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PERSISTENCE AND POINTWISE TOPOLOGICAL STABILITY FOR CONTINUOUS MAPS OF TOPOLOGICAL SPACES

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Abstract. In the paper, we prove that if a continuous map of a compact uniform space is equicontinuous and pointwise topologically stable, then it is persistent. We also show that if a sequence of uniformly expansive continuous maps of a compact uniform space has a uniform limit and the uniform shadowing property, then the limit is topologically stable. In addition, we introduce the concepts of shadowable points and topologically stable points for a continuous map of a compact topological space and obtain that every shadowable point of an expansive continuous map of a compact topological space is topologically stable.

1. Introduction

For the sake of description, we denote the sets of natural numbers (including 0), positive integers, integers, positive real numbers and real numbers in the paper by $\mathbb{Z}_0$, $\mathbb{Z}_+$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}$, respectively.

The concept of *topological stability* introduced by Walters in [24] for the studies of Anosov diffeomorphisms plays an important role in the general qualitative theory of dynamical systems. In [25], Walters proved that every *expansive* homeomorphism, which was introduced by Utz in [23], with the *pseudo-orbit shadowing property* (*shadowing property* for brief) of a compact metric space is topologically stable. Recently, Hua and Yin [11] introduced the notions of expansivity, shadowing property and topological stability for continuous maps of compact topological spaces from the viewpoint of open covers and extended *Walters's stability theorem* to continuous maps of disconnected compact topological spaces with *the first axiom of countability.* Variant basic notions of dynamical systems were studied from the pointwise viewpoint (see [4, 14, 17, 21]). For instance, Morales [21] introduced the concepts of *shadowable points* and

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proved that the shadowing property is equivalent to all points to be shadowable in the class of homeomorphisms of compact metric spaces; Koo et al. [17] decomposed the topological stability (in the sense of Walters in [19,25,31]) into the corresponding notion for points and showed that every shadowable point of an expansive homeomorphism of a compact metric space is topologically stable (also see [16]). Moreover, the notion of *pointwise topologically stable homeomorphism* introduced in [17] is the pointwise version of topological stability (see [15] for the case of group actions). The latter implies the former and the converse holds for certain homeomorphisms like the expansive ones of compact manifolds (see [17]). In 2023, Koo and Lee [16] introduced the notion of *uniformly expansive point* and proved that every uniformly expansive and shadowable point of each homeomorphism on a compact metric space is a topologically stable point. Arbieto and Rego [4] introduced the notion of *uniform shadowing property* for a sequence of continuous maps and obtained some sufficient conditions for the limit with positive entropy. Koo and Lee [16] introduced the notion of *uniformly expansiveness* for a sequence of continuous maps and proved that if a sequence $\{f_n\}_{n\in\mathbb{Z}_+}$ of continuous self-maps of a compact metric space X converges uniformly to a continuous self-map f on X and $\{f_n\}_{n\in\mathbb{Z}_+}$ is uniformly expansive and has the uniform shadowing property, then f is topologically stable. Lewowicz introduced the notion of *persistent homeomorphism* in [20]. Furthermore, the authors in [8] proved that every equicontinuous pointwise topologically stable homeomorphism of a compact metric space is persistent and they also introduced some notions of persistence (in the sense of Lewowicz [20]) for Borel probability measures with respect to homeomorphisms of compact metric spaces.

The concept of uniform space was introduced by Weil in [28], which includes some important topological spaces such as metric and pseudo-metrizable spaces, topological groups and function spaces. So far, some dynamical properties of continuous maps of uniform spaces have been studied by many authors. For example, see [5, 30] for the works related to expansivity, [3, 6, 27] for sensitivity and Devaney's chaos, [1, 2, 30] for shadowing property and topological stability, [7] for Spectral decomposition for topologically Anosov homeomorphism on noncompact and non-metrizable spaces.

Motivated by the idea of [10,30], many dynamical properties can be naturally extended to uniform spaces. In this paper, inspired by [4, 8], we introduce the concept of *pointwise topologically stable* continuous map of compact uniform spaces and we prove that pointwise topological stability implies persistence for equicontinuous continuous maps of compact uniform spaces with the first axiom of countability (see Theorem 4.5). The proof relies on the notion of *persistent measure* which has its own interest. Indeed, we decompose the persistence of a continuous map into the corresponding property for Borel probability measures. Furthermore, we extend some of the results of [8] to the case of continuous maps of compact uniform spaces. Concretely, we prove that every almost persistent measure is persistent for an equicontinuous continuous map of a compact uniform space with the first axiom of countability (see Lemma 4.2). We also prove that every equicontinuous pointwise persistent continuous map of a compact uniform space with the first axiom of countability is persistent (see Lemma 4.3).

In addition, in this paper, we give the notions of *uniform shadowing property* and *uniform expansivity* for a sequence of continuous maps of a compact uniform space and study the dynamical behaviors of the limit of a sequence of continuous maps with the expansivity and shadowing property. Concretely, let X be a compact uniform space with the first axiom of countability, $f : X \to X$ be a continuous map and $\{f_n\}_{n\in\mathbb{Z}_+}$ be a sequence of continuous self-maps of X which *converges uniformly* to f, we prove that if $\{f_n\}_{n\in\mathbb{Z}_+}$ has the uniform shadowing property, then so does f (see Lemma 5.1). We also prove that if $\{f_n\}_{n\in\mathbb{Z}_+}$ is uniformly expansive, then f is expansive (see Lemma 5.2). Furthermore, we show that if $\{f_n\}_{n\in\mathbb{Z}_+}$ is uniformly expansive and has the uniform shadowing property, then f is topologically stable (see Theorem 5.7).

