $D(g)=\sum_I h_I\theta^I$ for some functions $h_I\in C^{\infty}(U)$ (here $I=(s_1,s_2,\dots,s_k)\in\{0,1\}^k$). By the generalized Hadamard lemma there are functions g_μ such that

$$g(y)=\sum_{|\mu|\leq k}a_\mu(y-b)^\mu+\sum_{|\mu|=k+1}g_\mu(y)(y-b)^\mu.$$

Since $D(y^\alpha)=0$, then $D\left((y-b)^\mu\right)=0$ and therefore

$$\begin{aligned}D(g)&=D\left(\sum_{|\mu|\leq k}a_\mu(y-b)^\mu+\sum_{|\mu|=k+1}g_\mu(y)(y-b)^\mu\right)\\&=\sum_{|\mu|=k+1}D\left(g_\mu\right)\left(f^*(y-b)\right)^\mu\\&=\sum_{|\mu|=k+1}D\left(g_\mu\right)\prod_{\alpha=1}^{n}\left(f^*(y^\alpha-b^\alpha)\right)^{\mu_\alpha}.\qquad(11)\end{aligned}$$

Let $f^*\left(y^\alpha-b^\alpha\right)=r_\alpha+n_\alpha$ where $r_\alpha\in C^{\infty}(U)$ and n_α is the nilpotent (even) part. Therefore we have

$$\left(f^*(y^\alpha-b^\alpha)\right)^{\mu_\alpha}=\left(r_\alpha+n_\alpha\right)^{\mu_\alpha}=r_\alpha\ell_\alpha+n_\alpha^{\mu_\alpha}$$

for some function $\ell_\alpha\in C^{\infty}(\mathcal{M}|_U)$. Using this and (11) we get

$$D(g)=\sum_I h_I\theta^I=\sum_{|\mu|=k+1}D\left(g_\mu\right)\prod_{\alpha=1}^{n}\left(r_\alpha\ell_\alpha+n_\alpha^{\mu_\alpha}\right).$$

Since $v_a\left[f^*\left(y^\alpha-b^\alpha\right)\right]=0$, we have $r_\alpha(a)=0$. We define the *degree* of a monomial $z_I\theta^I$, with $z_I\in C^{\infty}(U)$ and $z_I\neq 0$, to be $|I|$; and we convene that 0 is of degree $k+1$. We can see that every monomial in $\sum_{|\mu|=k+1}D\left(g_\mu\right)\prod_{\alpha=1}^{n}(r_\alpha\ell_\alpha$ $+n_\alpha^{\mu_\alpha})$ has a factor r_α or is of degree $|\mu|=k+1$ (but a monomial of degree $k+1$ is equal to 0). Hence the monomial $h_I\theta^I$ has a factor r_α, and since r_α is a factor of h_I and $r_\alpha(a)=0$, we get $h_I(a)=0$. Since a is arbitrary, we conclude that $h_I=0$ for all I and therefore $D(g)=0$. If $g\in C^{\infty}(\mathcal{N}|_V)$, then $g=\sum_J g_J\eta^J$ for some functions $g_J\in C^{\infty}(V)$ (here $J=(t_1,t_2,\dots,t_p)\in\{0,1\}^p$). Therefore

$$D(g)=\sum_J D(g_J)f^*(\eta^J)=0.$$

Since $D(g_J)=0$ by the first part, and since this is true for all g then we conclude that $D=0$. If D is not necessarily homogeneous, then just apply the preceding argument to the homogeneous parts, and this completes the proof. □

Proposition 3.3. *Let $\varepsilon\in\{0,1\}$ and consider a map $m\colon\mathcal{M}_1\longrightarrow\mathcal{M}_2$ with coordinates (x^A) on $\mathcal{M}_1$ and coordinates (y^B) on $\mathcal{M}_2$, and suppose that we have an $\mathbb{R}$-bilinear map $W\colon C^{\infty}\left(\mathcal{M}_2\right)\times C^{\infty}\left(\mathcal{M}_2\right)\longrightarrow C^{\infty}\left(\mathcal{M}_1\right)$ such that for homogeneous functions,*

- $W(f,g) = -(-1)^{(\widetilde{f}+\varepsilon)(\widetilde{g}+\varepsilon)} W(g,f)$,
- $W(f,gh) = W(f,g)m^*(h) + (-1)^{(\widetilde{f}+\varepsilon)\widetilde{g}} m^*(g)W(f,h)$,
- $\widetilde{W(f,g)} = \widetilde{f} + \widetilde{g} + \varepsilon$.

Then W is locally given by

$$W(f,g) = \sum_{A,B} (-1)^{(\widetilde{A}+\widetilde{f})(\widetilde{B}+\varepsilon)} W\left(y^A, y^B\right) m^* \left(\frac{\partial f}{\partial y^A}\right) \; m^* \left(\frac{\partial g}{\partial y^B}\right),$$

where $\widetilde{A} = \widetilde{y^A}$.

Proof. We can see that

$$W(fg,h) = m^*(f)W(g,h) + (-1)^{\widetilde{f}\widetilde{g}} m^*(g)W(f,h)$$

for homogeneous functions $f, g, h \in C^\infty(\mathcal{M}_2)$.

For $f \in C^\infty(\mathcal{M}_2)$ let $N(f)\colon C^\infty(\mathcal{M}_2) \longrightarrow C^\infty(\mathcal{M}_1)$ be the map defined by $N(f)(g) = W(f,g)$. Therefore

$$N(f)(gh) = N(f)(g)m^*(h) + (-1)^{\widetilde{N(f)}\widetilde{g}} m^*(g)N(f)(h).$$

Hence $N(f)$ is an m-derivation, and therefore by Proposition 3.1 we have

$$N(f)(g) = \sum_B N^B(f) m^* \left(\frac{\partial g}{\partial y^B}\right), \tag{12}$$

where $N^B(f) = N(f)(y^B)$.

On the other hand we have

$$N^B(fg) = m^*(f)N^B(g) + (-1)^{\widetilde{f}\widetilde{g}} m^*(g)N^B(f),$$

with $\widetilde{N^B} = \widetilde{B} + \varepsilon$. If we let $\mathcal{N}^B(f) = (-1)^{(\widetilde{B}+\varepsilon)\widetilde{f}} N^B(f)$ then $\mathcal{N}^B$ is an m-derivation, and therefore, by Proposition 3.1, $\mathcal{N}^B(f) = \sum_A \mathcal{N}^{AB} m^* \left(\frac{\partial f}{\partial y^A}\right)$, with $\mathcal{N}^{AB} = \mathcal{N}^B(y^A)$. Therefore we get

$$N^B(f) = \sum_A (-1)^{(\widetilde{A}+f)(\widetilde{B}+\varepsilon)} W\left(y^A, y^B\right) m^* \left(\frac{\partial f}{\partial y^A}\right). \tag{13}$$

By (12) and (13) we conclude that

$$W(f,g) = \sum_{A,B} (-1)^{(\widetilde{A}+\widetilde{f})(\widetilde{B}+\varepsilon)} W\left(y^A, y^B\right) m^* \left(\frac{\partial f}{\partial y^A}\right) m^* \left(\frac{\partial g}{\partial y^B}\right). \qquad \square$$

This proposition will be used later to express Q-curvature, P-curvature and S-curvature in a general local representation.

4. Curvature

Definition. Let $\varepsilon \in \{0,1\}$ and $\mathcal{M}$ be a supermanifold with an $\mathbb{R}$-bilinear bracket $\{\ ,\ \}\colon C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \longrightarrow C^\infty(\mathcal{M})$ such that, for homogeneous functions,

- $\widetilde{\{f,g\}} = \widetilde{f} + \widetilde{g} + \varepsilon$,
- $\{f,g\} = -(-1)^{(\widetilde{f}+\varepsilon)(\widetilde{g}+\varepsilon)}\{g,f\}$,
- $\{f,\{g,h\}\} = \{\{f,g\},h\} + (-1)^{(\widetilde{f}+\varepsilon)(\widetilde{g}+\varepsilon)}\{g,\{f,h\}\}$,
- $\{f,gh\} = \{f,g\}h + (-1)^{(\widetilde{f}+\varepsilon)\widetilde{g}}g\{f,h\}$.

When $\varepsilon = 0$, then we say that $\mathcal{M}$ is a *Poisson supermanifold* (hence we have a Poisson bracket), and when $\varepsilon = 1$, then we say that $\mathcal{M}$ is an *odd Poisson* (or a *Schouten*) *supermanifold* (hence we have a Schouten bracket). For more details, see [9].

Here $\widetilde{f}$ denotes the parity of the function f. The third and fourth conditions in the definition are called the *Jacobi identity* and the *Leibniz rule* respectively.

Definition. A *Q-manifold* is a supermanifold $\mathcal{M}$ with a *homological vector field* Q, i.e., a vector field Q with $[Q,Q]=0$ and parity $\widetilde{Q}=1$. A Q-manifold will be denoted by $(\mathcal{M},Q)$, and a *morphism* between Q-manifolds $(\mathcal{M}_1,Q_1)$ and $(\mathcal{M}_2,Q_2)$ is defined to be a map $F\colon \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ such that Q_1 and Q_2 are *F-related*, i.e., $Q_1 \circ F^* = F^* \circ Q_2$.

Let $(E,M,\pi,a,[,])$ be a Lie algebroid, and let ΠE and ΠE^* be the supermanifolds given respectively by $\Pi E = (M,\Gamma\Lambda(E^*))$ and $\Pi E^* = (M,\Gamma\Lambda(E))$, where $\Lambda(E^*)$ and $\Lambda(E)$ are the exterior bundles of E^* and E respectively. Therefore we have the global sections $C^\infty(\Pi E^*) = \Gamma\Lambda(E)$ and $C^\infty(\Pi E) = \Gamma\Lambda(E^*)$. The symbol Π is called the *parity reversal functor* and this because what it does can be interpreted as changing the parity of the fibre coordinates.

