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SIGN CHANGES OF THE COEFFICIENTS OF TRIPLE PRODUCT L-FUNCTIONS

Huixue Lao and Fengjiao Qiao

Abstract. Let $f(z)$ be a primitive holomorphic cusp form and $g(z)$ be a Maass cusp form. In this paper, we give quantitative results for the sign changes of coefficients of triple product L-functions $L(f \times f \times f, s)$ and $L(f \times f \times g, s)$.

1. Introduction

Triple product L-functions are important automorphic L-functions. In this paper, our main objective is to study the sign changes of coefficients of triple product L-functions. We consider the holomorphic cusp forms or Maass cusp forms for the full modular group $SL_2(\mathbb{Z})$ which are eigenfunctions of all the Hecke operators T_n. We denote H_k^* by the set of all normalized primitive holomorphic cusp forms of weight k, where $k \geq 2$ is an even integer. More precisely, for $f \in H_k^*$, we have

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z},$$

where $\lambda_f(n) \in \mathbb{R}$ is Hecke eigenvalues of T_n. It is known that $\lambda_f(n)$ satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right), \tag{1}$$

where $m, n \geq 1$ are integers. In 1974, Deligne [4] proved the Ramanujan-Peterson conjecture

$$|\lambda_f(n)| \leq d(n) \ll n^{\varepsilon}, \tag{2}$$

where $d(n)$ is the Dirichlet divisor function.

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Let M_r^* be the set of normalized primitive Maass cusp forms of Laplace eigenvalue $\lambda = \frac{1}{4} + r^2$. For $g \in M_r^*$, we write its Fourier expansion at ∞ as

$$g(z) = \sum_{n=1}^{\infty} \lambda_g(n) \sqrt{y} K_{ir}(2\pi |n| y) e^{2\pi i n z},$$

where K_{ir} is the K-Bessel function and $\lambda_g(n) \in \mathbb{R}$ is the n-th eigenvalue of the Hecke operator. The current best estimate is

$$|\lambda_g(n)| \leq n^{\frac{7}{64}} d(n) \ll n^{\frac{7}{64}+\varepsilon}, \tag{3}$$

which is due to Kim and Sarnak [16, Appendix 2].

A series of articles in the literature are devoted to investigations in the number of sign changes of Fourier coefficients. The number of sign changes of the sequence of Fourier coefficients at prime numbers was first studied by Murty [31]. Moreover, there exists a small positive number θ such that the number of sign changes for $p \leq x$ is at least ax^θ for some $a > 0$. Meher and Murty [30] focused their attention on the sequence of Fourier coefficients of cusp forms and proved that the sequence $\{\lambda_f(n)\}$ has at least one sign change for $n \in (x, x + x^{\frac{43}{70}}]$. For two different non-trivial cusp forms $f \in H_{k_1}^*$ and $h \in H_{k_2}^*$, Kumari and Murty [20] got the lower bound of the number of sign changes of $\{\lambda_f(n)\lambda_h(n)\}$. They showed that the sequence $\{\lambda_f(n)\lambda_h(n)\}$ has at least one sign change for $n \in (x, x + x^{1-\delta}]$ for sufficiently large x and $\delta > \frac{7}{8}$. In addition, Banerjee and Pandey [1] and Lowry–Duda [27] studied sign changes of $\{\lambda_f(n)\}$ on indices which are sums of two squares. Our first aim is to prove the following two theorems for $f \in H_k^*$.

Theorem 1.1. *Suppose $f \in H_k^*$ and $\lambda_{f\times f\times f}(n)$ is the Dirichlet coefficient of $L(f \times f \times f, s)$. Then*

(i) *for any δ_1 with $\frac{5743}{5953} < \delta_1 < 1$, the sequence $\{\lambda_{f\times f\times f}(n)\}$ has at least one sign change for $n \in (x, x + x^{\delta_1}]$ for sufficiently large x. Moreover, the number of sign changes for $n \leq x$ is $\gg x^{1-\delta_1}$;*

(ii) *for any δ_2 with $\frac{2048}{2083} < \delta_2 < 1$, the sequence $\{\lambda_{f\times f\times f}(n) : n = c^2 + d^2, (c, d) \in \mathbb{N}^2\}$ has at least one sign change among indices $n = c^2 + d^2$ with $n \in (x, x + x^{r_2}]$ for sufficiently large x. Moreover, the number of sign changes for $n \leq x$ is $\gg x^{1-\delta_2}$.*

Remark 1.1. By comparison, $\frac{5743}{5953} < \frac{2903}{3008}$. Hence (i) of Theorem 1.1 improves the result of Theorem 1.4 in Hua [10].

Theorem 1.2. *Assume the Generalized Lindelöf Hypothesis. Suppose $f \in H_k^*$ and $\lambda_{f\times f\times f}(n)$ is the Dirichlet coefficient of $L(f \times f \times f, s)$. Then for any δ_3 with $\frac{1}{2} < \delta_3 < 1$, both the sequences $\{\lambda_{f\times f\times f}(n)\}$ and $\{\lambda_{f\times f\times f}(n) : n = c^2 + d^2, (c, d) \in \mathbb{N}^2\}$ have at least one sign change for $n \in (x, x + x^{\delta_3}]$ for sufficiently large x. Moreover, the numbers of sign changes for $n \leq x$ are both $\gg x^{1-\delta_3}$.*

Recently, much attention is drawn to Maass cusp forms. Jiang et al. [15] showed the mean value connected with Fourier coefficients of Maass cusp forms. And Liu et al. [26] investigated the power moments of automorphic L-function attached to Maass cusp forms. For $g \in M_r^*$, Tang [35] established the upper bound of a shifted convolution sum of $d_3(n)$ and $\lambda_g(n)$. Hafner and Ivić [7] proved that $\sum_{n\le x} \lambda_g(n) \ll x^{\frac{2}{5}}$. Further more, Lü [28] successfully improved $\sum_{n\le x} \lambda_g(n) \ll x^{\frac{1027}{2827}+\varepsilon}$. Lü's approach seems to be flexible enough for the study of more general correlations sums associated with Maass cusp forms. More recently, Kumari and Sengupta [21] proved the sequence $\{\lambda_g(n)\lambda_h(n)\}$ has infinitely many changes for g and h being two Maass cusp forms.

Our second aim is to obtain quantitative results for the sign changes of coefficients of triple product L-function $L(f\times f\times g, s)$ attached to $f \in H_k^*$ and $g \in M_r^*$.

Theorem 1.3. *Suppose $f \in H_k^*$, $g \in M_r^*$ and $\lambda_{f\times f\times g}(n)$ is the Dirichlet coefficient of $L(f\times f\times g, s)$. Then*

(i) *for any δ_4 with $\frac{497}{512} < \delta_4 < 1$, the sequence $\{\lambda_{f\times f\times g}(n)\}$ has at least one sign change for $n \in (x, x+x^{\delta_4}]$ for sufficiently large x. Moreover, the number of sign changes for $n \le x$ is $\gg x^{1-\delta_4}$;*

(ii) *for any δ_5 with $\frac{63}{64} < \delta_5 < 1$, the sequence $\{\lambda_{f\times f\times g}(n) : n = c^2 + d^2, (c,d) \in \mathbb{N}^2\}$ has at least one sign change among indices $n = c^2+d^2$ with $n \in (x, x+x^{\delta_5}]$ for sufficiently large x. Moreover, the number of sign changes for $n \le x$ is $\gg x^{1-\delta_5}$.*

This paper is organized as follows. In Section 2, we recall some fundamental facts of L-functions and give some definitions. In Section 3, we introduce analytic properties and individual and convexity bounds of L-functions. We also give the main tools which we need in the proofs. Inspired by Lowry–Duda [27, Theorem 3], we establish the general criteria to detect sign changes. In the last three sections, we will prove Theorem 1.1–Theorem 1.3 by Lemma 3.7. In order to fit the conditions of Lemma 3.7, it's necessary to calculate the bounds of partial sums of the coefficients $\lambda_{f\times f\times f}(n)r(n)$, $\lambda^2_{f\times f\times f}(n)r(n)$, $\lambda_{f\times f\times g}(n)r(n)$ and $\lambda^2_{f\times f\times g}(n)r(n)$. In Section 4, the main techniques are the factorization of an automorphic L-function into a product of L-functions of lower ranks. Then we use Perron's formula and Cauchy's residue theorem for the proof of Theorem 1.1. The detailed method was nicely and extensively discussed in Theorem 1.2 in [29]. In Section 5, under the Generalized Lindelöf Hypothesis, Lowry–Duda [27] assumed the strongest conjectured bounds. Based on this, the proof of Theorem 1.2 follows from the same line of proof of Theorem 2 in [27] and Theorem 1.2 in [29]. That means we also apply Perron's formula and Cauchy's residue theorem. The difference between the proof of Theorem 1.1 and Theorem 1.2 is that we will take advantage of the Generalized Lindelöf Hypothesis. In Section 6, we firstly decompose the corresponding Dirichlet series into a product of automorphic L-functions and a simpler Dirichlet series. Then we will prove Theorem 1.3 after applications of Landau's Lemma and Lemma 3.9.

2. Preliminaries on L-functions

In this section, we give definitions and recall some fundamental facts of L-functions. The L-functions related to $g \in M_r^*$ is the same as L-functions involving $f \in H_k^*$. For convenience, we just give the definitions of L-functions related to $f \in H_k^*$.