Besides, some concepts of dynamical systems, such as *transitivity*, *minimality* and *mixing*, can be defined in the sense of topology, namely, we can restate these concepts from the viewpoint of open covers but not distance. In this paper, we introduce the concepts of shadowable point and topologically stable point of continuous maps of compact topological spaces from the viewpoint of open covers and prove that every shadowable point of an expansive continuous map of a disconnected compact topological space with the first axiom of countability is topologically stable (see Theorem 6.2).

The detailed arrangement of this paper is as follows: In Section 2, we recall some necessary notions of this paper and introduce the concept of persistence for continuous maps of compact uniform spaces from the viewpoint of entourages. In Section 3, we introduce the notion of persistence for Borel probability measures with respect to continuous maps of compact uniform spaces and we prove that strong persistence for Borel probability measures and persistence for continuous maps are equivalent. In Section 4, we prove that every equicontinuous pointwise topologically stable continuous map of a compact uniform space with the first axiom of countability is persistent. Moreover, we obtain several general results related to persistence. In Section 5, we discuss the shadowing property, expansivity and topological stability of the limit of a sequence of continuous self-maps of a compact uniform space and prove that if a sequence of continuous self-maps of a compact uniform space with the first axiom of countability is uniformly expansive with the uniform shadowing property, then the limit is topologically stable. In Section 6, we introduce the concepts of shadowable point and topologically stable point for continuous maps of compact topological spaces from the viewpoint of open covers and prove that every shadowable point of an expansive continuous map of a disconnected compact topological space with the first axiom of countability is topologically stable.

2. Preliminaries

In this section, we will restate some basic notions of uniform spaces and dynamical systems.

2.1. Uniform space

Let X be a nonempty set and $X \times X = \{(x,y) : x,y \in X\}$ denote the *cartesian product* of X with itself. Denote the *diagonal* of $X \times X$ by $\triangle_X = \{(x,x) : x \in X\}$. For a subset U of $X \times X$, the inverse of U is denoted by $U^{-1} = \{(y,x) : (x,y) \in U\}$ and U is called *symmetric* if $U = U^{-1}$. Given two subsets U and V of $X \times X$, the *composition* of U and V is defined as

$$U \circ V = \{(x,z) : \text{ there exists } y \in X \text{ such that } (x,y) \in U \text{ and } (y,z) \in V\}.$$

Use nU to denote $\overbrace{U \circ U \circ \cdots \circ U}^{n \text{ times}}$. If $U \subseteq X \times X$ contains the diagonal $\triangle_X$ of $X \times X$, then we call U an *entourage* of the diagonal $\triangle_X$ (usually, an entourage for simplicity).

Definition 1 (see [29])**.** A uniformity $\mathcal{U}$ on a nonempty set X is a collection of entourages of the diagonal satisfying the following conditions:
(1) $U, V \in \mathcal{U} \Rightarrow U \cap V \in \mathcal{U}$;
(2) $U \in \mathcal{U}, U \subseteq V \Rightarrow V \in \mathcal{U}$;
(3) $U \in \mathcal{U} \Rightarrow V \circ V \subseteq U$ for some $V \in \mathcal{U}$;
(4) $U \in \mathcal{U} \Rightarrow U^{-1} \in \mathcal{U}$.

Meanwhile, we call the pair $(X, \mathcal{U})$ a *uniform space.* $\mathcal{U}$ is called *separating* if $\cap_{U \in \mathcal{U}} U = \triangle_X$, at the same time, we say that X is *separating.* A sub-collection $\mathcal{V}$ of $\mathcal{U}$ is called a *base* of $\mathcal{U}$ if for every $U \in \mathcal{U}$, there exists $V \in \mathcal{V}$ such that $V \subseteq U$. Furthermore, the family of all open (resp., closed) symmetric entourages in $\mathcal{U}$ is a base of $\mathcal{U}$ (see [13], Theorems 7 and 8). Obviously, if $\mathcal{U}$ is separating, then $\cap_{V \in \mathcal{V}} V = \triangle_X$ for each base $\mathcal{V}$ of $\mathcal{U}$.

Clearly, each base $\mathcal{V}$ of a uniformity $\mathcal{U}$ has the following properties:
(1) if $U_1, U_2 \in \mathcal{U}$, then there exists $V \in \mathcal{V}$ such that $V \subseteq U_1 \cap U_2$;
(2) $U \in \mathcal{U} \Rightarrow V \circ V \subseteq U$ for some $V \in \mathcal{V}$;
(3) $U \in \mathcal{U} \Rightarrow V^{-1} \subseteq U$ for some $V \in \mathcal{V}$.

For an entourage $U \in \mathcal{U}$ and a point $x \in X$, write $U[x] = \{y \in X : (x,y) \in U\}$. For each $x \in X$, the collection $\mathcal{U}_x = \{U[x] : U \in \mathcal{U}\}$ is a *neighborhood base* at x which induces a topology of X. This topology can also be induced by a base $\mathcal{V}$ of $\mathcal{U}$ in the same manner. This topology is *Hausdorff* if and only if $\mathcal{U}$ is separating.

Lemma 2.1 (see [9], Section 8.3.13)**.** *For a compact Hausdorff space X, there is a unique uniformity $\mathcal{U}$ which induces the topology of X.*

2.2. Pointwise topological stability and persistence for continuous maps of uniform spaces

Let $(X, \mathcal{U})$ be a compact uniform space, $f : X \to X$ be a continuous map of X (meanwhile, we call the pair (X, f) a dynamical system) and $\widetilde{\mathcal{U}}$ be the set of closed symmetric entourages of $\mathcal{U}$.