Let (e_i) be a local basis of ΓE, let (x^d) be coordinates on the base M, (ξ^i) be coordinates on ΠE, (p_i) coordinates on E^*, and (π_i) coordinates on ΠE^* corresponding to the local sections (e_i). Let

$$[e_i,e_j] = \sum_k c_{ij}^k e_k, \qquad a(e_i) = \sum_d a_i^d \frac{\partial}{\partial x^d}.$$

Then the corresponding homological vector field on ΠE is given locally (see [10, p. 284]) by

$$Q = \sum_{i,d} \xi^i a_i^d \frac{\partial}{\partial x^d} - \frac{1}{2}\sum_{i,j,k} \xi^i \xi^j c_{ij}^k \frac{\partial}{\partial \xi^k}.$$

The Poisson bracket on $C^\infty(E^*)$ is given locally (see [10, p. 285]) by

$$\{x^a,x^b\} = 0, \qquad \{p_i,x^d\} = a_i^d, \qquad \{p_i,p_j\} = \sum_k c_{ij}^k p_k.$$

The Schouten bracket on $C^\infty(\Pi E^*)$ is given locally (see [10, p. 285]) by

$$\{x^a, x^b\} = 0, \qquad \{\pi_i, x^d\} = a_i^d, \qquad \{\pi_i, \pi_j\} = \sum_k c_{ij}^k \pi_k.$$

In this section we introduce A-curvature, Q-curvature, P-curvature and S-curvature. The terminology is motivated by the fact that these curvatures are defined in the context of Lie algebroid, Q-manifold, Poisson and Schouten structures, respectively. In this section we let $(E_1, M_1, \pi_1, a_1, [,]_1)$ and $(E_2, M_2, \pi_2, a_2, [,]_2)$ be Lie algebroids over bases M_1 and M_2, and anchors a_1 and a_2 respectively, and consider a vector bundle morphism $(F, f)\colon (E_1, M_1, \pi_1, a_1, [,]_1) \longrightarrow (E_2, M_2, \pi_2, a_2, [,]_2)$ such that the map $f\colon M_1 \longrightarrow M_2$ is a diffeomorphism.

4.1. A-curvature

Let $(E_1, M_1, \pi_1, a_1, [,]_1)$ and $(E_2, M_2, \pi_2, a_2, [,]_2)$ be Lie algebroids over bases M_1 and M_2 and anchors a_1 and a_2 respectively. Let $(F, f)\colon (E_1, M_1, \pi_1, a_1, [,]_1) \longrightarrow (E_2, M_2, \pi_2, a_2, [,]_2)$ be a vector bundle morphism such that the map $f\colon M_1 \longrightarrow M_2$ is a diffeomorphism.

Let (e_i) and (v_α) be local bases of ΓE_1 and ΓE_2, respectively, and let (x^d) and (y^q) be coordinates on the base manifolds M_1 and M_2, respectively. We let as well

$$[e_i, e_j] = \sum_k c_{ij}^k e_k, \qquad [v_\alpha, v_\beta] = \sum_\gamma {c'}_{\alpha\beta}^{\gamma} v_\gamma,$$

$$a_1(e_i) = \sum_d a_i^d \frac{\partial}{\partial x^d}, \qquad a_2(v_\alpha) = \sum_q {a'}_\alpha^q \frac{\partial}{\partial y^q}.$$

We recall the map $R_1\colon \Gamma E_1 \times C^\infty(M_1) \longrightarrow C^\infty(M_1)$ given by

$$R_1(r,h) = f^* \bar{F}[r,h]_1 - f^*[\bar{F}(r), \bar{F}(h)]_2.$$

Equation (3) states that

$$R_1(r,h) = [r,h]_1 - \sum_i r^\alpha f^*[v_\alpha, g^*(h)]_2,$$

and therefore by letting $F \circ e_i = \sum_\alpha F_i^\alpha f^*(v_\alpha)$ we have

$$\begin{aligned}
R_1(e_i, h) &= [e_i, h]_1 - \sum_\gamma F_i^\gamma f^*[v_\gamma, g^*(h)]_2 \\
&= \sum_d a_i^d \frac{\partial h}{\partial x^d} - \sum_{q,\gamma} F_i^\gamma f^*({a'}_\gamma^q) f^*\left(\frac{\partial g^*(h)}{\partial y^q}\right) \\
&= \sum_{d,q} a_i^d f^*\left(\frac{\partial g^*(h)}{\partial y^q}\right) \frac{\partial f^*(y^q)}{\partial x^d} - \sum_{q,\gamma} F_i^\gamma f^*({a'}_\gamma^q) f^*\left(\frac{\partial g^*(h)}{\partial y^q}\right) \\
&= \sum_q \left(\sum_d a_i^d \frac{\partial f^*(y^q)}{\partial x^d} - \sum_\gamma F_i^\gamma f^*({a'}_\gamma^q)\right) f^*\left(\frac{\partial g^*(h)}{\partial y^q}\right)
\end{aligned}$$

$$= \sum_q A_i^q f^* \left(\frac{\partial g^*(h)}{\partial y^q} \right), \tag{14}$$

where

$$A_i^q = \sum_d a_i^d \frac{\partial f^*(y^q)}{\partial x^d} - \sum_\alpha F_i^\alpha f^*({a'}_\alpha^q). \tag{15}$$

Now let $R\colon \Gamma E_1 \times \Gamma E_1 \longrightarrow \Gamma f^* E_2$ be the map given by

$$R(r,z) = f^* \bar{F}[r,z]_1 - f^*[\bar{F}(r), \bar{F}(z)]_2,$$

and therefore by (7) we have (cf. [3, Section 4.2])

$$\begin{aligned} R(r,z) &= F \circ [r,z]_1 - \sum_{\alpha,\beta} r^\alpha z^\beta f^*[v_\alpha, w_\beta]_2 - \sum_j [r, z^\beta]_1 f^*(w_\beta) \\ &+ \sum_\alpha [z, r^\alpha]_1 f^*(v_\alpha) - \sum_i R_1(z, r^\alpha) f^*(v_\alpha) + \sum_\beta R_1(r, z^\beta) f^*(w_\beta). \end{aligned} \tag{16}$$

Have

$$\begin{aligned} & R(r,hz) \\ &= F \circ [r,hz]_1 - \sum_{i,j} r^i h z^j f^*[v_i, w_j]_2 - \sum_j [r, hz^j]_1 f^*(w_j) + \sum_i [hz, r^i]_1 f^*(v_i) \\ &\quad - \sum_i R_1(hz, r^i) f^*(v_i) + \sum_j R_1(r, hz^j) f^*(w_j) \\ &= h(F \circ [r,z]_1) + [r,h]_1 (F \circ z) - \sum_{i,j} r^i h z^j f^*[v_i, w_j]_2 - \sum_j h[r, z^j]_1 f^*(w_j) \\ &\quad - [r,h]_1 \sum_j z^j f^*(w_j) + \sum_i h[z, r^i]_1 f^*(v_i) - \sum_i r_1(hz, r^i) f^*(v_i) \\ &\quad + \sum_j R_1(r, hz^j) f^*(w_j) \\ &= h \left(F \circ [r,z]_1 - \sum_{i,j} r^i z^j f^*[v_i, w_j]_2 - \sum_j [r, z^j]_1 f^*(w_j) + \sum_i [z, r^i]_1 f^*(v_i) \right) \\ &\quad - \sum_i h R_1(z, r^i) f^*(v_i) + \sum_j h R_1(r, z^j) f^*(w_j) + \sum_j z^j R_1(r,h) f^*(w_j) \\ &= hR(r,z) + \sum_j z^j R_1(r,h) f^*(w_j) \\ &= hR(r,z) + R_1(r,h)(F \circ z). \end{aligned}$$

We can immediately see that R is an $\mathbb{R}$-bilinear and antisymmetric map, and for $r, z \in \Gamma E_1$ we have

$$R(r,hz) = hR(r,z) + R_1(r,h)(F \circ z). \tag{17}$$

Let $R_{ij} = R(e_i, e_j)$, and by using (1) and (16) we get

$$\begin{aligned}
R_{ij} &= F \circ [e_i, e_j]_1 - \sum_{\alpha\beta} F_i^\alpha F_j^\beta f^*[v_\alpha, v_\beta]_2 - \sum_\beta [e_i, F_j^\beta]_1 f^*(v_\beta) \\
&\quad + \sum_\alpha [e_j, F_i^\alpha]_1 f^*(v_\alpha) - \sum_\alpha R_1(e_j, F_i^\alpha) f^*(v_\alpha) + \sum_\beta R_1(e_i, F_j^\beta) f^*(v_\beta) \\
&= \sum_{k,\gamma} F_k^\gamma c_{ij}^k f^*(v_\gamma) - \sum_{\alpha\beta\gamma} F_i^\alpha F_j^\beta f^*\left({c'}^\gamma_{\alpha\beta}\right) f^*(v_\gamma) - \sum_{d,\gamma} a_i^d \frac{\partial F_j^\gamma}{\partial x^d} f^*(v_\gamma) \\
&\quad + \sum_{d,\gamma} a_j^d \frac{\partial F_i^\gamma}{\partial x^d} f^*(v_\gamma) \sum_\gamma -R_1(e_j, F_i^\gamma) f^*(v_\gamma) + \sum_\gamma R_1(e_i, F_j^\gamma) f^*(v_\gamma) \\
&= \sum_\gamma \Bigg(\sum_k F_k^\gamma c_{ij}^k - \sum_{\alpha\beta} F_i^\alpha F_j^\beta f^*\left({c'}^\gamma_{\alpha\beta}\right) - \sum_d a_i^d \frac{\partial F_j^\gamma}{\partial x^d} + \sum_d a_j^d \frac{\partial F_i^\gamma}{\partial x^d} \\
&\quad + R_1(e_i, F_j^\gamma) - R_1(e_j, F_i^\gamma) \Bigg) f^*(v_\gamma) \\
&= \sum_\gamma \Bigg(\sum_k F_k^\gamma c_{ij}^k - \sum_{\alpha\beta} F_i^\alpha F_j^\beta f^*\left({c'}^\gamma_{\alpha\beta}\right) - \sum_d a_i^d \frac{\partial F_j^\gamma}{\partial x^d} + \sum_d a_j^d \frac{\partial F_i^\gamma}{\partial x^d} \\
&\quad + \sum_q A_i^q f^*\left(\frac{\partial g^* F_j^\gamma}{\partial y^q}\right) - \sum_q A_j^q f^*\left(\frac{\partial g^* F_i^\gamma}{\partial y^q}\right) \Bigg) f^*(v_\gamma)
\end{aligned}$$