2.1. Hecke L-function

Let $L(f,s)$ be a Dirichlet series associated to $f \in H_k^*$ which admits an Euler product, given as

$$L(f,s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}$$

for $\Re e(s) > 1$. By the work of Deligne [4], we have

$$L(f,s) = \prod_p \big(1 - \frac{\alpha_f(p)}{p^s}\big)^{-1} \big(1 - \frac{\beta_f(p)}{p^s}\big)^{-1},$$

where $\alpha_f(p)$ and $\beta_f(p)$ are two complex numbers, such that

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p), \qquad \alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1. \tag{4}$$

For $g \in M_r^*$, by Kim and Sarnak [16, Appendix 2], we have

$$|\alpha_g(p)| \leq p^{\frac{7}{64}}, \quad |\beta_g(p)| \leq p^{\frac{7}{64}}, \tag{5}$$

where we use $\alpha_g(p)$ and $\beta_g(p)$ for the similar meanings as $\alpha_f(p)$ and $\beta_f(p)$.

2.2. Symmetric power L-function

We define the jth symmetric power L-function as

$$L(\mathrm{sym}^j f, s) = \prod_p \prod_{m=0}^{j} \big(1 - \frac{\alpha_f(p)^{j-m}\beta_f(p)^m}{p^s}\big)^{-1} \tag{6}$$

for $\Re e(s) > 1$. And $L(\mathrm{sym}^j f, s)$ can be represented as a Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^j f, s) &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} \\ &= \prod_p \big(1 + \sum_{k=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}}\big), \end{aligned}$$

where $\lambda_{\mathrm{sym}^j f}(n)$ is a real and multiplicative function. Then one checks that

$$\lambda_{\mathrm{sym}^j f}(p) = \sum_{m=0}^{j} \alpha_f(p)^{j-m}\beta_f(p)^m = \lambda_f(p^j). \tag{7}$$

Let χ be a Dirichlet's character modulo q. Then we define the twisted jth symmetric power L-functon as

$$\begin{aligned} &L(\mathrm{sym}^j f \times \chi, s) \\ &= \prod_p \prod_{m=0}^{j} (1 - \frac{\alpha_f(p)^{j-m}\beta_f(p)^m \chi(p)}{p^s})^{-1}. \end{aligned}$$

It's easy to see that

$$\begin{cases} L(\mathrm{sym}^0 f \times \chi, s) = L(\chi, s), \\ L(\mathrm{sym}^1 f \times \chi, s) = L(f \times \chi, s). \end{cases}$$

2.3. Rankin Selberg *L*-function

The Rankin–Selberg L-function associated with $\mathrm{sym}^i f$ and $\mathrm{sym}^j g$ is defined by

$$\begin{aligned} &L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) \\ &= \prod_p \prod_{m=0}^{i} \prod_{n=0}^{j} (1 - \frac{\alpha_f(p)^{i-m}\beta_f(p)^m \alpha_g(p)^{j-n}\beta_g(p)^n}{p^s})^{-1}. \end{aligned}$$

$L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ can also be written as

$$\begin{aligned} &L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) \\ &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)}{n^s} \\ &= \prod_p (1 + \sum_{k=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p^k)}{p^{ks}}). \end{aligned}$$

Then we get

$$\begin{aligned} &\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p) \\ &= \sum_{m=0}^{i} \sum_{n=0}^{j} \alpha_f(p)^{i-m}\beta_f(p)^m \alpha_g(p)^{j-n}\beta_g(p)^n \\ &= \lambda_{\mathrm{sym}^i f}(p)\lambda_{\mathrm{sym}^j g}(p). \end{aligned} \tag{8}$$

We define the Rankin–Selberg L-function of $\mathrm{sym}^i f$ and $\mathrm{sym}^j g \times \chi$ as

$$\begin{aligned} &L(\mathrm{sym}^i f \times \mathrm{sym}^j g \times \chi, s) \\ &= \prod_p \prod_{m=0}^{i} \prod_{n=0}^{j} (1 - \frac{\alpha_f(p)^{i-m}\beta_f(p)^m \alpha_g(p)^{j-n}\beta_g(p)^n \chi(p)}{p^s})^{-1}. \end{aligned}$$

2.4. Triple product L-function

For $\Re e(s) > 1$, the triple product L-function is defined as

$$\begin{aligned} &L(f\times f\times f,s)\\ &=\sum_{n=1}^{\infty}\frac{\lambda_{f\times f\times f}(n)}{n^s}\\ &=\prod_p\big(1-\frac{\alpha_f(p)^3}{p^s}\big)^{-1}\big(1-\frac{\alpha_f(p)}{p^s}\big)^{-3}\big(1-\frac{\beta_f(p)}{p^s}\big)^{-3}\big(1-\frac{\beta_f(p)^3}{p^s}\big)^{-1},\end{aligned}$$

where the coefficient $\lambda_{f\times f\times f}(n)$ is real and multiplicative. From (4), we know

$$\lambda_{f\times f\times f}(p)=\lambda_f^3(p). \tag{9}$$

(1), (2) and (9) show that

$$\lambda_{f\times f\times f}(n)\ll n^{\varepsilon}. \tag{10}$$

For $f\in H_k^*$ and $g\in M_r^*$, we define

$$\begin{aligned} &L(f\times f\times g,s)\\ &=\sum_{n=1}^{\infty}\frac{\lambda_{f\times f\times g}(n)}{n^s}\\ &=\prod_p\big(1-\frac{\alpha_f(p)^2\alpha_g(p)}{p^s}\big)^{-1}\big(1-\frac{\alpha_g(p)}{p^s}\big)^{-2}\big(1-\frac{\beta_f(p)^2\alpha_g(p)}{p^s}\big)^{-1}\\ &\quad\times\big(1-\frac{\alpha_f(p)^2\beta_g(p)}{p^s}\big)^{-1}\big(1-\frac{\beta_g(p)}{p^s}\big)^{-2}\big(1-\frac{\beta_f(p)^2\beta_g(p)}{p^s}\big)^{-1}.\end{aligned} \tag{11}$$

(11) implies that

$$\begin{aligned} &\lambda_{f\times f\times g}(p)\\ &=\alpha_f(p)^2\alpha_g(p)+2\alpha_g(p)+\beta_f(p)^2\alpha_g(p)+\alpha_f(p)^2\beta_g(p)+2\beta_g(p)+\beta_f(p)^2\beta_g(p).\end{aligned}$$

In view of (4) and $\lambda_f^2(p)=\alpha_f(p)^2+2+\beta_f(p)^2$, we show

$$\lambda_{f\times f\times g}(p)=\lambda_f^2(p)\lambda_g(p).$$

By (2), (3) and the multiplicative property of $\lambda_{f\times f\times g}(n)$, we find that

$$\lambda_{f\times f\times g}(n)\ll n^{\frac{7}{64}+\varepsilon}. \tag{12}$$

If we want to get the number of sign changes of the sequence $\{\lambda_{f\times f\times f}(n): n=c^2+d^2,(c,d)\in\mathbb{N}^2\}$, we need to consider the summations

$$S_1(x)=:\sum_{n=c^2+d^2\le x}\lambda_{f\times f\times f}(c^2+d^2),$$
$$S_2(x)=:\sum_{n=c^2+d^2\le x}\lambda^2_{f\times f\times f}(c^2+d^2)$$

for $x\ge 1$ and $c,d\in\mathbb{Z}$.

For integers $m \geq 0$ and $k \geq 2$, we define

$$r_k(m) = \sharp\{(n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k, n_1^2 + n_2^2 + \cdots + n_k^2 = m\},$$

which has been widely concerned. For $k = 2$, we have

$$r_2(m) = 4 \sum_{d|m} \chi_4(d),$$

where $\chi_4(d)$ stands for the non-trivial Dirichlet character modulo 4, and we have $\chi_4(m) = \sin \frac{m\pi}{2}$. As a consequence, $S_1(x)$ and $S_2(x)$ can be viewed as

$$S_3(x) = \sum_{n \leq x} \lambda_{f\times f\times f}(n) r_2(n),$$

$$S_4(x) = \sum_{n \leq x} \lambda^2_{f\times f\times f}(n) r_2(n),$$

respectively. We write $r(n)$ for

$$r(n) =: \frac{1}{4} r_2(n) = \sum_{d|n} \chi_4(d). \tag{13}$$

It is obvious that

$$r(p) = 1 + \chi_4(p), \qquad r(p^2) = 1 + \chi_4(p) + \chi_4(p^2). \tag{14}$$

3. Auxiliary lemmas

3.1. Analytic properties, mean values and subconvexity bounds for *L*-functions

Recently, Newton and Thorne [32, Theorem A] proved the automorphy of all symmetric powers for cuspidal Hecke eigenforms of level 1 and weight $k \geq 2$. More precisely, for $j \geq 1$ and $f \in H_k^*$, the L-function $L(\mathrm{sym}^j f, s)$ attached to $\mathrm{sym}^j f$ is automorphic. Then we derive the following Lemma.