In this subsection, we will recall and introduce some basic notions of dynamical systems in terms of entourages of uniform spaces.

Let $U \in \mathcal{U}$ be a symmetric entourage. We say that a sequence $\{x_n\}_{n\in\mathbb{Z}_0}$ of X is a *U-pseudo orbit* of f if for every $n \in \mathbb{Z}_0$, $(f(x_n), x_{n+1}) \in U$. We say that a sequence $\{x_n\}_{n\in\mathbb{Z}_0}$ of X is *U-shadowed* by $z \in X$ if for every $n \in \mathbb{Z}_0$, $(f^n(z), x_n) \in U$(see [30]).

Definition 2 (see [30])**.** We say that (X, f) has the shadowing property if for every symmetric entourage $U \in \mathcal{U}$, there exists a symmetric entourage $V \in \mathcal{U}$ such that every V-pseudo orbit of f can be U-shadowed by some point in X.

Definition 3 (see [30])**.** We say that f is expansive if for each $n \in \mathbb{Z}_0$,

$$(f^n(x), f^n(y)) \in U$$

for some $U \in \widetilde{\mathcal{U}}$, then $x = y$, i.e.,

$$\triangle_X = \bigcap_{n\in\mathbb{Z}_0} (f \times f)^{-n}U.$$

Such an entourage U is called an expansivity neighborhood of f.

Denote by $I : X \to X$ the identity map. If necessary we write I_X to indicate the dependence on X. In the case when $A \subset X$ we use $I_A : A \to X$ to denote the inclusion, i.e., $I_A(x) = x$ for $x \in A$ (see [8]). Given a continuous map $g : X \to X$ and $\emptyset \neq A \subset X$, we write

$$\Gamma_A(f, g) = \bigcap\{U \in \mathcal{U} : (f(x), g(x)) \in U \text{ for each } x \in A\},$$

and $\Gamma(f, g) = \Gamma_X(f, g)$.

Denote by $O_g(x) = \{g^n(x) : n \in \mathbb{Z}_0\}$ the *orbit* of $x \in X$ under a continuous self-map g of X.

Definition 4 (see [30])**.** We say that f is topologically stable if for every symmetric entourage $W \in \mathcal{U}$, there exists a symmetric entourage $V \in \mathcal{U}$ such that for every continuous map $g : X \to X$ with $\Gamma(f, g) \subset V$, there is a continuous map $h : X \to X$ satisfying
(1) $\Gamma(h, I) \subset W$;
(2) $f \circ h = h \circ g$.

Remark 2.2. The notions of expansivity, shadowing property and topological stability of a homeomorphism of a compact uniform space can be similarly defined.

Definition 5 (see [30])**.** We say that $x \in X$ is a topologically stable point of f if for each symmetric entourage $U \in \mathcal{U}$, there exists a symmetric entourage $V_x \in \mathcal{U}$ such that for every continuous map $g : X \to X$ with $\Gamma(f, g) \subset V_x$, there is a continuous map $h : \overline{O_g(x)} \to X$ satisfying

(1) $\Gamma_{\overline{O_g(x)}}(h, I_{\overline{O_g(x)}}) \subset U$;

(2) $f \circ h = h \circ g$.

Denote by $T(f)$ the set of topologically stable points of f.

Definition 6 (see [30])**.** We say that f is pointwise topologically stable if every point of X is topologically stable (i.e., $T(f) = X$).

Clearly, every topologically stable continuous map is pointwise topologically stable. But we have no idea about the converse even X is metrizable (see [30] for more details).

We say that f is *equicontinuous* if the associated iterated family $\{f^n : n \in \mathbb{Z}_0\}$ is equicontinuous, i.e., for each symmetric entourage $U \in \mathcal{U}$, there exists a symmetric entourage $V \in \mathcal{U}$ such that $(x_1, x_2) \in V$ implies that for every $n \in \mathbb{Z}_0$, $(f^n(x_1), f^n(x_2)) \in U$.

Definition 7. We say that f is persistent if for each $U \in \widetilde{\mathcal{U}}$, there exists $V \in \widetilde{\mathcal{U}}$ such that for every $x \in X$ and every continuous map $g : X \to X$ with $\Gamma(f, g) \subset V$, there is $y \in X$ satisfying that for every $n \in \mathbb{Z}_0$, $(f^n(x), g^n(y)) \in U$.

Remark 2.3. Let $(X, \mathcal{U})$ be a compact uniform space, $f : X \to X$ be a homeomorphism. The persistence of f can be similarly defined as: f is persistent if for each $U \in \widetilde{\mathcal{U}}$, there exists $V \in \widetilde{\mathcal{U}}$ such that for every $x \in X$ and every homeomorphism $g : X \to X$ with $\Gamma(f, g) \subset V$, there is $y \in X$ satisfying that for every $n \in \mathbb{Z}$, $(f^n(x), g^n(y)) \in U$.

Example 2.4. Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}_+\} \subset \mathbb{R}$ with the relative uniformity $\mathcal{U}_X$ of the uniform space $(\mathbb{R}, \mathcal{U})$, where $\mathcal{U}$ is the usual uniformity on $\mathbb{R}$, having the base collection of sets U_ε and $U_\varepsilon = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - y| < \varepsilon\}$ for each $\varepsilon > 0$. Then $(X, \mathcal{U}_X)$ is a compact uniform space. Let $\widetilde{\mathcal{U}}_X = \{V_\varepsilon : \varepsilon > 0\}$ be the set of closed symmetric entourages, where $V_\varepsilon = \{(x, y) \in X \times X : |x - y| \le \varepsilon\}$. It is not hard to verify that the identity $I : X \to X$ is persistent.