$$= \sum_\gamma \left(K_{ij}^\gamma + \sum_q A_i^q f^*\left(\frac{\partial g^* F_j^\gamma}{\partial y^q}\right) - \sum_q A_j^q f^*\left(\frac{\partial g^* F_i^\gamma}{\partial y^q}\right) \right) f^*(v_\gamma), \tag{18}$$

where

$$K_{ij}^\gamma = \sum_k F_k^\gamma c_{ij}^k - \sum_{\alpha,\beta} F_i^\alpha F_j^\beta f^*\left({c'}^\gamma_{\alpha\beta}\right) - \sum_d a_i^d \frac{\partial F_j^\gamma}{\partial x^d} + \sum_d a_j^d \frac{\partial F_i^\gamma}{\partial x^d}. \tag{19}$$

Therefore we have

$$\begin{aligned}
&R\left(\sum_i \alpha^i e_i, \sum_j \beta^j e_j\right) \\
&= \sum_{i,j} R_{ij} \alpha^i \beta^j + \sum_{i,j} \left(\alpha^j R_1\left(e_j, \beta^i\right) - \beta^j R_1\left(e_j, \alpha^i\right)\right) F \circ e_i.
\end{aligned} \tag{20}$$

The function R will be called the *A-curvature* of the vector bundle map (F, f). As can be seen the map (F, f) is a morphism of Lie algebroids if and only if $R = 0$.

4.2. Q-curvature

As before let $(E_1, M_1, \pi_1, a_1, [,]_1)$ and $(E_2, M_2, \pi_2, a_2, [,]_2)$ be Lie algebroids over bases M_1 and M_2 and anchors a_1 and a_2 respectively. Let

$$(F, f)\colon (E_1, M_1, \pi_1, a_1, [,]_1) \longrightarrow (E_2, M_2, \pi_2, a_2, [,]_2)$$

be a vector bundle morphism such that the map $f\colon M_1 \longrightarrow M_2$ is a diffeomorphism. This subsection is mainly based on [3, Section 4.2].

Let (e_i) and (v_α) be local bases of ΓE_1 and ΓE_2 respectively, and let (x^d) and (y^q) be coordinates on the base manifolds M_1 and M_2 respectively, (ξ^i) and (η^α) coordinates on ΠE_1 and ΠE_2 corresponding to the bases (e_i) and (v_α) respectively. Then the corresponding homological vector fields on ΠE_1 and ΠE_2 are given by

$$Q_1 = \sum_{d,i} \xi^i a_i^d \frac{\partial}{\partial x^d} - \sum_{i,j,k} \frac{1}{2} \xi^i \xi^j c_{ij}^k \frac{\partial}{\partial \xi^k}$$

and

$$Q_2 = \sum_{q,\alpha} \eta^\alpha {a'}_\alpha^q \frac{\partial}{\partial y^q} - \sum_{\alpha,\beta,\gamma} \frac{1}{2} \eta^\alpha \eta^\beta {c'}_{\alpha\beta}^\gamma \frac{\partial}{\partial \eta^\gamma},$$

where

$$[e_i, e_j] = \sum_k c_{ij}^k e_k, \qquad [v_\alpha, v_\beta] = \sum_\gamma {c'}_{\alpha\beta}^\gamma v_\gamma,$$

$$a_1(e_i) = \sum_d a_i^d \frac{\partial}{\partial x^d}, \quad a_2(v_\alpha) = \sum_q {a'}_\alpha^q \frac{\partial}{\partial y^q}.$$

From the map $F\colon E_1 \longrightarrow E_2$ we get the corresponding map $F^\Pi\colon \Pi E_1 \longrightarrow \Pi E_2$ between the Q-manifolds ΠE_1 and ΠE_2. In what follows the map F^Π will be denoted simply by F, and it will be clear from the context which of F and F^Π is intended. The Q-curvature of the map F denoted by D (which is called *field strength* as well) is defined to be the failure of the homological vector fields on ΠE_1 and ΠE_2 to be F-related, that is

$$D = Q_1 \circ F^* - F^* \circ Q_2.$$

Since D is an F-derivation, then it is determined by $D(h)$ and $D(\eta^\alpha)$ where $h \in C^\infty(M_2)$ (see Proposition 3.1).

Since $F \circ e_i = \sum_\alpha F_i^\alpha f^*(v_\alpha)$, then $F^*(\eta^\nu) = \sum_s F_s^\nu \xi^s$ and $F^*(h) = f^*(h)$ for $h \in C^\infty(M_2)$. Therefore we have (cf. [3, Section 4.2])

$$\begin{aligned} D(h) &= Q_1 F^*(h) - F^* Q_2(h) \\ &= \sum_i \left(\sum_d a_i^d \frac{\partial f^*(h)}{\partial x^d} - \sum_{q,\alpha} F_i^\alpha f^*\left({a'}_\alpha^q\right) f^*\left(\frac{\partial h}{\partial y^q}\right) \right) \xi^i \\ &= \sum_i \left(\sum_d a_i^d f^*\left(\frac{\partial h}{\partial y^q}\right) \frac{\partial f^*(y^q)}{\partial x^d} - \sum_{q,\alpha} F_i^\alpha f^*\left({a'}_\alpha^q\right) f^*\left(\frac{\partial h}{\partial y^q}\right) \right) \xi^i \end{aligned}$$

$$= A_i^q f^* \left(\frac{\partial h}{\partial y^q} \right) \xi^i, \tag{21}$$

where A_i^q is as given in (15).

It can be seen that $R_1 = 0$ if and only if $D(h) = 0$ for all $h \in C^\infty(M_2)$, that is $D(h) = 0$ for all h is equivalent to the anchor condition, i.e. $\mathrm{d}\, f \circ a_1 = a_2 \circ F$.

We have as well (see [3, Section 4.2])

$$\begin{aligned} D(\eta^\nu) &= Q_1 F^*(\eta^\nu) - F^* Q_2(\eta^\nu) \\ &= -\frac{1}{2} \sum_{i,j} K_{ij}^\nu \xi^i \xi^j, \end{aligned} \tag{22}$$

where K_{ij}^ν are as given in (19). From (21), (22), and Proposition 3.1 we conclude that

$$D = -\frac{1}{2} \sum_{i,j,\alpha} K_{ij}^\alpha \xi^i \xi^j \left(F^* \circ \frac{\partial}{\partial \eta^\alpha} \right) + \sum_{i,q} \xi^i A_i^q \left(F^* \circ \frac{\partial}{\partial y^q} \right), \tag{23}$$

where K_{ij}^γ, A_i^q are as given in equations (19) and (15) respectively.

We summarize all of this in the following theorem.

Theorem 4.1 (cf. [3, Section 4.2])**.** *We have the following*:

- $D(h) = 0$ *for all* $h \in C^\infty(M_2)$ *if and only if* $\mathrm{d}\, f \circ a_1 = a_2 \circ F$.
- $D = -\frac{1}{2} \sum_{i,j,\alpha} K_{ij}^\alpha \xi^i \xi^j \left(F^* \circ \frac{\partial}{\partial \eta^\alpha} \right) + \sum_{i,q} \xi^i A_i^q \left(F^* \circ \frac{\partial}{\partial y^q} \right)$.
- $D = 0$ *if and only if* (F, f) *is a morphism of Lie algebroids.*

As can be seen, the equation $D = 0$ is equivalent to saying that F is a morphism of Q-manifolds. Hence we recover the result of Vaintrob [7] which states that (F, f) is a morphism of Lie algebroids if and only if F is a morphism of Q-manifolds.