Lemma 3.1. *Let $f \in H_k^*$, and the jth symmetric power L-function $L(\mathrm{sym}^j f, s)$ is defined by* (6)*. For $j \geq 1$, $L(\mathrm{sym}^j f, s)$ has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$ and satisfies a certain functional equation of Riemann zeta-type of degree $j+1$ (see* Lemma 1 *in* [37]*).*

From the above Lemma and Section 2.2 in Jiang and Lü [13], we know $L(\mathrm{sym}^j f, s)$ and $L(\mathrm{sym}^j f \times \chi, s)$ are general L-functions in the sense of Perelli [34] for $j \geq 1$ and $f \in H_k^*$. By standard arguments in analytic number theory, the mean values and convexity bounds for $L(\mathrm{sym}^j f, s)$ and $L(\mathrm{sym}^j f \times \chi, s)$ were established.

Lemma 3.2. *Suppose that $f \in H_k^*$ and χ is a primitive character modulo q, then for any $\varepsilon > 0$, we have*

$$\int_T^{2T} |L(\mathrm{sym}^j f, \sigma + it)|^2 dt \ll T^{(j+1)(1-\sigma)+\varepsilon},$$

$$L(\mathrm{sym}^j f, \sigma+it) \ll (|t|+1)^{\frac{j+1}{2}(1-\sigma)+\varepsilon},$$

$$\int_T^{2T} |L(\mathrm{sym}^j f\times\chi, \sigma+it)|^2 dt \ll (qT)^{(j+1)(1-\sigma)+\varepsilon},$$

$$L(\mathrm{sym}^j f\times\chi, \sigma+it) \ll (q(|t|+1))^{\frac{j+1}{2}(1-\sigma)+\varepsilon}$$

uniformly for $\frac{1}{2}\le\sigma\le\ 1+\varepsilon$, $T\ge 1$ *and* $|t|\ge 1$.

Proof. The first two assertions followed from Lemma 3.2 in Lao and Luo [23] for $m=j+1$, and the last two assertions were seen in Lemma 2.4 in Jiang and Lü [13] for $J=1$, $n_J=0$, $m_J=j$ and $N=j+1$. □

For some small degree L-functions, we invoke individual or averaged subconvexity bounds.

Lemma 3.3. *For any* $\varepsilon>0$, *then we have*

$$\int_0^T \left|\zeta\left(\frac{5}{7}+it\right)\right|^{12} dt \ll T^{1+\varepsilon}$$

and

$$\zeta(\sigma+it) \ll (|t|+1)^{\frac{13}{42}(1-\sigma)+\varepsilon}$$

uniformly for $T\ge 1$, $\frac{1}{2}\le\sigma\le\ 1+\varepsilon$ *and* $|t|\ge 1$.

Proof. See Theorem 8.4 and (8. 87) in [11], Theorem 5 in [2], respectively. □

Lemma 3.4. *Let* $f\in H_k^*$ *and any* $\varepsilon>0$, *then we have*

$$\int_0^T \left|L\left(f,\frac{5}{8}+it\right)\right|^4 dt \ll T^{1+\varepsilon}$$

and

$$L(f,\sigma+it) \ll (|t|+1)^{\frac{2}{3}(1-\sigma)+\varepsilon}$$

uniformly for $T\ge 1$, $\frac{1}{2}\le\sigma\le\ 1+\varepsilon$ *and* $|t|\ge 1$.

Proof. These are Theorem 2, (1.8) in [12] and Corollary 3 in [6], respectively. □

Lemma 3.5. *Let* $f\in H_k^*$ *and any* $\varepsilon>0$, *then we have*

$$L(\mathrm{sym}^2 f,\sigma+it) \ll (|t|+1)^{\frac{6}{5}(1-\sigma)+\varepsilon}$$

uniformly for $\frac{1}{2}\le\sigma\le\ 1+\varepsilon$ *and* $|t|\ge 1$.

Proof. See Corollary 1.2 in [25]. □

On the basis of the subconvexity bounds given by Heath-Brown [9] and Kuan [19], using the phragmen-lindelöf principle, Xu [36] proved the following two results.

Lemma 3.6. *Let $f \in H_k^*$ and χ be a primitive character modulo q. For any $\varepsilon > 0$ and $t \geq 1$ with $q \ll t^2$, we have*

$$L(\chi, \sigma + it) \ll (q(|t| + 1))^{\frac{1}{3}(1-\sigma)+\varepsilon}$$

and

$$L(f \times \chi, \sigma + it) \ll (q(|t| + 1))^{\frac{2}{3}(1-\sigma)+\varepsilon}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \varepsilon$ and $|t| \geq 1$.

By Kim [17], π is an automorphic cuspidal representation of $GL_2(\mathbb{Q}_A)$. It's known that $\mathrm{sym}^k\pi$ is an automorphic representation of $GL_{k+1}(\mathbb{Q}_A)$. For $k = 2$, Gelbart and Jacquet [5] proved that $\mathrm{sym}^2\pi$ is cuspidal if and only if π is not monomial. For $k = 3$, by Kim and Shahidi [18], $\mathrm{sym}^3\pi$ is cuspidal if and only if π does not conform to a dihedral or tetrahedral Galois representation. For $k = 4$, $\mathrm{sym}^4\pi$ is either cuspidal or unitarily induced from cuspidal representations of $GL_2(\mathbb{Q}_A)$ and $GL_3(\mathbb{Q}_A)$ by Kim [16]. For $1 \leq i \leq j \leq 4$ and $g \in M_r^*$, $L(\mathrm{sym}^j g, s)$ and $L(\mathrm{sym}^i g \times \mathrm{sym}^j g, s)$ are automorphic. Thanks to the work of Lü and Sankaranarayanan [29], in which it is established that the triple product L-function is closely related to $GL_3(\mathbb{Q}_A) \times GL_2(\mathbb{Q}_A)$ L-function $L(\mathrm{sym}^2 f \times f, s)$, namely $L(f \times f \times f, s)$ and $L(f \times f \times g, s)$ are automorphic.

3.2. Main tools used in the proofs

In this part, we collect some Lemmas which will be used in our proofs. Based on the work of Lowry–Duda [27, Theorem 3], we give the following Lemma to detect sign changes.

Lemma 3.7. *Let $\omega(n) \geq 0$ denote a system of non-negative weights. Suppose a sequence of real numbers $\{a(n)\}$ satisfies*

(i) $a(n) = O(n^{\alpha+\varepsilon})$,
(ii) $\sum_{n \leq x} a(n)\omega(n) = O(x^{\beta+\varepsilon})$,
(iii) $\sum_{n \leq x} a^2(n)\omega(n) = x^\gamma P_m(\log x) + O(x^{\eta+\varepsilon})$,

where α, β, γ and η are positive real constants and $P_m(t)$ is a polynomial of degree m. Then for any δ with

$$\max(\alpha + \beta, \eta) - (\gamma - 1) < \delta < 1, \tag{15}$$

the sequence $\{a(n)\}$ has at least one sign change for n in the interval $(x, x+x^\delta]$ and all $x \gg 1$. Moreover, the number of sign changes for $n \leq x$ is greater or equal to $x^{1-\delta}$.

Proof. Suppose $a(n)$ is positive for all $n \in (x, x + x^\delta]$. We choose

$$\varepsilon < \frac{\delta + (\gamma - 1) - \max(\alpha + \beta, \eta)}{2}. \tag{16}$$

By conditions (i) and (ii), we conclude

$$\sum_{x \leq n \leq x+x^\delta} a(n)^2\omega(n) \ll x^{\alpha+\varepsilon} \sum_{x \leq n \leq x+x^\delta} a(n)\omega(n) \ll x^{\alpha+\beta+2\varepsilon}.$$

Then (15) and condition (iii) imply that

$$\begin{aligned}
&\sum_{x\le n\le x+x^{\delta}} a(n)^2\omega(n)\\
&= (x+x^{\delta})^{\gamma}P_m(\log(x+x^{\delta})) - x^{\gamma}P_m(\log x) + O(x^{\eta+\varepsilon})\\
&= (x^{\gamma}+\gamma x^{\gamma-1+\delta}+O(x^{\gamma-2+2\delta}))\cdot P_m(\log(x+x^{\delta})) - x^{\gamma}P_m(\log x) + O(x^{\eta+\varepsilon})\\
&= a_0\gamma x^{\gamma-1+\delta}(\log x)^m + O(x^{\gamma-1+\delta}(\log x)^{m-1}) + O(x^{\eta+\varepsilon})\\
&\gg x^{\gamma-1+\delta},
\end{aligned}$$

where a_0 is the coefficient of t^m in $P_m(t)$ and $\delta+(\gamma-1)>\eta+\varepsilon$ according to (16). These two inequalities imply $x^{\gamma-1+\delta}\ll x^{\alpha+\beta+2\varepsilon}$, which is in conflict with the original assumption. Thus $a(n)$ changes sign at least once for $n\in(x,x+x^{\delta}]$. The number of sign changes is obvious and can be obtained immediately. □

Lemma 3.8. *Let $g\in M_r^*$, then*

$$\prod_p\big(1+\frac{|\alpha_g(p)|^n+|\beta_g(p)|^n}{p^{\sigma}}\big)$$

converges for $\sigma>1$ and $0\le n\le 8$.

Proof. See Lemma 2.2 in [24]. □

The following Lemma is a special circumstance of Theorem 4.1 in [3] in the case when

$$\mu_n=\lambda_n=n,\quad a_n=b_n=C(n),\quad A=4,\quad \rho=8,\quad \delta=1,\quad q=-\infty.$$

Lemma 3.9. *For $f\in H_k^*$, $g\in M_r^*$ and $\vartheta\ge 0$, we have*

$$\sum_{n\le x}C(n)=O\big(x^{\frac{7}{16}+\frac{7}{2}\vartheta}\big)+O\Big(\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}}|C(n)|\Big),$$

where we take $C(n)$ as $\lambda_{f\times f\times g}(n)$ and $\lambda_{f\times f\times g}(n)r_2(n)$.