Example 2.5. The finite set $X_i = \{0, 1\}$ is fixed with the discrete topology for $i \in \mathbb{Z}$. Consider $X = \Pi_{i=-\infty}^{\infty} X_i$ with the usual metric d and the usual uniformity $\mathcal{U}$ with a base consisting of sets $V_\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}$ for each $\varepsilon > 0$, where the metric d is defined as follows: For each $x = (\ldots, x_{-j}, \ldots, x_{-1}, x_0, x_1, \ldots, x_j, \ldots)$, $y = (\ldots, y_{-j}, \ldots, y_{-1}, y_0, y_1, \ldots, y_j, \ldots) \in X$, $d(x, y) = 2^{-n}$ if n is the largest positive integer with $x_j = y_j$ for all $-n < j < n$, and $d(x, y) = 1$ if $x_0 \ne y_0$. Then $(X, \mathcal{U})$ is a compact uniform space. Define the shift map $\sigma : X \to X$ as $(\sigma(x))_j = x_{j+1}$ for all $j \in \mathbb{Z}$. It is well known that σ is topologically stable (since it is expansive and has the shadowing property). Now we show that σ is not persistent. Let $\widetilde{\mathcal{U}} = \{D_\varepsilon : \varepsilon > 0\}$ be the set of closed symmetric entourages, where $D_\varepsilon = \{(x,y) \in X \times X : d(x,y) \le \varepsilon\}$. Put $D_{\frac{1}{8}} \in \widetilde{\mathcal{U}}$ and for each $D_{\varepsilon_1} \in \widetilde{\mathcal{U}}$, there is $n > 0$ such that $\frac{1}{2^n} < \varepsilon_1$. Define a homeomorphism $g : X \to X$ as

$$(1) \qquad (g(x))_j = \begin{cases} x_j, & \text{if } \quad j > n \text{ or } j < -n; \\ x_{j+1}, & \text{if } \quad -n \le j < n; \\ x_{-n}, & \text{if } \quad j = n \end{cases}$$

for each $x = (\ldots, x_{-j}, \ldots, x_{-1}, x_0, x_1, \ldots, x_j, \ldots) \in X$. Clearly, we obtain that $\Gamma(\sigma, g) \subset D_{\varepsilon_1}$ and $g^{2n+1}(z) = z$ for each $z \in X$. Next consider

$$a = (\cdots 0, \ldots, 0, 1, 0, 0, 1, 0, \dot{1}, 0, 1, 0, 0, 1, 0, \ldots, 0, \ldots) \in X.$$

Then for all $z \in X$ with $(a, z) \in D_{\frac{1}{8}}$, it is not hard to verify that

$$(\sigma^{2n+1}(a), g^{2n+1}(z)) \notin D_{\frac{1}{8}};$$

for all $z \in X$ with $(a, z) \notin D_{\frac{1}{8}}$, there exists $k \in \mathbb{Z}_0$ with $k \le 2$ such that $(\sigma^k(a), g^k(z)) \notin D_{\frac{1}{8}}$. Therefore, σ is not persistent (this example comes from [22]).

3. Persistent measures for continuous maps of uniform spaces

In this section we introduce the notion of persistence for a Borel probability measure with respect to a continuous map of a compact uniform space. For the sake of description, we denote by $\overline{A}$ the closure of $A \subset X$ and $A - B$ the difference set of $A, B \subset X$.

Definition 8. Let (X, f) be a dynamical system. We say that

(1) A nonempty set $K \subset X$ is a *persistent set* of f if for each $U \in \widetilde{\mathcal{U}}$, there exists $V \in \widetilde{\mathcal{U}}$ such that for every $x \in K$ and every continuous map $g : X \to X$ with $\Gamma(f, g) \subset V$, there is $y \in X$ satisfying that for every $n \in \mathbb{Z}_0$, $(f^n(x), g^n(y)) \in U$;

(2) $x \in X$ is a *persistent point* of f if $\{x\}$ is a persistent set of f;

(3) f is persistent if X is a persistent set of f;

(4) f is pointwise persistent if every point of X is persistent.

If the set of persistent points of f is denoted by $Persi(f)$, then f is pointwise persistent if and only if $Persi(f) = X$. Clearly, $Persi(f)$ is an f-invariant set, i.e., $f(Persi(f)) \subset Persi(f)$.

It is natural to answer whether every pointwise persistent continuous map of a compact uniform space is persistent. We have no idea about this question but it holds for equicontinuous maps (see Lemma 4.3).

Next we will prove that $Persi(f)$ is a measurable set. For this we use the following elementary lemma. For simplicity, for given continuous maps $f, g : X \to X$ of a compact uniform space $(X, \mathcal{U})$, $x \in X$ and $U \in \widetilde{\mathcal{U}}$, we write

$$\Gamma_U^{f,g}(x) = \{y \in X : (f^n(x), g^n(y)) \in U \text{ for every } n \in \mathbb{Z}_0\}$$

$$= \bigcap_{n\in\mathbb{Z}_0} g^{-n}(U[f^n(x)])$$

and

$$B(U,f,g) = \{x \in X : \Gamma_U^{f,g}(x) \neq \emptyset\}.$$

Lemma 3.1. *Let* $(X, \mathcal{U})$ *be a compact uniform space with the first axiom of countability and* $f, g : X \to X$ *be continuous maps. Then* $B(U,f,g)$ *is closed* (*hence measurable*) *for every* $U \in \widetilde{\mathcal{U}}$.