4.3. P-curvature

As before let $(E_1, M_1, \pi_1, a_1, [,]_1)$ and $(E_2, M_2, \pi_2, a_2, [,]_2)$ be Lie algebroids over bases M_1 and M_2 and anchors a_1 and a_2 respectively. Let

$$(F, f) \colon (E_1, M_1, \pi_1, a_1, [,]_1) \longrightarrow (E_2, M_2, \pi_2, a_2, [,]_2)$$

be a vector bundle morphism such that the map $f \colon M_1 \longrightarrow M_2$ is a diffeomorphism. Then we get the diagram

$$\begin{array}{ccc} E_2^* & \xrightarrow{F^*} & E_1^* \\ \downarrow & & \downarrow \\ M_2 & \xrightarrow{g} & M_1 \end{array} \tag{24}$$

where $g = f^{-1}$. Let (e_i) and (v_α) be local bases of ΓE_1 and ΓE_2 respectively, and let (x^d) and (y^q) be coordinates on the base manifolds M_1 and M_2 respectively, (p_i) and (u_α) coordinates on E_1^* and E_2^* corresponding to the bases (e_i) and (v_α) respectively. Since (v_α) is a local basis of ΓE_2, then (v_α^*) is a local basis of ΓE_2^*. Let $x = g(y)$, then $F[e_i(x)] = \sum_\alpha F_i^\alpha(x)v_\alpha(y)$. Let $G_i^\alpha(y) = F_i^\alpha(g(y))$. Then we get the map $F^*\colon E_2^* \longrightarrow E_1^*$ defined by $F^*(\ell) = \ell \circ F$. For $\ell = \sum_\alpha u_\alpha v_\alpha^*(y)$, we get

$$F^*\left(\sum_\alpha u_\alpha v_\alpha^*(y)\right) = \sum_\alpha p_\alpha F^*(v_\alpha^*(y)) \\ = \sum_{j,\alpha} u_\alpha G_j^\alpha\,(y)\, e_j^*(g(y)).$$

We put $H = F^*$, and we get the map $H\colon E_2^* \longrightarrow E_1^*$, which is given locally by

$$H\left((y, u_\alpha)_\alpha\right) = \left(g(y), \sum_\alpha G_j^\alpha(y)u_\alpha\right)_j.$$

Therefore we get the pullback $H^*\colon C^\infty\left(E_1^*\right) \longrightarrow C^\infty\left(E_2^*\right)$ given locally by

$$H^*(x^a) = g^*(x^a) \qquad \text{and} \qquad H^*(p_j) = \sum_\alpha G_j^\alpha u_\alpha.$$

On ΓE_1, we have

$$[e_i, e_j]_1 = \sum_k c_{ij}^k e_k \qquad \text{and} \qquad a_1\left(e_i\right) = \sum_d a_i^d \frac{\partial}{\partial x^d}.$$

On ΓE_2, we have

$$[v_\alpha, v_\beta]_2 = \sum_\alpha {c'}_{\alpha\beta}^{\gamma} v_\gamma \qquad \text{and} \qquad a_2\left(v_\alpha\right) = \sum_q {a'}_\alpha^q \frac{\partial}{\partial y^q}.$$

The Poisson bracket on $C^\infty({E_1}^*)$ is given locally by

$$\{x^a, x^b\}_1 = 0, \qquad \{p_i, x^d\}_1 = a_i^d, \qquad \{p_i, p_j\}_1 = \sum_k c_{ij}^k p_k.$$

Therefore by Proposition 3.3 we have

$$\{v, w\}_1 = \sum_{i,j,k} c_{ij}^k p_k \frac{\partial v}{\partial p_i}\frac{\partial w}{\partial p_j} + \sum_{i,d} a_i^d \frac{\partial v}{\partial p_i}\frac{\partial w}{\partial x^d} - \sum_{i,d} a_i^d \frac{\partial v}{\partial x^d}\frac{\partial w}{\partial p_i}.$$

The Poisson bracket on $C^\infty({E_2}^*)$ is given locally by

$$\{y^q, y^s\}_2 = 0, \qquad \{u_\alpha, y^q\}_2 = {a'}_\alpha^q, \qquad \{u_\alpha, u_\beta\}_2 = \sum_\gamma {c'}_{\alpha\beta}^\gamma u_\gamma.$$

Therefore by Proposition 3.3 we get

$$\{v, w\}_2 = \sum_{\alpha,\beta,\gamma} {c'}_{\alpha\beta}^\gamma u_\gamma \frac{\partial v}{\partial u_\alpha}\frac{\partial w}{\partial u_\beta} + \sum_{q,\alpha} {a'}_\alpha^q \frac{\partial v}{\partial u_\alpha}\frac{\partial w}{\partial y^q} - \sum_{q,\alpha} {a'}_\alpha^q \frac{\partial v}{\partial y^q}\frac{\partial w}{\partial u_\alpha}.$$

Let $P\colon C^\infty\left(E_1^*\right) \times C^\infty\left(E_1^*\right) \longrightarrow C^\infty\left(E_2^*\right)$ be the map given by

$$P(v, w) = \{H^*(v), H^*(w)\}_2 - H^*\{v, w\}_1.$$

The map P is called the *P-curvature* associated with the map H. As can be seen the map P represents the failure of the map H to be a morphism of Poisson manifolds.

By Proposition 3.3 we have

$$P(v,w) = \sum_{i,j} P^{ij} H^*\left(\frac{\partial v}{\partial p_i}\right) H^*\left(\frac{\partial w}{\partial p_j}\right) + \sum_{i,d} L_i^d H^*\left(\frac{\partial v}{\partial p_i}\right) H^*\left(\frac{\partial w}{\partial x^d}\right)$$

$$- \sum_{i,d} L_i^d H^*\left(\frac{\partial v}{\partial x^d}\right) H^*\left(\frac{\partial w}{\partial p_i}\right), \tag{25}$$

where $P^{ij} = P(p_i, p_j)$ and $L_i^d = P(p_i, x^d)$. Using equation (15) we get

$$\begin{aligned} L_i^d &= P(p_i, x^d) \\ &= \{H^*(p_i), H^*(x^d)\}_2 - H^*\{p_i, x^d\}_1 \\ &= \left\{\sum_\alpha G_i^\alpha u_\alpha, g^*(x^d)\right\}_2 - H^*(a_i^d) \\ &= \sum_{q,\alpha} G_i^\alpha {a'}_\alpha^q \frac{\partial g^*(x^d)}{\partial y^q} - g^*(a_i^d) \\ &= -g^* \sum_s (Jf)^{-1}{}_{ds} A_i^s. \end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned} P^{ij} &= \{H^*(p_i), H^*(p_j)\}_2 - H^*\{p_i, p_j\}_1 \\ &= \left\{\sum_\alpha G_i^\alpha u_\alpha, \sum_\beta G_j^\beta u_\beta\right\}_2 - H^*\left(\sum_k c_{ij}^k p_k\right) \\ &= -\sum_\gamma \left(\sum_k g^*(c_{ij}^k) G_k^\gamma - \sum_{\alpha,\beta} G_i^\alpha G_j^\beta {c'}_{\alpha\beta}^\gamma - \sum_{q,\alpha} G_i^\alpha {a'}_\alpha^q \frac{\partial G_j^\gamma}{\partial y^q} \right. \\ &\quad \left. + \sum_{q,\alpha} G_j^\alpha {a'}_\alpha^q \frac{\partial G_i^\gamma}{\partial y^q}\right) u_\gamma \\ &= -\sum_\gamma \left(\sum_k g^*(c_{ij}^k) g^*(F_k^\gamma) - \sum_{\alpha,\beta} g^*(F_i^\alpha) g^*(F_j^\beta) {c'}_{\alpha\beta}^\gamma \right. \\ &\quad \left. - \sum_d \left(L_i^d + g^*(a_i^d)\right) g^*\left(\frac{\partial F_j^\gamma}{\partial x^d}\right) + \sum_d \left(L_j^d + g^*(a_j^d)\right) g^*\left(\frac{\partial F_i^\gamma}{\partial x^d}\right)\right) u_\gamma \\ &= -\sum_\gamma \left(\sum_k g^*(c_{ij}^k) g^*(F_k^\gamma) - \sum_{\alpha,\beta} g^*(F_i^\alpha) g^*(F_j^\beta) {c'}_{\alpha\beta}^\gamma \right. \end{aligned}$$

$$-\sum_d g^*(a_i^d) g^*\left(\frac{\partial F_j^\gamma}{\partial x^d}\right) + \sum_d g^*(a_j^d) g^* \frac{\partial F_i^\gamma}{\partial x^d} - \sum_d L_i^d g^*\left(\frac{\partial F_j^\gamma}{\partial x^d}\right)$$

$$+\sum_d L_j^d g^*\left(\frac{\partial F_i^\gamma}{\partial x^d}\right)\Bigg) u_\gamma$$

$$= -\sum_\gamma \left(g^*(K_{ij}^\gamma) - \sum_d L_i^d g^*\left(\frac{\partial F_j^\gamma}{\partial x^d}\right) + \sum_d L_j^d g^*\left(\frac{\partial F_i^\gamma}{\partial x^d}\right)\right) u_\gamma$$

$$= -\sum_\gamma \Bigg(g^*(K_{ij}^\gamma) - \sum_{d,s} g^*\left((Jf)^{-1}{}_{ds} A_i^s\right) g^*\left(\frac{\partial F_j^\gamma}{\partial x^d}\right)$$

$$-\sum_{d,s} g^*\left((Jf)^{-1}{}_{ds} A_j^s\right) g^*\left(\frac{\partial F_i^\gamma}{\partial x^d}\right)\Bigg) u_\gamma$$

$$= -\sum_\gamma g^* \left(K_{ij}^\gamma - \sum_{d,s} (Jf)^{-1}{}_{ds} A_i^s \frac{\partial F_j^\gamma}{\partial x^d} - \sum_{d,s} (Jf)^{-1}{}_{ds} A_j^s \frac{\partial F_i^\gamma}{\partial x^d}\right) u_\gamma, \tag{27}$$

where K_{ij}^γ, A_j^s are as given by equations (19) and (15) respectively.