After the application of Landau's Lemma, we will make use of this Lemma to calculate the mean value of $\lambda_{f\times f\times g}(n)$ and $\lambda_{f\times f\times g}(n)r_2(n)$.

4. Proof of Theorem 1.1

In this section, we calculate summations of coefficients of triple product L-functions to satisfy the conditions of Lemma 3.7.

Our first goal, however, is to replace a complicated Dirichlet series with a simpler one for a controllable discrepancy. For $\Re e(s)>1$, we write

$$L_1(s)=\sum_{n=1}^{\infty}\frac{\lambda^2_{f\times f\times f}(n)}{n^s}=\prod_p\big(1+\sum_{k\ge 1}\frac{\lambda^2_{f\times f\times f}(p^k)}{p^{ks}}\big).$$

For $f \in H_k^*$, Lü and Sankaranarayanan [29] proved

$$L(f \times f \times f, s) = L(f, s)^2 L(\mathrm{sym}^3 f, s)$$

and

$$L_1(s) = \zeta(s)^5 L(\mathrm{sym}^2 f, s)^8 L(\mathrm{sym}^4 f, s)^4 L(\mathrm{sym}^2 f \times \mathrm{sym}^4 f, s) U_1(s),$$

where the function $U_1(s)$ is a Dirichlet series and absolutely convergent for $\Re e(s) > \frac{1}{2}$ and $U_1(s) \neq 0$ for $\Re e(s) = 1$.

Theorem A in [32] implies that the L-function $L(\mathrm{sym}^j f, s)$ is automorphic for $j \geq 1$ and $f \in H_k^*$. And one should note that for a holomorphic cusp form $f(z)$, $L(\mathrm{sym}^2 f \times \mathrm{sym}^4 f, s) = L(\mathrm{sym}^2 f, s) L(\mathrm{sym}^4 f, s) L(\mathrm{sym}^6 f, s)$. So we have

$$L_1(s) = \zeta(s)^5 L(\mathrm{sym}^2 f, s)^9 L(\mathrm{sym}^4 f, s)^5 L(\mathrm{sym}^6 f, s) U_1(s), \tag{17}$$

where the function $U_1(s)$ is a Dirichlet series and absolutely convergent for $\Re e(s) > \frac{1}{2}$ and $U_1(s) \neq 0$ for $\Re e(s) = 1$. Motivated by this and [22], we can determine the following proposition.

Proposition 4.1. *Let $f \in H_k^*$ and $\chi_4(n)$ be the non-trivial Dirichlet character modulo 4, we define $L_2(s)$ and $L_3(s)$ as*

$$L_2(s) =: \sum_{n=1}^{\infty} \frac{\lambda_{f \times f \times f}(n) r(n)}{n^s}$$

and

$$L_3(s) =: \sum_{n=1}^{\infty} \frac{\lambda^2_{f \times f \times f}(n) r(n)}{n^s}$$

for $\Re e(s) > 1$. Then we have

$$L_2(s) = L(f, s)^2 L(f \times \chi_4, s)^2 L(\mathrm{sym}^3 f, s) L(\mathrm{sym}^3 f \times \chi_4, s) U_2(s)$$

and

$$\begin{aligned} L_3(s) = \zeta(s)^5 L(\chi_4, s)^5 L(\mathrm{sym}^2 f, s)^9 L(\mathrm{sym}^2 f \times \chi_4, s)^9 L(\mathrm{sym}^4 f, s)^5 \\ \times L(\mathrm{sym}^4 f \times \chi_4, s)^5 L(\mathrm{sym}^6 f, s) L(\mathrm{sym}^6 f \times \chi_4, s) U_3(s), \end{aligned}$$

where $U_2(s)$ and $U_3(s)$ are Dirichlet series and absolutely convergent in the half plane $\Re e(s) \geq \frac{1}{2} + \varepsilon$.

Proof. Since $\lambda_{f \times f \times f}(p)$ and $\lambda^2_{f \times f \times f}(p)$ are multiplicative and satisfy the trivial upper bound $O(n^{\varepsilon})$, we have

$$L_2(s) = \prod_p \big(1 + \frac{\lambda_{f \times f \times f}(p) r(p)}{p^s} + \frac{\lambda_{f \times f \times f}(p^2) r(p^2)}{p^{2s}} + \cdots\big)$$

and

$$L_3(s) = \prod_p \big(1 + \frac{\lambda^2_{f \times f \times f}(p) r(p)}{p^s} + \frac{\lambda^2_{f \times f \times f}(p^2) r(p^2)}{p^{2s}} + \cdots\big)$$

for $\Re e(s) \geq \frac{1}{2} + \varepsilon$.

(4), (7) and (9) imply that

$$\lambda_{f\times f\times f}(p) = \lambda_f^3(p) = \alpha_f^3(p) + 3\alpha_f(p) + 3\beta_f(p) + \beta_f^3(p) = 2\lambda_f(p) + \lambda_{\mathrm{sym}^3 f}(p).$$

By (4), (7) and (8), we can easily find that

$$\begin{aligned}\lambda_{f\times f\times f}(p)r(p) &= (2\lambda_f(p) + \lambda_{\mathrm{sym}^3 f}(p))(1+\chi_4(p))\\ &= 2\lambda_f(p) + 2\lambda_f(p)\times\chi_4(p) + \lambda_{\mathrm{sym}^3 f}(p) + \lambda_{\mathrm{sym}^3 f}(p)\times\chi_4(p)\\ &=: b_1(p)\end{aligned}$$

and

$$\begin{aligned}\lambda^2_{f\times f\times f}(p)r(p) &= (2\lambda_f(p) + \lambda_{\mathrm{sym}^3 f}(p))^2\cdot(1+\chi_4(p))\\ &= (4\lambda_f^2(p) + \lambda^2_{\mathrm{sym}^3 f}(p) + 4\lambda_f(p)\lambda_{\mathrm{sym}^3 f}(p))\cdot(1+\chi_4(p))\\ &= 4(1+\lambda_{\mathrm{sym}^2 f}(p))\cdot(1+\chi_4(p))\\ &\quad + (1+\lambda_{\mathrm{sym}^2 f}(p) + \lambda_{\mathrm{sym}^4 f}(p) + \lambda_{\mathrm{sym}^6 f}(p))\cdot(1+\chi_4(p))\\ &\quad + 4(\lambda_{\mathrm{sym}^2 f}(p) + \lambda_{\mathrm{sym}^4 f}(p))\cdot(1+\chi_4(p))\\ &= 5 + 5\chi_4(p) + 9\lambda_{\mathrm{sym}^2 f}(p) + 9\lambda_{\mathrm{sym}^2 f}(p)\times\chi_4(p) + 5\lambda_{\mathrm{sym}^4 f}(p)\\ &\quad + 5\lambda_{\mathrm{sym}^4 f}(p)\times\chi_4(p) + \lambda_{\mathrm{sym}^6 f}(p) + \lambda_{\mathrm{sym}^6 f}(p)\times\chi_4(p)\\ &=: b_2(p),\end{aligned}$$

where

$$L(f,s)^2 L(f\times\chi_4,s)^2 L(\mathrm{sym}^3 f,s) L(\mathrm{sym}^3 f\times\chi_4,s) =: \sum_{n=1}^{\infty}\frac{b_1(n)}{n^s}$$

and

$$\zeta(s)^5 L(\chi_4,s)^5 L(\mathrm{sym}^2 f,s)^9 L(\mathrm{sym}^2 f\times\chi_4,s)^9 L(\mathrm{sym}^4 f,s)^5 =: \sum_{n=1}^{\infty}\frac{b_2(n)}{n^s}.$$

Then we have

$$\begin{aligned}L_2(s) &= L(f,s)^2 L(f\times\chi_4,s)^2 L(\mathrm{sym}^3 f,s) L(\mathrm{sym}^3 f\times\chi_4,s)\\ &\quad\times\prod_p\big(1+\frac{\lambda_{f\times f\times f}(p^2)r(p^2) - b_1(p^2)}{p^{2s}} + \cdots\big)\\ &= L(f,s)^2 L(f\times\chi_4,s)^2 L(\mathrm{sym}^3 f,s) L(\mathrm{sym}^3 f\times\chi_4,s) U_2(s)\end{aligned}$$

and

$$\begin{aligned}&L_3(s)\\ &= \zeta(s)^5 L(\chi_4,s)^5 L(\mathrm{sym}^2 f,s)^9 L(\mathrm{sym}^2 f\times\chi_4,s)^9 L(\mathrm{sym}^4 f,s)^5 L(\mathrm{sym}^4 f\times\chi_4,s)^5\\ &\quad\times L(\mathrm{sym}^6 f,s) L(\mathrm{sym}^6 f\times\chi_4,s)\prod_p\big(1+\frac{\lambda^2_{f\times f\times f}(p^2)r(p^2) - b_2(p^2)}{p^{2s}} + \cdots\big)\\ &= \zeta(s)^5 L(\chi_4,s)^5 L(\mathrm{sym}^2 f,s)^9 L(\mathrm{sym}^2 f\times\chi_4,s)^9 L(\mathrm{sym}^4 f,s)^5 L(\mathrm{sym}^4 f\times\chi_4,s)^5\\ &\quad\times L(\mathrm{sym}^6 f,s) L(\mathrm{sym}^6 f\times\chi_4,s) U_3(s).\end{aligned}$$