Proof. Fix $U \in \widetilde{\mathcal{U}}$, it suffices to show that $\overline{B(U,f,g)} \subset B(U,f,g)$. Take $x \in \overline{B(U,f,g)}$. Since X is a compact uniform space with the first axiom of countability, there exists a sequence $\{x_k\}_{k\in\mathbb{Z}_+} \subset B(U,f,g)$ which converges to x. And since for each $k \in \mathbb{Z}_+$, $x_k \in B(U,f,g)$, there exists $y_k \in X$ such that for every $n \in \mathbb{Z}_0$, $(f^n(x_k), g^n(y_k)) \in U$. Without loss of generality, we assume that $y_k \to y$ for some $y \in X$. Then, by fixing $n \in \mathbb{Z}_0$ and letting $k \to \infty$ we obtain $(f^n(x), g^n(y)) \in U$. It follows that $y \in \Gamma_U^{f,g}(x)$ and hence $\Gamma_U^{f,g}(x) \neq \emptyset$, i.e., $x \in B(U,f,g)$. Therefore, $B(U,f,g)$ is closed and the proof is completed. □

Recall that the *Borel σ-algebra* of X is the σ-algebra generated by all open subsets of X. Every element of the Borel σ-algebra of X is referred to as a *Borelian* of X. A Borel probability measure of X is a σ-additive measure, defined in the Borel σ-algebra of X, which takes the value 1 at X (see [26]).

Recall that a subset of a topological space Y is a *G_δ-set* if it is a countable intersection of open sets; an *F_σ-set* if it is a countably union of closed sets; a *$G_{\delta\sigma}$-set* if it is a countably union of G_δ-sets and an *$F_{\sigma\delta}$-set* if it is a countably intersection of F_σ-sets. These classes form part of the so-called *Borel hierarchy* (see [12]). All such classes are formed by measurable sets, namely, they belong to the Borel σ-algebra of Y.

Corollary 3.2. *Let* $(X, \mathcal{U})$ *be a compact uniform space with the first axiom of countability and* $f : X \to X$ *be a continuous map. Then* $Persi(f)$ *is an* $F_{\sigma\delta}$*-subset* (*hence measurable*) *of* X.

Proof. Obviously,

$$X - Persi(f) = \bigcup_{U\in\widetilde{\mathcal{U}}} \bigcap_{V\in\widetilde{\mathcal{U}}} C(U,V),$$

where

$$C(U,V)$$
$$= \{x \in X : \Gamma_U^{f,g}(x) = \emptyset \text{ for some continuous map } g : X \to X \text{ with } \Gamma(f,g) \subset V\}.$$

It follows from Lemma 3.1 that $C(U,V)$ is open for any $U, V \in \widetilde{\mathcal{U}}$. Then, $X - Persi(f)$ is a $G_{\delta\sigma}$-subset of X and so $Persi(f)$ is an $F_{\sigma\delta}$-subset of X. The proof is finished. □

The first natural attempt to extend the concept of persistence from continuous maps to Borel probability measures of a compact uniform space is through the following concept (based on Corollary 3.2).

Definition 9. Let $(X, \mathcal{U})$ be a compact uniform space with the first axiom of countability and $f : X \to X$ be a continuous map. A Borel probability measure μ of X is called almost persistent (or pointwise persistent) if $\mu(Persi(f)) = 1$.

Proposition 3.3. *Let $(X, \mathcal{U})$ be a compact uniform space with the first axiom of countability and $f : X \to X$ be a continuous map. Suppose that μ is a Borel probability measure of X satisfying the following property: For each $U \in \widetilde{\mathcal{U}}$, there exists $V \in \widetilde{\mathcal{U}}$ and a full measure subset B of X such that for every continuous map $g : X \to X$ with $\Gamma(f, g) \subset V$ and every $x \in B$, there is $y \in X$ satisfying that for every $n \in \mathbb{Z}_0$, $(f^n(x), g^n(y)) \in U$. Then μ is almost persistent.*

Proof. Take a sequence $\{U_k\}_{k=1}^{\infty}$ of decreasing closed symmetric entourages with $\cap_{k=1}^{\infty} U_k = \triangle_X$. By the given condition, there exists a sequence $\{V_k\}_{k=1}^{\infty}$ of closed symmetric entourages and a sequence $\{B_k\}_{k=1}^{\infty}$ of full measure sets such that for every continuous map $g : X \to X$ with $\Gamma(f, g) \subset V_k$ and every $x \in B_k$, there is $y \in X$ satisfying that for every $k \in \mathbb{Z}_+$ and every $n \in \mathbb{Z}_0$, $(f^n(x), g^n(y)) \in U_k$.

Write $B = \cap_{k=1}^{\infty} B_k$. It is easy to prove that B has full measure. Now take $U \in \widetilde{\mathcal{U}}$. Fix a large enough $k \in \mathbb{Z}_+$ such that $U_k \subset U$ and choose a continuous map $g : X \to X$ with $\Gamma(f, g) \subset V_k$. For each $x \in B$ (in fact, $x \in B_k$), there is $y \in X$ such that for every $n \in \mathbb{Z}_0$, $(f^n(x), g^n(y)) \in U_k \subset U$. Hence x is a persistent point and this proves that $B \subset Persi(f)$. Then, $\mu(Persi(f)) \geq \mu(B) = 1$ so $\mu(Persi(f)) = 1$ and the proof is finished. □

This proposition motivates us to introduce the second notion of persistence for Borel probability measures.