We endow the manifold $f^*E_2^*$ with a Poisson bracket induced by the Poisson bracket on $C^\infty(E_2^*)$ as follows:

$$\{\hat{u}_\alpha, \hat{u}_\beta\}_3 = f^*\{u_\alpha, u_\beta\}_2 = \sum_\gamma f^*(c'^\gamma_{\alpha\beta}) \hat{u}_\gamma,$$

$$\{x^a, x^b\}_3 = 0,$$

$$\{\hat{u}_\alpha, x^b\}_3 = f^*\{u_\alpha, g^*(x^b)\}_2 = \sum_q f^*(a'^q_\alpha) f^*\left(\frac{\partial g^*(x^b)}{\partial y^q}\right), \tag{28}$$

where $(x^a, \hat{u}_\alpha)$ are local coordinates on $f^*E_2^*$, in particular the coordinates $(\hat{u}_\alpha)$ are the coordinates associated with the coordinates (u_α). Consider the diagram

$$f^*E_2^* \xrightarrow{\ \varphi\ } E_2^* \xrightarrow{\ H\ } E_1^*, \tag{29}$$

where the map $\varphi\colon f^*E_2^* \longrightarrow E_2^*$ is given by

$$\varphi\left((x, \hat{u}_\alpha)_\alpha\right) = (f(x), \hat{u}_\alpha)_\alpha. \tag{30}$$

We define the map $\widehat{H}\colon f^*E_2^* \longrightarrow E_1^*$ and $\widehat{P}\colon C^\infty(E_1^*) \times C^\infty(E_1^*) \longrightarrow C^\infty(f^*E_2^*)$ by

$$\widehat{H} = H \circ \varphi \qquad \text{and} \qquad \widehat{P} = \varphi^* \circ P.$$

Therefore the map $\widehat{H}$ is given locally by

$$\widehat{H}((x, \hat{u}_\alpha)_\alpha) = \left(x, \sum_\alpha F_j^\alpha(x) \hat{u}_\alpha\right)_j. \tag{31}$$

Therefore we have

$$\widehat{P}(v,w) = \sum_{i,j} \varphi^*(P^{ij})\widehat{H}^*\left(\frac{\partial v}{\partial p_i}\right)\widehat{H}^*\left(\frac{\partial w}{\partial p_j}\right) + \sum_{d,i} \varphi^*(L_i^d)\widehat{H}^*\left(\frac{\partial v}{\partial p_i}\right)\widehat{H}^*\left(\frac{\partial w}{\partial x^d}\right) - \sum_{d,i} \varphi^*(L_i^d)\widehat{H}^*\left(\frac{\partial v}{\partial x^d}\right)\widehat{H}^*\left(\frac{\partial w}{\partial p_i}\right). \tag{32}$$

By (26) we get

$$\varphi^*(L_i^d) = -\sum_s (Jf)^{-1}_{ds} A_i^s. \tag{33}$$

and

$$\varphi^*(P^{ij}) = -\sum_\gamma \left(K_{ij}^\gamma - \sum_{d,s}(Jf)^{-1}_{ds}A_i^s\frac{\partial F_j^\gamma}{\partial x^d} - \sum_{d,s}(Jf)^{-1}_{ds}A_j^s\frac{\partial F_i^\gamma}{\partial x^d}\right)\hat{u}_\gamma. \tag{34}$$

We could have considered $\widehat{P}$ instead of P as the definition of P-curvature, in particular $\widehat{P}$ and P are equivalent in the sense that $\widehat{P} = 0$ if and only if $P = 0$. The advantage of considering $\widehat{P}$ is that when the anchor condition is satisfied that is $A_i^s = 0$, then there is no f^{-1} involved in its expression, and this allows us to extend $\widehat{P}$ to the case when f is not necessarily a diffeomorphism. In that case $\widehat{P}$ becomes

$$\widehat{P}(v,w) = \sum_{i,j} \varphi^*(P^{ij})\widehat{H}^*\left(\frac{\partial v}{\partial p_i}\right)\widehat{H}^*\left(\frac{\partial w}{\partial p_j}\right).$$

So then (that is when the anchor condition is satisfied and f is not necessarily a diffeomorphism) we have to verify that it is well-defined, that is, it does not depend on the choice of local coordinates. Hence if $(\bar{e}_i)$ and $(\bar{v}_\alpha)$ are other local bases of ΓE_1 and ΓE_2 respectively with $\bar{e}_i = \sum_k m_{ki}e_k$ and $\bar{v}_\alpha = \sum_\beta \tau_{\beta\alpha}v_\beta$, then we get

$$\bar{K}_{ij}^\gamma = \sum_{s,t,\sigma} m_{si}m_{tj}f^*\left(\tau^{-1}_{\gamma\sigma}\right)K_{st}^\sigma, \qquad \bar{p}_i = \sum_k m_{ki}p_k, \qquad \bar{u}_\alpha = \sum_\beta \tau_{\beta\alpha}u_\beta$$

for the corresponding quantities. Therefore

$$\begin{aligned}&\sum_{i,j} \varphi^*(P^{ij})\widehat{H}^*\left(\frac{\partial v}{\partial p_i}\right)\widehat{H}^*\left(\frac{\partial w}{\partial p_j}\right)\\ =&\sum_{i,j,\gamma} K_{ij}^\gamma\hat{u}_\gamma\widehat{H}^*\left(\frac{\partial v}{\partial p_i}\right)\widehat{H}^*\left(\frac{\partial w}{\partial p_j}\right)\\ =&\sum_{\substack{s,i,t,j,\gamma,\\ \sigma,\delta,r,z}} m^{-1}_{si}m^{-1}_{tj}f^*\left(\tau_{\gamma\sigma}\right)\bar{K}_{st}^\sigma f^*\left(\tau^{-1}_{\delta\gamma}\right)\hat{u}_\delta m_{ir},\end{aligned}$$

$$\widehat{H}^*\left(\frac{\partial v}{\partial \bar{p}_r}\right) m_{jz}\widehat{H}^*\left(\frac{\partial w}{\partial \bar{p}_z}\right) = \sum_{s,t}\varphi^*(\bar{P}^{st})\widehat{H}^*\left(\frac{\partial v}{\partial \bar{p}_s}\right)\widehat{H}^*\left(\frac{\partial w}{\partial \bar{p}_t}\right),$$

which means that it is well-defined.

4.4. S-curvature

As before let $(E_1, M_1, \pi_1, a_1, [,]_1)$ and $(E_2, M_2, \pi_2, a_2, [,]_2)$ be Lie algebroids over bases M_1 and M_2 and anchors a_1 and a_2, respectively. Let

$$(F, f)\colon (E_1, M_1, \pi_1, a_1, [,]_1) \longrightarrow (E_2, M_2, \pi_2, a_2, [,]_2)$$

be a vector bundle morphism such that the map $f\colon M_1 \longrightarrow M_2$ is a diffeomorphism.

Let (e_i) and (v_α) be local bases of ΓE_1 and ΓE_2 respectively, and let (x^d) and (y^q) be coordinates on the base manifolds M_1 and M_2 respectively, (π_i) and (θ_α) coordinates on ΠE_1^* and ΠE_2^* corresponding to the bases (e_i) and (v_α) respectively. Let $g = f^{-1}$ and $x = g(y)$, then $F[e_i(x)] = \sum_\alpha F_i^\alpha(x) v_\alpha(y)$. From the map $H\colon E_2^* \longrightarrow E_1^*$, we get the corresponding map $T\colon \Pi E_2^* \longrightarrow \Pi E_1^*$ given locally by

$$T((y, \theta_\alpha)_\alpha) = \left(g(y), \sum_\alpha G_j^\alpha(y)\theta_\alpha\right)_j,$$

where $G_j^\alpha(y) = F_j^\alpha\left(g(y)\right)$. And we get the pullback

$$T^*\colon C^\infty\left(\Pi E_1^*\right) \longrightarrow C^\infty\left(\Pi E_2^*\right)$$

given locally by

$$T^*(x^a) = g^*\left(x^a\right) \qquad \text{and} \qquad T^*(\pi_j) = \sum_\alpha G_j^\alpha \theta_\alpha.$$

On ΓE_1, we have

$$[e_i, e_j]_1 = \sum_k c_{ij}^k e_k \qquad \text{and} \qquad a_1\left(e_i\right) = \sum_d a_i^d \frac{\partial}{\partial x^d}.$$

On ΓE_2, we have

$$[v_\alpha, v_\beta]_2 = \sum_\gamma {c'}_{\alpha\beta}^{\gamma} v_\gamma \qquad \text{and} \qquad a_2\left(v_\alpha\right) = \sum_q {a'}_\alpha^q \frac{\partial}{\partial y^q}.$$

The Schouten bracket on $C^\infty(\Pi E_1^*)$ is given locally by

$$\{x^a, x^b\}_1 = 0, \qquad \{\pi_i, x^d\}_1 = a_i^d, \qquad \{\pi_i, \pi_j\}_1 = \sum_k c_{ij}^k \pi_k.$$

Therefore, by Proposition 3.3, we get

$$\{v, w\}_1 = \sum_{i,j,k} c_{ij}^k(x)\pi_k \frac{\partial v}{\partial \pi_i}\frac{\partial w}{\partial \pi_j} - \sum_{d,i}(-1)^{\widetilde{v}} a_i^d \frac{\partial v}{\partial \pi_i}\frac{\partial w}{\partial x^d} - \sum_{d,i} a_i^d \frac{\partial v}{\partial x^d}\frac{\partial w}{\partial \pi_i}.$$

The Schouten bracket on $C^\infty(\Pi E_2{}^*)$ is given locally by

$$\{y^q, y^s\}_2 = 0, \qquad \{\theta_\alpha, y^q\}_2 = {a'}^q_\alpha, \qquad \{\theta_\alpha, \theta_\beta\}_2 = \sum_\gamma {c'}^\gamma_{\alpha\beta}\theta_\gamma.$$

Therefore, by Proposition 3.3, we get

$$\{v, w\}_2 = \sum_{\alpha,\beta,\gamma} {c'}^\gamma_{\alpha\beta}\theta_\gamma \frac{\partial v}{\partial\theta_\alpha}\frac{\partial w}{\partial\theta_\beta} - \sum_{q,\alpha}(-1)^{\tilde{v}}{a'}^q_\alpha \frac{\partial v}{\partial\theta_\alpha}\frac{\partial w}{\partial y^q} - \sum_{q,\alpha}{a'}^q_\alpha \frac{\partial v}{\partial y^q}\frac{\partial w}{\partial\theta_\alpha}.$$

We define the map $S\colon C^\infty(\Pi E_1^*) \times C^\infty(\Pi E_1^*) \longrightarrow C^\infty(\Pi E_2^*)$ by

$$S(v, w) = \{T^*(v), T^*(w)\}_2 - T^*\{v, w\}_1.$$

The map S is called the *S-curvature* associated with the map T. As can be seen the map S represents the failure of the map T to be a morphism of Schouten supermanifolds.