With the help of (2), we know $U_2(s)$ and $U_3(s)$ are Dirichlet series and absolutely convergent in the half plane $\Re e(s) \geq \frac{1}{2}+\varepsilon$. Then Proposition 4.1 follows from the above identities. □

Then by Perron's formula and Cauchy's residue theorem, we arrive at

$$\begin{aligned}
&\sum_{n\leq x}\lambda_{f\times f\times f}(n)r_2(n)\\
&=\frac{4}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}L_2(s)\frac{x^s}{s}ds+O\big(\frac{x^{1+\varepsilon}}{T}\big)\\
&=\frac{4}{2\pi i}\big(\int_{\frac{5}{8}+\varepsilon-iT}^{\frac{5}{8}+\varepsilon+iT}+\int_{\frac{5}{8}+\varepsilon+iT}^{1+\varepsilon+iT}+\int_{1+\varepsilon-iT}^{\frac{5}{8}+\varepsilon-iT}\big)L_2(s)\frac{x^s}{s}ds+O\big(\frac{x^{1+\varepsilon}}{T}\big)\\
&=H_1+H_2+H_3+O\big(\frac{x^{1+\varepsilon}}{T}\big),
\end{aligned}$$

where $1\leq T\leq x$ is a parameter to be chosen later. Using Cauchy's inequality and Proposition 4.1, H_1 becomes

$$\begin{aligned}
&H_1\\
&\ll x^{\frac{5}{8}+\varepsilon}\int_1^T\big|L\big(f,\frac{5}{8}+\varepsilon+it\big)^2L\big(f\times\chi_4,\frac{5}{8}+\varepsilon+it\big)^2L\big(\mathrm{sym}^3f,\frac{5}{8}+\varepsilon+it\big)\\
&\quad\times L\big(\mathrm{sym}^3f\times\chi_4,\frac{5}{8}+\varepsilon+it\big)U_2\big(\frac{5}{8}+\varepsilon+it\big)\big|t^{-1}dt+x^{\frac{5}{8}+\varepsilon}\\
&\ll x^{\frac{5}{8}+\varepsilon}\log T\max_{T_1\leq T}\big\{\frac{1}{T_1}L\big(f\times\chi_4,\frac{5}{8}+\varepsilon+iT_1\big)^2L\big(\mathrm{sym}^3f,\frac{5}{8}+\varepsilon+iT_1\big)\\
&\quad\times\big(\int_{\frac{T_1}{2}}^{T_1}\big|L\big(f,\frac{5}{8}+\varepsilon+it\big)\big|^4dt\big)^{\frac{1}{2}}\big(\int_{\frac{T_1}{2}}^{T_1}\big|L\big(\mathrm{sym}^3f\times\chi_4,\frac{5}{8}+\varepsilon+it\big)\big|^2dt\big)^{\frac{1}{2}}\big\}\\
&\quad+x^{\frac{5}{8}+\varepsilon}.
\end{aligned}$$

Now depending on Lemma 3.2, Lemma 3.4 and Lemma 3.6, we deduce that

$$H_1\ll x^{\frac{5}{8}+\varepsilon}T^{\frac{1}{2}+\frac{3}{4}+\frac{1}{2}+\frac{3}{4}-1+\varepsilon}+x^{\frac{5}{8}+\varepsilon}\ll x^{\frac{5}{8}+\varepsilon}T^{\frac{3}{2}+\varepsilon}+x^{\frac{5}{8}+\varepsilon}\ll x^{\frac{5}{8}+\varepsilon}T^{\frac{3}{2}+\varepsilon}.$$

For H_2+H_3, we have

$$\begin{aligned}
H_2+H_3&\ll\int_{\frac{5}{8}+\varepsilon}^{1+\varepsilon}x^\sigma|L(f,\sigma+iT)^2L(f\times\chi_4,\sigma+iT)^2L(\mathrm{sym}^3f,\sigma+iT)\\
&\qquad\times L(\mathrm{sym}^3f\times\chi_4,\sigma+iT)|T^{-1}d\sigma\\
&\ll\max_{\frac{5}{8}+\varepsilon\leq\sigma\leq1+\varepsilon}x^\sigma T^{\frac{20}{3}(1-\sigma)+\varepsilon}T^{-1}\ll\frac{x^{1+\varepsilon}}{T}+x^{\frac{5}{8}+\varepsilon}T^{\frac{3}{2}+\varepsilon},
\end{aligned}$$

which is bounded by Lemma 3.2, Lemma 3.4 and Lemma 3.6.

Now taking $T=x^{\frac{3}{20}}$, $\sum_{n\leq x}\lambda_{f\times f\times f}(n)r_2(n)$ turns into

$$\sum_{n\leq x}\lambda_{f\times f\times f}(n)r_2(n)\ll x^{\frac{17}{20}+\varepsilon}. \tag{18}$$

By Perron's formula and Cauchy's residue theorem, we have

$$\begin{aligned}
&\sum_{n\le x}\lambda^2_{f\times f\times f}(n)r_2(n)\\
&=\frac{4}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}L_3(s)\frac{x^s}{s}ds+O\big(\frac{x^{1+\varepsilon}}{T}\big)\\
&=xP_4(\log x)+\frac{4}{2\pi i}\Big(\int_{\frac57+\varepsilon-iT}^{\frac57+\varepsilon+iT}+\int_{\frac57+\varepsilon+iT}^{1+\varepsilon+iT}+\int_{1+\varepsilon-iT}^{\frac57+\varepsilon-iT}\Big)L_3(s)\frac{x^s}{s}ds\\
&\quad+O\big(\frac{x^{1+\varepsilon}}{T}\big)\\
&=xP_4(\log x)+J_1+J_2+J_3+O\big(\frac{x^{1+\varepsilon}}{T}\big),
\end{aligned}\tag{19}$$

where $1\le T\le x$ is a parameter to be chosen later and $P_4(t)$ is a polynomial of degree 4. For J_1, by using Hölder's inequality and Proposition 4.1, it follows that

$$\begin{aligned}
&J_1\\
&\ll x^{\frac57+\varepsilon}\int_1^T\big|\zeta\big(\frac57+\varepsilon+it\big)^5L\big(\chi_4,\frac57+\varepsilon+it\big)^5L\big(\mathrm{sym}^2f,\frac57+\varepsilon+it\big)^9\\
&\quad\times L\big(\mathrm{sym}^2f\times\chi_4,\frac57+\varepsilon+it\big)^9L\big(\mathrm{sym}^4f,\frac57+\varepsilon+it\big)^5\\
&\quad\times L\big(\mathrm{sym}^4f\times\chi_4,\frac57+\varepsilon+it\big)^5L\big(\mathrm{sym}^6f,\frac57+\varepsilon+it\big)\\
&\quad\times L\big(\mathrm{sym}^6f\times\chi_4,\frac57+\varepsilon+it\big)U_1\big(\frac57+\varepsilon+it\big)\big|t^{-1}dt+x^{\frac57+\varepsilon}\\
&\ll x^{\frac57+\varepsilon}\log T\max_{T_1\le T}\big\{\frac{1}{T_1}L\big(\chi_4,\frac57+\varepsilon+iT_1\big)^5L\big(\mathrm{sym}^2f,\frac57+\varepsilon+iT_1\big)^9\\
&\quad\times L\big(\mathrm{sym}^2f\times\chi_4,\frac57+\varepsilon+iT_1\big)^9L\big(\mathrm{sym}^4f\times\chi_4,\frac57+\varepsilon+iT_1\big)^5\\
&\quad\times L\big(\mathrm{sym}^6f,\frac57+\varepsilon+iT_1\big)\big(\int_{\frac{T_1}{2}}^{T_1}\big|L\big(\mathrm{sym}^4f,\ \frac57+\varepsilon+it\big)\big|^{60}dt\big)^{\frac{1}{12}}\\
&\quad\times\big(\int_{\frac{T_1}{2}}^{T_1}\big|\zeta\big(\frac57+\varepsilon+it\big)\big|^{12}dt\big)^{\frac{5}{12}}\big(\int_{\frac{T_1}{2}}^{T_1}\big|L\big(\mathrm{sym}^6f\times\chi_4,\frac57+\varepsilon+it\big)\big|^2dt\big)^{\frac12}\big\}\\
&\quad+x^{\frac57+\varepsilon}.
\end{aligned}$$

In virtue of Lemma 3.2, Lemma 3.3, Lemma 3.5 and Lemma 3.6, we estimate the above terms and deduce that

$$\begin{aligned}
J_1&\ll x^{\frac57+\varepsilon}T^{\frac{10}{21}+\frac{108}{35}+\frac{27}{7}+\frac{25}{7}+1+\frac{5}{12}+\frac{25}{7}+1-1+\varepsilon}+x^{\frac57+\varepsilon}\\
&\ll x^{\frac57+\varepsilon}T^{\frac{6711}{420}+\varepsilon}+x^{\frac57+\varepsilon}.
\end{aligned}$$