Definition 10. A Borel probability measure μ of X is called strongly persistent if for each $U \in \widetilde{\mathcal{U}}$, there exists $V \in \widetilde{\mathcal{U}}$ and a full measure set B with respect to μ such that for every continuous map $g : X \to X$ with $\Gamma(f, g) \subset V$ and every $x \in B$, there is $y \in X$ satisfying that for every $n \in \mathbb{Z}_0$, $(f^n(x), g^n(y)) \in U$.

We can therefore reformulate Proposition 3.3 by saying that every strongly persistent measure is almost persistent.

Remark 3.4. Let δ_x be the Dirac measure supported on $x \in X$, i.e., δ_x is the Borel probability measure of X such that

$$\delta_x(A) = \begin{cases} 1, & \text{if} \quad x \in A; \\ 0, & \text{if} \quad x \notin A. \end{cases} \tag{2}$$

Then, for every continuous map $f : X \to X$,

$$Persi(f) = \{x : \delta_x \text{ is a strongly persistent measure of } f\}.$$

Remark 3.5. If $\mu_1, \dots, \mu_k$ are strongly persistent measures of a continuous map $f : X \to X$, and $t_1, \dots, t_k \in \mathbb{R}^+$ satisfy $\sum_{i=1}^k t_i = 1$, then $\sum_{i=1}^k t_i\mu_i$ is also a strongly persistent measure of f.

Finally, based on Lemma 3.1, we give the third notion of persistence for Borel probability measures.

Definition 11. Let $(X, \mathcal{U})$ be a compact uniform space with the first axiom of countability and $f : X \to X$ be a continuous map. A Borel probability measure μ of X is called persistent if for each $U \in \widetilde{\mathcal{U}}$, there exists $V \in \widetilde{\mathcal{U}}$ such that $\mu(B(U, f, g)) = 1$ for every continuous map $g : X \to X$ with $\Gamma(f, g) \subset V$.

Remark 3.6. Every strongly persistent measure is persistent (it is clear if note that the full measure set B in Definition 10 is contained in $B(U, f, g)$). Therefore, Remarks 3.4 and 3.5 are valid if we replace "strongly persistent measure" by "persistent measure" in the corresponding statements.

Let $(X, \mathcal{U})$ be a compact uniform space with the first axiom of countability and $f : X \to X$ be a continuous map. The following statements illustrate the relationship between the aforementioned concepts of measure persistence.

(1) Every Borel probability measure of X is strongly persistent.
(2) Every Borel probability measure of X is almost persistent.
(3) Every Borel probability measure of X is persistent.

Then, (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). In Lemma 4.2 we will prove that (2) $\Rightarrow$ (3) for equicontinuous continuous maps.

Lemma 3.7. *Let $(X, \mathcal{U})$ be a compact uniform space with the first axiom of countability and $f : X \to X$ be a continuous map. Then the following statements are equivalent*:

(1) *Every Borel probability measure of X is strongly persistent*;
(2) *Every Borel probability measure of X is persistent*;
(3) *f is persistent.*

Proof. Since (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are obvious, it remains to prove that (2) $\Rightarrow$ (3).

We assume by contradiction that every Borel probability measure of X is persistent but f is not. It turns out that there exists $U \in \widetilde{\mathcal{U}}$ such that for a decreasing sequence $\{V_k\}_{k=1}^\infty$ of closed symmetric entourages with $\cap_{k\in\mathbb{Z}_+} V_k = \triangle_X$, there is a sequence $\{g_k\}_{k=1}^\infty$ of continuous maps of X with $\Gamma(f, g_k) \subset V_k$ and a sequence $\{x_k\}_{k=1}^\infty \subset X$ with

$$\bigcap_{n\in\mathbb{Z}_0} g_k^{-n}(U[f^n(x_k)]) = \emptyset, \quad \forall k \in \mathbb{Z}_+. \tag{3}$$

Denote by δ_z the Dirac measure supported on $z \in X$. Define

$$\mu = \sum_{k=1}^{\infty} 2^{-k}\delta_{x_k}.$$

Then, it follows from the hypothesis that μ is a persistent measure of f. Let $V \in \widetilde{\mathcal{U}}$ be given by this property for $U \in \widetilde{\mathcal{U}}$ as above. Pick $k_0 \in \mathbb{Z}_+$ large enough such that $g_{k_0} : X \to X$ satisfies $\Gamma(f, g_{k_0}) \subset V_{k_0} \subset V$. Then, there is a full measure subset B_{k_0} of X such that

$$\bigcap_{n\in\mathbb{Z}_0} g_{k_0}^{-n}(U[f^n(z)]) \neq \emptyset, \quad \forall z \in B_{k_0}. \tag{4}$$

Since $\mu(B_{k_0}) = 1$ we obtain $\delta_{x_{k_0}}(B_{k_0}) = 1$, and so $x_{k_0} \in B_{k_0}$. Replacing z by x_{k_0} in (4), we get

$$\bigcap_{n\in\mathbb{Z}_0} g_{k_0}^{-n}(U[f^n(x_{k_0})]) \neq \emptyset,$$

which contradicts (3). This contradiction proves the result. □

4. Persistence of equicontinuous maps of uniform spaces

In this section, we discuss the relationships of several kinds of persistence of equicontinuous maps of uniform spaces. First of all, we give some necessary lemmas.