By Proposition 3.3 and proceeding as in the case of P, we get

$$\begin{aligned} S(v,w) = &\sum_{i,j} S^{ij}T^*\left(\frac{\partial v}{\partial\pi_i}\right)T^*\left(\frac{\partial w}{\partial\pi_j}\right) - \sum_{d,i}(-1)^{\tilde{v}}L_i^d T^*\left(\frac{\partial v}{\partial\pi_i}\right)T^*\left(\frac{\partial w}{\partial x^d}\right) \\ &- \sum_{d,i} L_i^d T^*\left(\frac{\partial v}{\partial x^d}\right)T^*\left(\frac{\partial w}{\partial\pi_i}\right), \end{aligned} \tag{35}$$

where

$$\begin{aligned} S^{ij} &= \{T^*(\pi_i), T^*(\pi_j)\}_2 - T^*\{\pi_i, \pi_j\}_1 \\ &= \left\{\sum_\alpha G_i^\alpha\theta_\alpha, \sum_\beta G_j^\beta\theta_\beta\right\}_2 - \sum_k T^*(c_{ij}^k\pi_k) \\ &= -\sum_\gamma\left(\sum_k g^*(c_{ij}^k)G_k^\gamma - \sum_{\alpha,\beta}G_i^\alpha G_j^\beta {c'}^\gamma_{\alpha\beta} - \sum_{q,\alpha}G_i^\alpha {a'}^q_\alpha\frac{\partial G_j^\gamma}{\partial y^q}\right. \\ &\qquad\left. + \sum_{q,\alpha}G_j^\alpha {a'}^q_\alpha \frac{\partial G_i^\gamma}{\partial y^q}\right)\theta_\gamma \\ &= -\sum_\gamma g^*\left(K_{ij}^\gamma - \sum_{d,s}(Jf)^{-1}_{ds}A_i^s\frac{\partial F_j^\gamma}{\partial x^d} - \sum_{d,s}(Jf)^{-1}_{ds}A_j^s\frac{\partial F_i^\gamma}{\partial x^d}\right)\theta_\gamma, \end{aligned} \tag{36}$$

and L_i^d as given in (26).

We endow the supermanifold $\Pi f^* E_2^*$ with a Schouten bracket induced by the Schouten bracket on $C^\infty(\Pi E_2^*)$ as follows:

$$\{\hat\theta_\alpha, \hat\theta_\beta\}_3 = f^*\{\theta_\alpha, \theta_\beta\}_2 = \sum_\gamma f^*({c'}^\gamma_{\alpha\beta})\hat\theta_\gamma,$$

$$\{x^a, x^b\}_3 = 0,$$

$$\{\hat{\theta}_\alpha, x^b\}_3 = f^*\{\theta_\alpha, g^*(x^b)\}_2 = \sum_q f^*(a'^q_\alpha) f^*\left(\frac{\partial g^*(x^b)}{\partial y^q}\right), \tag{37}$$

where $(x^a, \hat{\theta}_\alpha)$ are local coordinates on $f^*E_2^*$, in particular the coordinates $(\hat{\theta}_\alpha)$ are the coordinates associated with the coordinates (θ_α). Now consider the diagram

$$\Pi f^* E_2^* \xrightarrow{\ \rho\ } \Pi E_2^* \xrightarrow{\ T\ } \Pi E_1^*, \tag{38}$$

where the map $\rho\colon \Pi f^* E_2^* \longrightarrow \Pi E_2^*$ is given by

$$\rho\left((x, \hat{\theta}_\alpha)_\alpha\right) = (f(x), \hat{\theta}_\alpha)_\alpha. \tag{39}$$

Then we define the maps $\widehat{T}\colon \Pi f^* E_2^* \longrightarrow \Pi E_1^*$ and $\widehat{S}\colon C^\infty(\Pi E_1^*) \times C^\infty(\Pi E_1^*) \longrightarrow C^\infty(\Pi f^* E_2^*)$ by

$$\widehat{T} = T \circ \rho \qquad \text{and} \qquad \widehat{S} = \rho^* \circ S.$$

Therefore the map $\widehat{T}$ is given locally by

$$\widehat{T}((x, \hat{\theta}_\alpha)_\alpha) = \left(x, \sum_\alpha F_j^\alpha(x)\hat{\theta}_\alpha\right)_j. \tag{40}$$

We have

$$\begin{aligned}\widehat{S}(v,w) = &\sum_{i,j} \rho^*(S^{ij})\widehat{T}^*\left(\frac{\partial v}{\partial \pi_i}\right)\widehat{T}^*\left(\frac{\partial w}{\partial \pi_j}\right) + \sum_{d,i} \rho^*(L_i^d)\widehat{T}^*\left(\frac{\partial v}{\partial \pi_i}\right)\widehat{T}^*\left(\frac{\partial w}{\partial x^d}\right)\\ &- \sum_{d,i} \rho^*(L_i^d)\widehat{T}^*\left(\frac{\partial v}{\partial x^d}\right)\widehat{T}^*\left(\frac{\partial w}{\partial \pi_i}\right).\end{aligned} \tag{41}$$

By (26) we have

$$\rho^*(L_i^d) = -\sum_s (Jf)^{-1}_{ds} A_i^s. \tag{42}$$

and

$$\rho^*(S^{ij}) = -\sum_\gamma \left(K_{ij}^\gamma - \sum_{d,s}(Jf)^{-1}_{ds} A_i^s \frac{\partial F_j^\gamma}{\partial x^d} - \sum_{d,s}(Jf)^{-1}_{ds} A_j^s \frac{\partial F_i^\gamma}{\partial x^d}\right)\hat{\theta}_\gamma. \tag{43}$$

We could have considered $\widehat{S}$ instead of S as the definition of S-curvature, in particular $\widehat{S}$ and S are equivalent in the sense that $\widehat{S} = 0$ if and only if $S = 0$. The advantage of considering $\widehat{S}$ is that when the anchor condition is satisfied that is $A_i^s = 0$, then there is no f^{-1} involved in its expression, and this allows us to extend $\widehat{P}$ to the case when f is not necessarily a diffeomorphism. In that case $\widehat{S}$ becomes

$$\widehat{S}(v,w) = \sum_{i,j} \rho^*(S^{ij})\widehat{T}^*\left(\frac{\partial v}{\partial \pi_i}\right)\widehat{T}^*\left(\frac{\partial w}{\partial \pi_j}\right).$$

So then (that is when the anchor condition is satisfied and f is not necessarily a diffeomorphism) we have to verify that it is well-defined, that is, it does not depend on the choice of local coordinates. Hence if $(\bar{e}_i)$ and $(\bar{v}_\alpha)$ are other local bases of ΓE_1 and ΓE_2 respectively with $\bar{e}_i = \sum_k m_{ki} e_k$ and $\bar{v}_\alpha = \sum_\beta \tau_{\beta\alpha} v_\beta$, then we get

$$\bar{K}_{ij}^\gamma = \sum_{s,t,\sigma} m_{si} m_{tj} f^*\left(\tau_{\gamma\sigma}^{-1}\right) K_{st}^\sigma, \qquad \bar{\pi}_i = \sum_k m_{ki}\pi_k, \qquad \bar{\theta}_\alpha = \sum_\beta \tau_{\beta\alpha}\theta_\beta$$

for the corresponding quantities. Therefore

$$\begin{aligned} &\sum_{i,j} \rho^*(S^{ij}) \widehat{T}^*\left(\frac{\partial v}{\partial \pi_i}\right) \widehat{T}^*\left(\frac{\partial w}{\partial \pi_j}\right) \\ &= \sum_{i,j} K_{ij}^\gamma \hat{\theta}_\gamma \widehat{T}^*\left(\frac{\partial v}{\partial \pi_i}\right) \widehat{T}^*\left(\frac{\partial w}{\partial \pi_j}\right) \\ &= \sum_{s,i,t,j,\gamma,\sigma,\delta,r,z} m_{si}^{-1} m_{tj}^{-1} f^*\left(\tau_{\gamma\sigma}\right) \bar{K}_{st}^\sigma f^*\left(\tau_{\delta\gamma}^{-1}\right) \hat{\bar{\theta}}_\delta m_{ir} \widehat{T}^*\left(\frac{\partial v}{\partial \bar{\pi}_r}\right) m_{jz} \widehat{T}^*\left(\frac{\partial w}{\partial \bar{\pi}_z}\right) \\ &= \sum_{s,t} \rho^*(\bar{S}^{st}) \widehat{H}^*\left(\frac{\partial v}{\partial \bar{\pi}_s}\right) \widehat{T}^*\left(\frac{\partial w}{\partial \bar{\pi}_t}\right) \end{aligned}$$

which means that it is well-defined.

We summarize the results of this section in the following theorem.