For J_2 and J_3, by Lemma 3.2, Lemma 3.3, Lemma 3.5 and Lemma 3.6, we have

$$\begin{aligned}
&J_2 + J_3 \\
\ll &\int_{\frac{5}{7}+\varepsilon}^{1+\varepsilon} x^\sigma |\zeta(\sigma+iT)^5 L(\chi_4, \sigma+iT)^5 L(\mathrm{sym}^2 f, \sigma+iT)^9 \\
&\times L(\mathrm{sym}^2 f \times \chi_4, \sigma+iT)^9 L(\mathrm{sym}^4 f, \sigma+iT)^5 L(\mathrm{sym}^4 f \times \chi_4, \sigma+iT)^5 \\
&\times L(\mathrm{sym}^6 f, \sigma+iT) L(\mathrm{sym}^6 f \times \chi_4, \sigma+iT) U_3(\sigma+iT)| T^{-1} d\sigma \\
\ll &\max_{\frac{5}{7}+\varepsilon \le \sigma \le 1+\varepsilon} x^\sigma T^{\frac{2083}{35}(1-\sigma)+\varepsilon} T^{-1} \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{5}{7}+\varepsilon} T^{\frac{3921}{245}+\varepsilon}.
\end{aligned}$$

Now taking $T = x^{\frac{35}{2083}}$, $\sum_{n\le x} \lambda^2_{f\times f\times f}(n) r_2(n)$ turns into

$$\sum_{n\le x} \lambda^2_{f\times f\times f}(n) r_2(n) = x P_4(\log x) + O\left(x^{\frac{2048}{2083}+\varepsilon}\right). \tag{20}$$

By (10), (18) and (20), we show that $\alpha = 0$, $\beta = \frac{17}{20}$, $\gamma = 1$ and $\eta = \frac{2048}{2083}$. Therefore, $\max(\alpha+\beta, \eta) - (\gamma - 1) = \frac{2048}{2083} < 1$. Thus we can apply Lemma 3.7 with $a(n) = \lambda_{f\times f\times f}(n)$ and $\omega(n) = r(n)$ to obtain the number of sign changes of $\{\lambda_{f\times f\times f}(n) : n = c^2 + d^2, (c,d) \in \mathbb{N}^2\}$.

For the first result of Theorem 1.1, we also need to consider the summations of the coefficients $\lambda_{f\times f\times f}(n)$ and $\lambda^2_{f\times f\times f}(n)$. Lü and Sankaranarayanan [29] proved

$$\sum_{n\le x} \lambda_{f\times f\times f}(n) \ll x^{\frac{7}{10}+\varepsilon} \tag{21}$$

and

$$\sum_{n\le x} \lambda^2_{f\times f\times f}(n) = x P_4(\log x) + O\left(x^{\frac{175}{181}+\varepsilon}\right). \tag{22}$$

Noting (17) and by the similar argument in [29], we can improve (22). By Perron's formula and Cauchy's residue theorem, one can easily find that

$$\begin{aligned}
&\sum_{n\le x} \lambda^2_{f\times f\times f}(n) \\
&= \frac{4}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} L_1(s) \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right) \\
&= x P_4(\log x) + \frac{4}{2\pi i} \left(\int_{\frac{5}{7}+\varepsilon-iT}^{\frac{5}{7}+\varepsilon+iT} + \int_{\frac{5}{7}+\varepsilon+iT}^{1+\varepsilon+iT} + \int_{1+\varepsilon-iT}^{\frac{5}{7}+\varepsilon-iT} \right) L_1(s) \frac{x^s}{s} ds \\
&\quad + O\left(\frac{x^{1+\varepsilon}}{T}\right) \\
&= x P_4(\log x) + I_1 + I_2 + I_3 + O\left(\frac{x^{1+\varepsilon}}{T}\right).
\end{aligned}$$

where $1 \le T \le x$ is a parameter to be chosen later and $P_4(t)$ is a polynomial of degree 4. For I_1, Hölder's inequality and Proposition 4.1 allow us to write

$$\begin{aligned}
I_1 &\ll x^{\frac{5}{7}+\varepsilon} \int_1^T \big|\zeta\big(\frac{5}{7}+\varepsilon+it\big)^5 L\big(\mathrm{sym}^2 f, \frac{5}{7}+\varepsilon+it\big)^9 L\big(\mathrm{sym}^4 f, \frac{5}{7}+\varepsilon+it\big)^5 \\
&\quad \times L\big(\mathrm{sym}^6 f, \frac{5}{7}+\varepsilon+it\big) U_1\big(\frac{5}{7}+\varepsilon+it\big)\big| t^{-1} dt + x^{\frac{5}{7}+\varepsilon} \\
&\ll x^{\frac{5}{7}+\varepsilon} \log T \max_{T_1 \le T} \Big\{ \frac{1}{T_1} L\big(\mathrm{sym}^2 f, \frac{5}{7}+\varepsilon+iT_1\big)^9 \Big(\int_{\frac{T_1}{2}}^{T_1} \big|\zeta\big(\frac{5}{7}+\varepsilon+it\big)\big|^{12} dt\Big)^{\frac{5}{12}} \\
&\quad \times \Big(\int_{\frac{T_1}{2}}^{T_1} \big|L\big(\mathrm{sym}^4 f, \frac{5}{7}+\varepsilon+it\big)\big|^{60} dt\Big)^{\frac{1}{12}} \\
&\quad \times \Big(\int_{\frac{T_1}{2}}^{T_1} \big|L\big(\mathrm{sym}^6 f, \frac{5}{7}+\varepsilon+it\big)\big|^{2} dt\Big)^{\frac{1}{2}} \Big\}.
\end{aligned}$$

In virtue of Lemma 3.2, Lemma 3.3 and Lemma 3.5, one has

$$I_1 \ll x^{\frac{5}{7}+\varepsilon} T^{\frac{108}{35}+\frac{5}{12}+\frac{25}{7}+1-1+\varepsilon} + x^{\frac{5}{7}+\varepsilon} \ll x^{\frac{5}{7}+\varepsilon} T^{\frac{2971}{420}+\varepsilon} + x^{\frac{5}{7}+\varepsilon}.$$

For I_2 and I_3, we apply Lemma 3.2, Lemma 3.3 and Lemma 3.5 to arrive at

$$\begin{aligned}
&I_2 + I_3 \\
&\ll \int_{\frac{5}{7}+\varepsilon}^{1+\varepsilon} x^{\sigma} |\zeta(\sigma+iT)^5 L(\mathrm{sym}^2 f, \sigma+iT)^9 L(\mathrm{sym}^4 f, \sigma+iT)^5 L(\mathrm{sym}^6 f, \sigma+iT) \\
&\quad \times U_1(\sigma+iT)| T^{-1} d\sigma \\
&\ll \max_{\frac{5}{7}+\varepsilon \le \sigma \le 1+\varepsilon} x^{\sigma} T^{\frac{5953}{210}(1-\sigma)+\varepsilon} T^{-1} \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{5}{7}+\varepsilon} T^{\frac{10436}{1470}+\varepsilon}.
\end{aligned}$$

By taking $T = x^{\frac{210}{5953}}$, $\sum_{n \le x} \lambda^2_{f\times f\times f}(n)$ turns into

$$\sum_{n \le x} \lambda^2_{f\times f\times f}(n) = x P_4(\log x) + O\left(x^{\frac{5743}{5953}+\varepsilon}\right). \tag{23}$$

(10), (21) and (23) satisfy the conditions of Lemma 3.7, i.e., $\alpha = 0$, $\beta = \frac{7}{10}$, $\gamma = 1$ and $\eta = \frac{5743}{5953}$. Hence $\max(\alpha+\beta, \eta) - (\gamma - 1) = \frac{5743}{5953} < 1$. Then we can get the first result of Theorem 1.1 by taking $\omega(n) \equiv 1$.

5. Proof of Theorem 1.2

To prepare for the applications of Lemma 3.7, we will also calculate the summations of coefficients of triple product L-functions. Suppose the parameters β_1, β_2, η_1, η_2 satisfy

$$\begin{aligned}
&\frac{1}{T}\int_1^T |L(\mathrm{sym}^j f, \frac{1}{2}+\varepsilon+it)|^2 dt \ll T^{\beta_1+\varepsilon}, \\
&L(\mathrm{sym}^j f, \frac{1}{2}+\varepsilon+it) \ll |t|^{\beta_2+\varepsilon},
\end{aligned} \tag{24}$$

$$\frac{1}{T}\int_1^T |L(\mathrm{sym}^j f \times \chi_4, \frac{1}{2}+\varepsilon+it)|^2 dt \ll T^{\eta_1+\varepsilon}, \tag{25}$$
$$L(\mathrm{sym}^j f \times \chi_4, \frac{1}{2}+\varepsilon+it) \ll |t|^{\eta_2+\varepsilon}.$$

Using (24) and (25), we have a new representation for the upper estimate of $\sum_{n\le x}\lambda^2_{f\times f\times f}(n)r_2(n)$ in (19). For J_1, (24), (25) and Proposition 4.1 yield

$$J_1 \ll x^{\frac{1}{2}+\varepsilon}T^{19\beta_2+19\eta_2+\frac{\beta_1}{2}+\frac{\eta_1}{2}+\varepsilon}.$$

For J_2+J_3, one has

$$J_2+J_3 \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon}T^{20\beta_2+20\eta_2-1+\varepsilon}.$$

Let

$$\delta = \max\left(19\beta_2+19\eta_2+\frac{\beta_1}{2}+\frac{\eta_1}{2}, 20\beta_2+20\eta_2-1\right).$$

Taking $T = x^{\frac{1}{2(1+\delta)}}$, we obtain

$$J_1+J_2+J_3 \ll x^{1-\frac{1}{2(1+\delta)}+\varepsilon}.$$

By the Generalized Lindelöf Hypothesis and Theorem 1 in [33, pp.63], we have $\beta_1=\beta_2=\eta_1=\eta_2=0$, which means $\delta=0$. Hence we derive

$$J_1+J_2+J_3 \ll x^{\frac{1}{2}+\varepsilon}. \tag{26}$$

Taking (26) into (19), we finally deduce

$$\sum_{n\le x}\lambda^2_{f\times f\times f}(n)r_2(n) = xP_4(\log x) + O\left(x^{\frac{1}{2}+\varepsilon}\right). \tag{27}$$

According to Perron's formula and Cauchy's residue theorem, we conclude

$$\sum_{n\le x}\lambda_{f\times f\times f}(n)r_2(n) \ll x^{\frac{1}{2}+\varepsilon}. \tag{28}$$

Combining (10), (27) and (28), we know $\alpha=0$, $\beta=\frac{1}{2}$, $\gamma=1$ and $\eta=\frac{1}{2}$. These results satisfy $max(\alpha+\beta,\eta)-(\gamma-1)=\frac{1}{2}<1$. Hence the second result of Theorem 1.2 follows from Lemma 3.7 by choosing $\omega(n)=r(n)$. In order to eliminate repetitive typing, for the other result, we shall not give the proof.