Lemma 4.1. *Let $(X, \mathcal{U})$ be a compact uniform space with the first axiom of countability. Then the set of persistent points of an equicontinuous continuous map of X is closed.*

Proof. Let $f : X \to X$ be an equicontinuous continuous map. It suffices to show that $\overline{Persi(f)} \subset Persi(f)$. Take $x \in \overline{Persi(f)}$. Since X is compact and satisfies the first axiom of countability, there exists a sequence $\{x_k\}_{k\in\mathbb{Z}_+} \subset Persi(f)$ which converges to x. For a given $U \in \widetilde{\mathcal{U}}$, we choose $V \in \widetilde{\mathcal{U}}$ such that $V \circ V \subset U$. For this $V \in \widetilde{\mathcal{U}}$, there exists a symmetric entourage $W \in \mathcal{U}$ satisfying the definition of equicontinuity of f. Since $x_k \to x$, we can choose $k \in \mathbb{Z}_+$ such that $(x_k, x) \in W$. Hence for each $n \in \mathbb{Z}_0$, $(f^n(x_k), f^n(x)) \in V = V^{-1}$.

On the other hand, since for each $k \in \mathbb{Z}_+$, $x_k \in Persi(f)$, for $V \in \widetilde{\mathcal{U}}$, we can choose $D_k \in \widetilde{\mathcal{U}}$ satisfying the definition of persistence of x_k. If $g : X \to X$ is a continuous map with $\Gamma(f, g) \subset D_k$, then there is $y \in X$ such that for each $n \in \mathbb{Z}_0$, $(f^n(x_k), g^n(y)) \in V$. Thus, for every $n \in \mathbb{Z}_0$, $(f^n(x), g^n(y)) \in V \circ V \subset U$, which shows that $x \in Persi(f)$. Therefore, $Persi(f)$ is closed and the proof is finished. □

Recall that the *support* of a Borel probability measure ν of a compact uniform space $(X, \mathcal{U})$ is the set of all points $x \in X$ with $\nu(U[x]) > 0$ for every $U \in \mathcal{U}$ (see [26]). Denote by $supp(\nu)$ the support of ν.

Lemma 4.2. *Let $(X, \mathcal{U})$ be a compact uniform space with the first axiom of countability. Then every almost persistent measure of an equicontinuous continuous map of X is persistent.*

Proof. Let $f : X \to X$ be an equicontinuous continuous map. Assume to the contrary that f admits an almost persistent measure μ which is not persistent. Then there exists $U \in \widetilde{\mathcal{U}}$ satisfying the opposite side of persistence of μ. Take a decreasing sequence $\{V_k\}_{k=1}^{\infty}$ of closed symmetric entourages with $\cap_{k \in \mathbb{Z}_+} V_k = \triangle_X$, and then there exists a sequence $\{g_k\}_{k=1}^{\infty}$ of continuous maps of X with $\Gamma(f, g_k) \subset V_k$ and a sequence $\{A_k\}_{k=1}^{\infty}$ of positive measure sets satisfying that for all $k \in \mathbb{Z}_+$,

$$\bigcap_{n \in \mathbb{Z}_0} g_k^{-n}(U[f^n(x)]) = \emptyset, \quad \forall x \in A_k. \tag{5}$$

Without loss of generality, we choose $D \in \widetilde{\mathcal{U}}$ such that $D \circ D \subset U$. Since for each $k \in \mathbb{Z}_+$, A_k has positive measure with respect to μ, and so $A_k \cap supp(\mu) \neq \emptyset$.

On the other hand, μ is almost persistent and so $supp(\mu) \subset \overline{Persi(f)}$. But f is equicontinuous, $Persi(f)$ is closed by Lemma 4.1, and thus $supp(\mu) \subset Persi(f)$, which yields that $\emptyset \neq A_k \cap supp(\mu) \subset A_k \cap Persi(f)$, that is,

$$A_k \cap Persi(f) \neq \emptyset, \quad \forall k \in \mathbb{Z}_+.$$

Therefore for each $k \in \mathbb{Z}_+$, we can choose $x_k \in A_k \cap Persi(f)$. Since X is compact and satisfies the first axiom of countability, we assume that $x_k \to x$ for some $x \in X$ and as $Persi(f)$ is closed, we get $x \in Persi(f)$. For $D \in \widetilde{\mathcal{U}}$ as above, there exists a symmetric entourage $W \in \mathcal{U}$ satisfying the definition of equicontinuity of f. Since $x_k \to x$, we can fix k such that $(x_k, x) \in W$. Hence for each $n \in \mathbb{Z}_0$, $(f^n(x_k), f^n(x)) \in D$.

Since x is persistent, let $V \in \widetilde{\mathcal{U}}$ be given by this property for $D \in \widetilde{\mathcal{U}}$ as above. Fix a large enough $k \in \mathbb{Z}_+$ such that $\Gamma(f, g_k) \subset V_k \subset V$. Then, there is $y \in X$ satisfying that for each $n \in \mathbb{Z}_0$, $(f^n(x), g_k^n(y)) \in D$. Thus, for every $n \in \mathbb{Z}_0$, $(f^n(x_k), g_k^n(y)) \in D \circ D \subset U$, which yields that

$$y \in \bigcap_{n \in \mathbb{Z}_0} g_k^{-n}(U[f^n(x_k)]).$$

This together with $x_k \in A_k$ contradicts (5). The proof is completed. □

Lemma 4.3. *Let $(X, \mathcal{U})$ be a compact uniform space with the first axiom of countability. Then every equicontinuous pointwise persistent continuous map of X is persistent.*

Proof. Let $f : X \to X$ be an equicontinuous pointwise persistent continuous map. It follows that every Borel probability measure of X is almost persistent and so persistent by Lemma 4.2. Then, f is persistent by Lemma 3.7 and the proof is finished. □

Let $(X, \mathcal{U})$ be a compact uniform space and $f : X \to X$ be a continuous map. $x \in X$ is called an *equicontinuous point* of f if for each symmetric entourage $U \in \mathcal{U}$, there exists a symmetric entourage $V \in \mathcal{U}$ such that if $y \in X$ and $(x, y) \in V$, then for every $n \in \mathbb{Z}_0$, $(f^n(x), f^n(y)) \in U$.