Theorem 4.2. *Let* $(F,f)\colon (E_1, M_1, \pi_1, a_1, [,]_1) \longrightarrow (E_2, M_2, \pi_2, a_2, [,]_2)$ *be a vector bundle morphism, where* $(E_1, M_1, \pi_1, a_1, [,]_1)$ *and* $(E_2, M_2, \pi_2, a_2, [,]_2)$ *are Lie algebroids over bases* M_1 *and* M_2 *and anchors* a_1 *and* a_2 *respectively, such that the map* $f\colon M_1 \longrightarrow M_2$ *is a diffeomorphism. Then we have the following*:

- $R\left(\sum_i \alpha^i e_i, \sum_j \beta^j e_j\right) = \sum_{i,j} R_{ij}\alpha^i\beta^j + \sum_{i,j}\left(\alpha^j R_1\left(e_j, \beta^i\right) - \beta^j R_1\left(e_j, \alpha^i\right)\right) F \circ e_i,$
- $D = -\frac{1}{2}\sum_{i,j,\alpha} K_{ij}^\alpha \xi^i \xi^j \left(F^* \circ \frac{\partial}{\partial \eta^\alpha}\right) + \sum_{i,q} \xi^i A_i^q \left(F^* \circ \frac{\partial}{\partial y^q}\right),$
- $P(v,w) = \sum_{i,j} P^{ij} H^*\left(\frac{\partial v}{\partial p_i}\right) H^*\left(\frac{\partial w}{\partial p_j}\right) + \sum_{d,i} L_i^d H^*\left(\frac{\partial v}{\partial p_i}\right) H^*\left(\frac{\partial w}{\partial x^d}\right) - \sum_{d,i} L_i^d H^*\left(\frac{\partial v}{\partial x^d}\right) H^*\left(\frac{\partial w}{\partial p_i}\right),$
- $S(v,w) = \sum_{i,j} S^{ij} T^*\left(\frac{\partial v}{\partial \pi_i}\right) T^*\left(\frac{\partial w}{\partial \pi_j}\right) - \sum_{d,i}(-1)^{\tilde{v}} L_i^d T^*\left(\frac{\partial v}{\partial \pi_i}\right) T^*\left(\frac{\partial w}{\partial x^d}\right) - \sum_{d,i} L_i^d T^*\left(\frac{\partial v}{\partial x^d}\right) T^*\left(\frac{\partial w}{\partial \pi_i}\right),$

where R_{ij}, $R_1(e_i,\cdot)$, K_{ij}^γ, A_i^q, L_i^d, P^{ij} *and* S^{ij} *are as given by equations* (18), (14), (19), (15), (26), (27) *and* (36).

Corollary 4.3. *The map (F, f) is a Lie algebroid morphism if and only if one of the following equivalent conditions is satisfied:*

- $H\colon E_2^* \longrightarrow E_1^*$ *is a morphism of Poisson manifolds.*
- $T\colon \Pi E_2^* \longrightarrow \Pi E_1^*$ *is a morphism of Schouten supermanifolds.*

The restriction that f should be a diffeomorphism as can be seen is not necessary in the cases of R and D, but necessary for P and S. The maps R, D, P and S are uniquely determined by the quantities K_{ij}^{γ} and A_i^d, and when one of them is annulled then all the others are annulled. This allows us to say that the maps P, S, R and D are equivalent characterizations of curvature and extend the usual notion of curvature of a Lie algebroid connection. In the case when $M_1 = M_2 = M$ and $f = \mathrm{id}$, we get the following corollary.

Corollary 4.4. *If* $\mathrm{d}\, f \circ a_1 = a_2 \circ F$, *then we have the following*:

- $R = \frac{1}{2} \sum_{i,j,\alpha} K_{ij}^{\alpha}\, e_i^* \wedge e_j^*\, f^*(v_\alpha)$.
- $D = -\frac{1}{2} \sum_{i,j,\alpha} K_{ij}^{\alpha} \xi^i \xi^j \left(F^* \circ \frac{\partial}{\partial \eta^\alpha} \right)$.
- $P = -\sum_{i,j,\alpha} g^*(K_{ij}^{\alpha}) u_\alpha \left(H^* \circ \frac{\partial}{\partial p_i} \right) \left(H^* \circ \frac{\partial}{\partial p_j} \right)$.
- $S = -\sum_{i,j,\alpha} g^*(K_{ij}^{\alpha}) \theta_\alpha \left(T^* \circ \frac{\partial}{\partial \pi_i} \right) \left(T^* \circ \frac{\partial}{\partial \pi_j} \right)$.

In the case when E_1 and E_2 are Lie algebroids over the same base M, i.e. $M_1 = M_2 = M$ and $f = \mathrm{id}$, with anchors a_1 and a_2 respectively then $R\colon \Gamma E_1 \times \Gamma E_1 \longrightarrow \Gamma E_2$ is given by $R\,(X, Y) = F\,[X, Y] - [FX, FY]$ and $R_1 = a_1 - a_2 \circ F$. The functions K_{ij}^{α} are given by $R\,(e_i, e_j) = \sum_\alpha K_{ij}^{\alpha} v_\alpha$.

Example 4.5. Let E be a vector bundle over M. We define the *Lie algebroid of derivations* $\mathfrak{D}(E)$ associated to E as follows (see [6, Chapter 3] and [3, Chapter 2]):

Let $D_x\colon \Gamma E \longrightarrow E_x$ be an $\mathbb{R}$-linear map from the space of sections ΓE to the fibre E_x at x . Then, we say that $D_x \in \mathfrak{D}(E)$ if there is a tangent vector $X_x \in T_x M$, such that

$$D_x(fs) = f(x) D_x(s) + X_x(f) s(x)$$

for any function $f \in C^\infty(M)$ and any section $s \in \Gamma E$ (the definition mimics that of a tangent vector at a point x). We define the anchor $a : \mathfrak{D}(E) \to TM$ by $a(D_x) = X_x$. Then $\mathfrak{D}(E)$ is a vector bundle over M by defining $\pi\colon \mathfrak{D}(E) \longrightarrow M$ by $\pi(D_x) = x$.

A section $D \in \Gamma\mathfrak{D}(E)$ will be identified with the $\mathbb{R}$-linear map $\widetilde{D} : \Gamma E \to \Gamma E$ given by $\widetilde{D}(s)(x) = D(x)(s)$, and therefore a section of $\mathfrak{D}(E)$ is any $\mathbb{R}$-linear map $\widetilde{D} : \Gamma E \to \Gamma E$ for which there is a vector field X such that

$$\widetilde{D}(fs) = f \widetilde{D}(s) + X(f) s$$

for $f \in C^\infty(M)$ and $s \in \Gamma E$ (in this case $a(\widetilde{D}) = X$). Then the Lie bracket on $\Gamma\mathfrak{D}(E)$ is given by

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1.$$

The vector bundle $\mathfrak{D}(E)$ as defined above is a Lie algebroid on M.

The tangent bundle TM is a Lie algebroid for which the anchor $a : TM \to TM$ is just the identity map, and the Lie bracket is the usual Lie bracket of vector fields.

Let $\nabla\colon \mathfrak{X}(M) \times \Gamma E \longrightarrow \Gamma E$ be a Koszul connection on a vector bundle E. Then ∇ can be reinterpreted as a vector bundle map $\widehat{\nabla}\colon TM \longrightarrow \mathfrak{D}(E)$ by letting $\widehat{\nabla}(X_x)(s) = \nabla_{X_x} s$ for $X_x \in T_xM$. This can be seen from the fact that

$$\widehat{\nabla}(X_x)(fs) = f(x)\widehat{\nabla}(X_x)(s) + X_x(f)s(x).$$

Since $a \circ \widehat{\nabla} = \mathrm{id}_{TM}$, then the map $\widehat{\nabla}$ is a (Lie algebroid) connection (see the introduction for the definition of a Lie algebroid connection).

In the notation of Section 4.1 if we take $E_1 = TM$, $E_2 = \mathfrak{D}(E)$, $M_1 = M_2 = M$, $a_1 = \mathrm{id}_{TM}$, $a_2 = a$ and $F = \widehat{\nabla}$, then the A-curvature R is given by

$$R(X, Y) = \widehat{\nabla} \circ [X, Y] - [\widehat{\nabla} \circ X, \widehat{\nabla} \circ X].$$

Let R^{∇} be the curvature of the Koszul connection ∇. Then R^{∇} is given by

$$R^{\nabla}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Therefore by the identifications above we have $R(X, Y) = -R^{\nabla}(X, Y)$. Hence the A-curvature of the vector bundle map $\widehat{\nabla}$ and the curvature of the Koszul connection ∇ are the same up to sign.

Appendix A.

In this appendix we give some background material on the theory of supermanifolds necessary for understanding the material of this paper. It is mainly based on [3, Chapter 2], more detailed treatment of the subject can be found in [5].

A.1. Supermanifolds

Let $C^\infty(U)$ be the commutative $\mathbb{R}$-algebra of smooth functions on an open subset U of a smooth manifold M. The exterior algebra over $C^\infty(U)$ with indeterminates $(\theta^1, \theta^2, \dots, \theta^n)$, which is denoted by $C^\infty(U)[\theta^1, \theta^2, \dots, \theta^n]$, is defined by

$$C^\infty(U)[\theta^1, \theta^2, \dots, \theta^n] = \left\{ f\colon f = \sum_I f_I \theta^I,\ f_I \in C^\infty(U),\ I \in B^n \right\}$$

where $B = \{0, 1\}$, and if $I = (i_1, i_2, \dots, i_n)$, then $\theta^I = (\theta^1)^{i_1}(\theta^2)^{i_2} \cdots (\theta_n)^{i_n}$, with $(\theta^j)^0 = 1$, $(\theta^j)^1 = \theta^j$, $\theta^i\theta^i = 0$, $\theta^i\theta^j = -\theta^j\theta^i$. The exterior algebra $C^\infty(U)[\theta^1, \theta^2, \dots, \theta^n]$ is a commutative $\mathbb{R}$-superalgebra (associative with unity). If $f = f_0 + \sum_{I \neq 0} f_I\theta^I$, then we define the value of the function f at $x \in U$ by $v_x(f) = f_0(x)$.