6. Proof of Theorem 1.3

For Theorem 1.3, we also need Lemma 3.7. To invoke our conditions, we shall calculate $\sum_{n\le x}\lambda_{f\times f\times g}(n)r_2(n)$ and $\sum_{n\le x}\lambda^2_{f\times f\times g}(n)r_2(n)$ for the number of sign changes of $\{\lambda_{f\times f\times g}(n) : n=c^2+d^2, (c,d)\in\mathbb{N}^2\}$. Since the bound of $\lambda_g(n)$ is not n^ε in the case of Maass cusp forms, Perron's formula is not available at all. To prove Theorem 1.3, we will take advantage of Landau's Lemma.

Proposition 6.1. *For $f \in H_k^*$, $g \in M_r^*$ and $\chi_4(n)$ being the non-trivial Dirichlet character modulo 4, we define*

$$\tilde{L}_1(s) =: \sum_{n=1}^{\infty} \frac{\lambda_{f\times f\times g}^2(n)}{n^s}$$

and

$$\tilde{L}_2(s) =: \sum_{n=1}^{\infty} \frac{\lambda_{f\times f\times g}^2(n) r(n)}{n^s}.$$

Then

$$\begin{aligned}\tilde{L}_1(s) = {}& \zeta(s)^2 L(\mathrm{sym}^2 f, s)^3 L(\mathrm{sym}^4 f, s) L(\mathrm{sym}^2 g, s)^2 L(\mathrm{sym}^2 f \times \mathrm{sym}^2 g, s)^3 \\ & \times L(\mathrm{sym}^4 f \times \mathrm{sym}^2 g, s) U_5(s)\end{aligned}$$

and

$$\begin{aligned}&\tilde{L}_2(s) \\ &= \zeta(s)^2 L(\chi_4, s)^2 L(\mathrm{sym}^2 f, s)^3 L(\mathrm{sym}^2 f \times \chi_4, s)^3 L(\mathrm{sym}^4 f, s) L(\mathrm{sym}^4 f \times \chi_4, s) \\ &\quad \times L(\mathrm{sym}^2 g, s)^2 L(\mathrm{sym}^2 g \times \chi_4, s)^2 L(\mathrm{sym}^2 f \times \mathrm{sym}^2 g, s)^3 L(\mathrm{sym}^4 f \times \mathrm{sym}^2 g, s) \\ &\quad \times L(\mathrm{sym}^2 f \times \mathrm{sym}^2 g \times \chi_4, s)^3 L(\mathrm{sym}^4 f \times \mathrm{sym}^2 g \times \chi_4, s) U_6(s),\end{aligned}$$

where $U_5(s)$ and $U_6(s)$ are Dirichlet series and absolutely convergent for $\Re e(s) > \frac{1}{2}$.

Proof. By the multiplicative property of $\lambda_{f\times f\times g}^2(n) r(n)$, $\tilde{L}_2(s)$ has an Euler product

$$\tilde{L}_2(s) = \prod_p \left(1 + \frac{\lambda_{f\times f\times g}^2(p) r(p)}{p^s} + \frac{\lambda_{f\times f\times g}^2(p^2) r(p^2)}{p^{2s}} + \cdots\right). \tag{29}$$

Applying (4), (7) and (8) again yields the following equations

$$\begin{aligned}&\lambda_{f\times f\times g}^2(p) r(p) \\ &= (\lambda_g(p) + \lambda_{\mathrm{sym}^2 f\times g}(p))^2 (1 + \chi_4(p)) \\ &= (\lambda_g^2(p) + \lambda_{\mathrm{sym}^2 f\times g}^2(p) + 2\lambda_g(p)\lambda_{\mathrm{sym}^2 f\times g}(p))(1 + \chi_4(p)) \\ &= (2 + 3\lambda_{\mathrm{sym}^2 f}(p) + \lambda_{\mathrm{sym}^4 f}(p) + 2\lambda_{\mathrm{sym}^2 g}(p) + 3\lambda_{\mathrm{sym}^2 f\times \mathrm{sym}^2 g}(p) \\ &\quad + \lambda_{\mathrm{sym}^4 f\times \mathrm{sym}^2 g}(p)) \times (1 + \chi_4(p)) \\ &= 2 + 2\chi_4(p) + 3\lambda_{\mathrm{sym}^2 f}(p) + 3\lambda_{\mathrm{sym}^2 f}(p)\chi_4(p) + \lambda_{\mathrm{sym}^4 f}(p) \\ &\quad + \lambda_{\mathrm{sym}^4 f}(p)\chi_4(p) + 2\lambda_{\mathrm{sym}^2 g}(p) + 2\lambda_{\mathrm{sym}^2 g}(p)\chi_4(p) \\ &\quad + 3\lambda_{\mathrm{sym}^2 f\times \mathrm{sym}^2 g}(p) + \lambda_{\mathrm{sym}^4 f\times \mathrm{sym}^2 g}(p) + 3\lambda_{\mathrm{sym}^2 f\times \mathrm{sym}^2 g}(p)\chi_4(p) \\ &\quad + \lambda_{\mathrm{sym}^4 f\times \mathrm{sym}^2 g}(p)\chi_4(p) \\ &=: c(p),\end{aligned} \tag{30}$$

where

$$\begin{aligned}
&\zeta(s)^2L(\chi_4,s)^2L(\mathrm{sym}^2f,s)^3L(\mathrm{sym}^2f\times\chi_4,s)^3L(\mathrm{sym}^4f,s)L(\mathrm{sym}^4f\times\chi_4,s)\\
&\times L(\mathrm{sym}^2g,s)^2L(\mathrm{sym}^2g\times\chi_4,s)^2L(\mathrm{sym}^2f\times\mathrm{sym}^2g,s)^3\\
&\times L(\mathrm{sym}^4f\times\mathrm{sym}^2g,s)L(\mathrm{sym}^2f\times\mathrm{sym}^2g\times\chi_4,s)^3\\
&\times L(\mathrm{sym}^4f\times\mathrm{sym}^2g\times\chi_4,s)\\
&=\tilde{L}_3(s)\\
&=:\sum_{n=1}^{\infty}\frac{c(n)}{n^s}.
\end{aligned}$$

For brevity, we write $\alpha_f(p)=\alpha_f$, $\beta_f(p)=\beta_f$, $\alpha_g(p)=\alpha_g$ and $\beta_g(p)=\beta_g$. After a short calculation, we arrive at the following result with (11) and (14)

$$\begin{aligned}
&\lambda^2_{f\times f\times g}(p^2)r(p^2)\\
&=\big(\alpha_f^8\alpha_g^4+4\alpha_f^6\alpha_g^4+\alpha_f^8\alpha_g^2+10\alpha_f^4\alpha_g^4+16\alpha_f^2\alpha_g^4+\cdots+\alpha_f^8\beta_g^4+4\alpha_f^6\beta_g^4\\
&\quad+\alpha_f^8\beta_g^2+10\alpha_f^4\beta_g^4+16\alpha_f^2\beta_g^4\big)(1+\chi_4(p)+\chi_4(p^2)).
\end{aligned}$$

By (7), (8), (14) and the definition of $c(n)$, we have the equality

$$\begin{aligned}
&c(p^2)\\
&=\big(\alpha_f^8\alpha_g^4+\alpha_f^6\alpha_g^4+\alpha_f^8\alpha_g^2+\cdots+\alpha_f^8\beta_g^4+\alpha_f^6\beta_g^4+\alpha_f^8\beta_g^2\big)(1+\chi_4(p)+\chi_4(p^2)).
\end{aligned}$$

To go further, one derives that

$$\begin{aligned}
&\lambda^2_{f\times f\times g}(p^2)r(p^2)-c(p^2)\\
&=(-\alpha_f^8+3\alpha_f^6\alpha_g^4+\cdots)\times(1+\chi_4(p)+\chi_4(p^2)).
\end{aligned}\tag{31}$$

(29), (30) and (31) yield

$$\begin{aligned}
&\tilde{L}_2(s)\\
&=\zeta(s)^2L(\chi_4,s)^2L(\mathrm{sym}^2f,s)^3L(\mathrm{sym}^2f\times\chi_4,s)^3L(\mathrm{sym}^4f,s)\\
&\quad\times L(\mathrm{sym}^4f\times\chi_4,s)L(\mathrm{sym}^2g,s)^2L(\mathrm{sym}^2g\times\chi_4,s)^2\\
&\quad\times L(\mathrm{sym}^2f\times\mathrm{sym}^2g,s)^3L(\mathrm{sym}^4f\times\mathrm{sym}^2g,s)\\
&\quad\times L(\mathrm{sym}^2f\times\mathrm{sym}^2g\times\chi_4,s)^3L(\mathrm{sym}^4f\times\mathrm{sym}^2g\times\chi_4,s)\\
&\quad\times\prod_p\left(1+\frac{\lambda^2_{f\times f\times g}(p^2)r(p^2)-c(p^2)}{p^{2s}}+\cdots\right)\\
&=\tilde{L}_3(s)U_6(s).
\end{aligned}\tag{32}$$

Due to (2), (5) and (31), we show

$$U_6(s)\ll\prod_p\left(1+\frac{|\alpha_g|^4+|\beta_g|^4}{p^{2\sigma}}\right).$$

Lemma 3.8 means that $U_6(s)$ is absolutely convergent for $\Re e(s) > \frac{1}{2}$. This completes the proof of the second result of Proposition 6.1. We can apply the similar method to deal with $\tilde{L}_1(s)$. $\square$

Since $L(\mathrm{sym}^i g, s)$ and $L(\mathrm{sym}^i g \times \mathrm{sym}^j g, s)$ are automorphic with $1 \le i \le j \le 4$, L-function $\tilde{L}_3(s)$ is entire except for a double pole at $s = 1$. Lemma 2 in [33, pp.512] and (1.12) in [8] imply that

$$\sum_{n\le x} d^2(n) \ll x(\log x)^3, \qquad \sum_{n\le x} \lambda^2_{\mathrm{sym}^2 f}(n) = cx + O(x^{\frac{13}{17}+\varepsilon}),$$

where c is a constant. Then by the refined version of Landau's Lemma (see Corollary 1.4 in [14]) with $d = 128$ and $\sigma_0 = \frac{1}{2}$, we have

$$\sum_{n\le x} \lambda^2_{f\times f\times g}(n) r_2(n) = xQ_1(\log x) + O\left(x^{\frac{63}{64}+\varepsilon}\right), \tag{33}$$

where $Q_1(t)$ is a polynomial of degree 1. By Lemma 3.9, we know

$$\begin{aligned}&\sum_{n\le x} \lambda_{f\times f\times g}(n) r_2(n)\\ &= O\left(x^{\frac{7}{16}+\frac{7}{2}\vartheta}\right) + O\left(\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} |\lambda_{f\times f\times g}(n) r_2(n)|\right).\end{aligned} \tag{34}$$

Then we need to consider the bound of $\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} |\lambda_{f\times f\times g}(n) r_2(n)|$. By Cauchy's inequality, we have

$$\begin{aligned}&\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} |\lambda_{f\times f\times g}(n) r_2(n)|\\ \le&\left(\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} \lambda^2_{f\times f\times g}(n) r_2(n)\right)^{\frac{1}{2}} \times \left(\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} r_2(n)\right)^{\frac{1}{2}}.\end{aligned}$$

Firstly, we shall estimate the first summation $\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} \lambda^2_{f\times f\times g}(n) r_2(n)$ by (33). Since

$$\begin{aligned}\left(\log(x + x^{\frac{7}{8}-\vartheta})\right)^j &= \left(\log x + \log(1 + x^{-\frac{1}{8}-\vartheta})\right)^j\\ &= (\log x)^j + O\left((\log x)^{j-1} \cdot x^{-\frac{1}{8}-\vartheta}\right),\end{aligned}$$

we have

$$\left(x + x^{\frac{7}{8}-\vartheta}\right) \cdot Q_1\left(\log(x + x^{\frac{7}{8}-\vartheta})\right) - xQ_1(\log x) \ll x^{\frac{7}{8}-\vartheta+\varepsilon}. \tag{35}$$

Noting that $\frac{63}{64} > \frac{7}{8}$, by (33) we can compute

$$\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} \lambda^2_{f\times f\times g}(n) r_2(n) \ll x^{\frac{63}{64}+\varepsilon} + x^{\frac{7}{8}-\vartheta+\varepsilon} \ll x^{\frac{63}{64}+\varepsilon}.$$

Next, for $\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} r_2(n)$, (13) implies

$$\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} r_2(n) = 4 \sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} \sum_{d|n} \chi_4(d).$$

Recall that [33]

$$\sum_{n\le x} d(n) = x(\log x + 2\gamma - 1) + O\left(x^{\frac{1}{3}-\frac{1}{246}} \log^2 x\right),$$

thus we deduce that

$$\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} \sum_{d|n} \chi_4(d) \ll \sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} d(n) \ll x^{\frac{7}{8}-\vartheta+\varepsilon}.$$

Furthermore, we have

$$\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} |\lambda_{f\times f\times g}(n) r_2(n)| \ll x^{\frac{63}{64}\times\frac{1}{2}+(\frac{7}{8}-\vartheta)\times\frac{1}{2}+\varepsilon} = x^{\frac{119}{128}-\frac{\vartheta}{2}+\varepsilon}.$$

Substituting these back into (34), we conclude

$$\sum_{n\le x} \lambda_{f\times f\times g}(n) r_2(n) \ll x^{\frac{7}{16}+\frac{7}{2}\vartheta+\varepsilon} + x^{\frac{119}{128}-\frac{\vartheta}{2}+\varepsilon}.$$

Then we take $\vartheta = \frac{63}{512}$ to obtain

$$\sum_{n\le x} \lambda_{f\times f\times g}(n) r_2(n) \ll x^{\frac{889}{1024}+\varepsilon}. \tag{36}$$

With (12), (33) and (36), the second result of Theorem 1.3 follows from Lemma 3.7 by $\alpha = \frac{7}{64}$, $\beta = \frac{889}{1024}$, $\gamma = 1$, $\eta = \frac{63}{64}$ and $\omega(n) = r(n)$, which means $\max(\alpha+\beta, \eta) - (\gamma - 1) = \frac{63}{64} < 1$.

Similar to (33), we can also use Corollary 1.4 in [14] with $d = 64$ and $\sigma_0 = \frac{1}{2}$ to conclude

$$\sum_{n\le x} \lambda^2_{f\times f\times g}(n) = xP_1(\log x) + O\left(x^{\frac{31}{32}+\varepsilon}\right). \tag{37}$$

Furthermore, we infer by Lemma 3.9 that

$$\sum_{n\le x} \lambda_{f\times f\times g}(n) = O\left(x^{\frac{7}{16}+\frac{7}{2}\vartheta}\right) + O\left(\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} |\lambda_{f\times f\times g}(n)|\right). \tag{38}$$

By using Cauchy's inequality, we write $\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} |\lambda_{f\times f\times g}(n)|$ as

$$\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} |\lambda_{f\times f\times g}(n)| \le \left(\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} \lambda^2_{f\times f\times g}(n)\right)^{\frac{1}{2}} \times \left(\sum_{x<n\le x+x^{\frac{7}{8}-\vartheta}} 1\right)^{\frac{1}{2}}.$$

Noting that $\frac{31}{32} > \frac{7}{8}$, by (35) and (37), we find that

$$\sum_{x<n\leq x+x^{\frac{7}{8}-\vartheta}} \lambda^2_{f\times f\times g}(n) \ll x^{\frac{31}{32}+\varepsilon} + x^{\frac{7}{8}-\vartheta+\varepsilon} \ll x^{\frac{31}{32}}.$$

So we have

$$\sum_{x<n\leq x+x^{\frac{7}{8}-\vartheta}} |\lambda_{f\times f\times g}(n)| \ll x^{\frac{31}{32}\times\frac{1}{2}+(\frac{7}{8}-\vartheta)\times\frac{1}{2}+\varepsilon} = x^{\frac{59}{64}-\frac{\vartheta}{2}+\varepsilon}.$$

By putting these bounds into (38), it is evident that

$$\sum_{n\leq x} \lambda_{f\times f\times g}(n) \ll x^{\frac{7}{16}+\frac{7}{2}\vartheta+\varepsilon} + x^{\frac{59}{64}-\frac{\vartheta}{2}+\varepsilon}.$$

Then we choose $\vartheta = \frac{31}{256}$ to deduce

$$\sum_{n\leq x} \lambda_{f\times f\times g}(n) \ll x^{\frac{441}{512}+\varepsilon}. \tag{39}$$

With (12), (37) and (39), we show that $\alpha = \frac{7}{64}$, $\beta = \frac{441}{512}$, $\gamma = 1$ and $\eta = \frac{31}{32}$, which means $\max(\alpha+\beta, \eta) = \frac{497}{512} < 1$. Thus we finish the proof of the first result of Theorem 1.3 by taking $\omega(n) \equiv 1$ in Lemma 3.7.

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Huixue Lao
School of Mathematics and Statistics
Shandong Normal University
Jinan 250358, P. R. China
Email address: lhxsdnu@163.com

Fengjiao Qiao
School of Mathematics and Statistics
Shandong Normal University
Jinan 250358, P. R. China
Email address: 1542460774@qq.com