A function $f \in C^\infty(U)[\theta^1, \theta^2, \dots, \theta^n]$ can be considered a polynomial on the indeterminates $(\theta^1, \theta^2, \dots, \theta^n)$. A function $f = \sum_I f_I\theta^I$ is said to be of *parity*

0 (or *even*) if it is the sum of monomials of even degree, and we write $\widetilde{f} = 0$. Likewise a function $f = \sum_I f_I \theta^I$ is said to be of *parity* 1 (or *odd*) if it is the sum of monomials of odd degree, and we write $\widetilde{f} = 1$. The function f is said to be *homogeneous* if it is even or odd.

Definition. Let m and n be nonnegative integers. A *supermanifold* $\mathcal{M}$ of dimension $(m|n)$ is a pair $(M, \mathcal{A})$ where M is a smooth manifold of dimension m and $\mathcal{A}$ is a sheaf of $\mathbb{R}$-superalgebras on M such that there is an open cover $\{U_\alpha\}$ of M and $\mathcal{A}(U_\alpha) \cong C^\infty(U_\alpha)[\theta^1, \dots, \theta^n]$.

The manifold M is called the *support* of $\mathcal{M}$ or the *underlying manifold*, a section $f \in \mathcal{A}(M)$ is called a *function*, and the $\mathbb{R}$-superalgebra $\mathcal{A}(M)$ will be denoted $C^\infty(\mathcal{M})$ as well. We can see that a function $f \in \mathcal{A}(M)$ is not determined by its values at its points. A supermanifold can be viewed as an ordinary manifold for which the sheaf of functions is enriched with odd ones.

A.2. Some examples of supermanifolds

An example of a supermanifold of dimension $(m|n)$ is the *affine superspace* $\mathbb{R}^{m|n} = (\mathbb{R}^m, \mathcal{A}^{m|n})$ where the sheaf $\mathcal{A}^{m|n}$ is given by

$$\mathcal{A}^{m|n}(U) = C^\infty(U)[\theta^1, \dots, \theta^n]$$

for any open subset U of $\mathbb{R}^m$. The supermanifold $U^{m|n} = (U, \mathcal{A}^{m|n}|_U)$ where U is an open subset of $\mathbb{R}^m$ and the sheaf $\mathcal{A}^{m|n}|_U$ is the restriction of the sheaf $\mathcal{A}^{m|n}$ to U is called a *superdomain*.

If we consider $\Omega^\bullet$ the sheaf of differential forms on a smooth manifold M, then $(M, \Omega^\bullet)$ is a supermanifold called the *antitangent bundle* and is denoted by ΠTM. Likewise if we consider $\mathfrak{X}^\bullet$ the sheaf of multivector fields on M, then $(M, \mathfrak{X}^\bullet)$ is a supermanifold called the *anticotangent bundle* and is denoted by $\Pi T^\star M$.

A.3. Local coordinates for supermanifolds

If $\mathcal{M} = (M, \mathcal{A})$ is a supermanifold, then we have an open cover $\{U_\alpha\}$ of M such that

$$\mathcal{A}(U_\alpha) \cong C^\infty(U_\alpha)[\theta^1, \dots, \theta^n].$$

Suppose that $(x^1, x^2, \dots, x^m)$ are local coordinates on U_α, then

$$(x^1, x^2, \dots, x^m, \theta^1, \theta^2, \dots, \theta^n)$$

will be called *local coordinates* for $\mathcal{M}$. If we restrict the sheaf $\mathcal{A}$ to an open subset U of M, then we get the supermanifold $\mathcal{U} = (U, \mathcal{A}|_U)$, which will be called a *submanifold* of $\mathcal{M}$, where $\mathcal{A}|_U$ is the restriction sheaf.

Definition. If $\mathcal{M} = (M, \mathcal{A})$ and $\mathcal{N} = (N, \mathcal{B})$ are supermanifolds, then a *morphism* $F \colon \mathcal{M} \longrightarrow \mathcal{N}$ is a pair (f, f^*) such that $f \colon M \longrightarrow N$ is a smooth map and $f^* \colon \mathcal{B} \longrightarrow f_* \mathcal{A}$ is a morphism of sheaves where $(f_*\mathcal{A})(U) = \mathcal{A}(f^{-1}(U))$.

Consider maps $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ and $G\colon \mathcal{N} \longrightarrow \mathcal{P}$ such that $F = (f, f^*)$ and $G = (g, g^*)$ are morphisms of supermanifolds. Then the composition of F and G is denoted by $G \circ F$ and is given by

$$G \circ F = (g \circ f, f^* \circ g^*).$$

The identity morphism denoted by $I_{\mathcal{M}} = (i, i^*)$ is the morphism $I_{\mathcal{M}}\colon \mathcal{M} \longrightarrow \mathcal{M}$ given by $i(x) = x$ and $i^*(f) = f$. A morphism $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ is said to be a *a diffeomorphism* if there is a morphism $G\colon \mathcal{N} \longrightarrow \mathcal{M}$ such that $F \circ G = I_{\mathcal{N}}$ and $G \circ F = I_{\mathcal{M}}$.

Theorem A.1 (**Chart theorem**, see [8, p. 140])**.** *Suppose that $\mathcal{M}$ is a supermanifold and $U^{m|n} \subseteq \mathbb{R}^{m|n}$ a superdomain with coordinates (y^i, ξ^j). Let (f^i) be m even functions and (η^j) be n odd functions on $\mathcal{M}$ such that*

$$(v_x(f^1), v_x(f^2), \dots, v_x(f^m)) \in U.$$

Then, there is a unique morphism $(f, f^*)\colon \mathcal{M} \longrightarrow U^{m|n}$ such that

$$f^*(y^i) = f^i \qquad \textit{and} \qquad f^*(\xi^j) = \eta^j.$$

This means that a morphism between supermanifolds is uniquely determined by the pullback of the coordinate functions (y^i, ξ^j). Hence, we will write a morphism $f\colon \mathcal{M} \longrightarrow \mathcal{N}$ with coordinates (x^i, θ^j) on $\mathcal{M}$ and coordinates (y^α, ξ^β) on $\mathcal{N}$ as

$$f(x^i, \theta^j) = \big(y^\alpha(x^i, \theta^j), \xi^\beta(x^i, \theta^j)\big)$$

to mean that

$$f(x) = \big(v_x(y^1), v_x(y^2), \dots, v_x(y^p)\big)$$

and

$$f^*(y^\alpha) = y^\alpha\left(x^i, \theta^j\right), \qquad f^*(\xi^\beta) = \xi^\beta(x^i, \theta^j),$$

where $x = (x^1, x^2, \dots, x^n)$. This abuse of notation is very convenient in the supermanifold setting since it makes the theory of supermanifolds similar to that of ordinary manifolds.

A.4. Vector fields on supermanifolds

Consider a supermanifold $\mathcal{M} = (M, \mathcal{A})$, and an open subset U of M. Let $X\colon \mathcal{A}(U) \longrightarrow \mathcal{A}(U)$ be an $\mathbb{R}$-linear map. Then we say that X is *even* if it preserves parity, that is, $\widetilde{X(f)} = \widetilde{f}$ for homogeneous f; and we say that X as *odd* if it reverses it, that is, $\widetilde{X(f)} = 1 + \widetilde{f}$ for homogeneous f. We say that X is *homogeneous* if it is even or odd. In this case its parity $\widetilde{X}$ is defined to be equal to 0 if it is even and equal to 1 if it is odd. Hence for a homogeneous map X we have $\widetilde{X(f)} = \widetilde{X} + \widetilde{f}$ for any homogeneous function f.

A *homogeneous derivation* on $\mathcal{A}(U)$ is a homogeneous $\mathbb{R}$-linear map $X\colon \mathcal{A}(U) \longrightarrow \mathcal{A}(U)$ such that

$$X(fg) = X(f)g + (-1)^{\widetilde{X}\widetilde{f}} f X(g)$$

for homogeneous functions f and g. Then the *space of derivations* on $\mathcal{A}(U)$ which we denote by $\mathrm{Der}(U)$ is given by $\mathrm{Der}(U) = \mathrm{Der}_0(U) \oplus \mathrm{Der}_1(U)$, where $\mathrm{Der}_0(U)$ and $\mathrm{Der}_1(U)$ are the spaces of even and odd derivations respectively. We can see that Der is a sheaf with respect to restrictions, and will be called the *tangent sheaf.* The space of derivations $\mathrm{Der}(M)$ which is a $C^\infty(\mathcal{M})$-module will be called the *space of vector fields* on $\mathcal{M}$, and will be denoted by $\mathfrak{X}(\mathcal{M})$ as well. If (x^i, θ^j) are local coordinates on $\mathcal{M}$, then a vector field X on $\mathcal{M}$ is expressed locally as

$$X = \sum_i a^i \frac{\partial}{\partial x^i} + \sum_j b^j \frac{\partial}{\partial \theta^j},$$

with functions a^i, b^j, where the operators $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial \theta^j}$ are the $\mathbb{R}$-linear operators given as follows: $\frac{\partial}{\partial x^i}(f\theta^I) = \frac{\partial f}{\partial x^i}\theta^I$ where $f \in C^\infty(M)$, and $\frac{\partial}{\partial \theta^j}(\theta^j f) = f$, $\frac{\partial}{\partial \theta^j}(f) = 0$ when f does not contain θ^j. The space $\mathfrak{X}(\mathcal{M})$ is a locally free module, and it is a super Lie algebra for which the Lie bracket is given by

$$[X, Y] = X \circ Y - (-1)^{\widetilde{X}\widetilde{Y}} Y \circ X$$

for homogeneous vector fields X and Y.

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