Denote by $Eq(f)$ the set of equicontinuous points of f. It is easy to see that if f is equicontinuous, then every point of X is equicontinuous.

Lemma 4.4. *Let $(X, \mathcal{U})$ be a compact uniform space and $f : X \to X$ be a continuous map. Then $Eq(f) \cap T(f) \subset Persi(f)$.*

Proof. Suppose $x \in Eq(f) \cap T(f)$ and $U \in \widetilde{\mathcal{U}}$. We choose $V \in \widetilde{\mathcal{U}}$ such that $V \circ V \subset U$. For this V, there exists a symmetric entourage $W \in \mathcal{U}$ with $W \subset V$ satisfying the definition of equicontinuous point of f. For the above $W \in \mathcal{U}$, there exists a symmetric entourage $D \in \mathcal{U}$ satisfying the definition of topologically stable points. Since $\widetilde{\mathcal{U}}$ is a base of $\mathcal{U}$, there exists $D^\star \in \widetilde{\mathcal{U}}$ such that $D^\star \subset D$.

Assume that $g : X \to X$ is a continuous map with $\Gamma(f, g) \subset D^\star$. Then, there is a continuous map $h : \overline{O_g(x)} \to X$ satisfying

$$\Gamma_{\overline{O_g(x)}}\left(h, I_{\overline{O_g(x)}}\right) \subset W$$

and $f \circ h = h \circ g$. In particular for each $y \in \overline{O_g(x)}$, $(h(y), y) \in W$.

Therefore for every $n \in \mathbb{Z}_0$, we obtain $(f^n(x), f^n(h(x))) \in V^{-1} = V$ and $(f^n(h(x)), g^n(x)) = (h(g^n(x)), g^n(x)) \in W \subset V$. Hence $(f^n(x), g^n(x)) \in V \circ V \subset U$, which shows that $x \in Persi(f)$. This ends the proof. □

Now by virtue of the above lemmas, we can prove the main result of this section.

Theorem 4.5. *Let $(X, \mathcal{U})$ be a compact uniform space with the first axiom of countability. Then every equicontinuous pointwise topologically stable continuous map of X is persistent.*

Proof. Let $f : X \to X$ be an equicontinuous pointwise topologically stable continuous map. Then $Eq(f) = X = T(f)$ and so $Eq(f) \cap T(f) = X$, which together with Lemma 4.4 implies $Persi(f) = X$. Therefore, f is pointwise persistent and so persistent by Lemma 4.3. The proof is completed. □

5. Shadowing property, expansivity and topological stability of uniform limits

In this section we discuss the shadowing property, expansivity and topological stability of the limit of a sequence of continuous self-maps of a compact uniform space.

Let $(X,\mathcal{U})$ be a compact uniform space, $f : X \to X$ be a continuous map, $\{f_n\}_{n\in\mathbb{Z}_+}$ be a sequence of continuous self-maps of X.

Definition 12. We say that $\{f_n\}_{n\in\mathbb{Z}_+}$ has the uniform shadowing property if for each sequence $\{U_k\}_{k\in\mathbb{Z}_+}$ of closed symmetric entourages, there exists $V \in \widetilde{\mathcal{U}}$ such that for each $n \in \mathbb{Z}_+$, every V-pseudo-orbit of f_n can be U_k-shadowed by some point in X for f_n.

Definition 13. We say that $\{f_n\}_{n\in\mathbb{Z}_+}$ is uniformly expansive if for every $n \in \mathbb{Z}_+$, f_n is expansive with the same expansivity neighborhood $W \in \widetilde{\mathcal{U}}$.

We say that $\{f_n\}_{n\in\mathbb{Z}_+}$ *converges uniformly* to f if for each $U \in \widetilde{\mathcal{U}}$, there exists $N \in \mathbb{Z}_+$ such that for each $n \geq N$, $(f_n(x), f(x)) \in U$ for every $x \in X$.

Lemma 5.1. *Let $(X,\ \mathcal{U})$ be a compact uniform space with the first axiom of countability, $f : X \to X$ be a continuous map and $\{f_n\}_{n\in\mathbb{Z}_+}$ a sequence of continuous self-maps of X which converges uniformly to f. If $\{f_n\}_{n\in\mathbb{Z}_+}$ has the uniform shadowing property, then f has the shadowing property.*

Proof. Given $U \in \widetilde{\mathcal{U}}$, let $U_1 \in \widetilde{\mathcal{U}}$ with $U_1 \circ U_1 \circ U_1 \subset U$. Since $\{f_n\}_{n\in\mathbb{Z}_+}$ has the uniform shadowing property, choose $V \in \widetilde{\mathcal{U}}$ such that for each $n \in \mathbb{Z}_+$, every V-pseudo-orbit of f_n can be U_1-shadowed by some point $y_n \in X$ for f_n.

Pick $V_1 \in \widetilde{\mathcal{U}}$ such that $V_1 \circ V_1 \subset V$. Let $\{x_i\}_{i\in\mathbb{Z}_0}$ be a V_1-pseudo-orbit of f. We claim that $\{x_i\}_{i\in\mathbb{Z}_0}$ is a V-pseudo-orbit of f_n if n is sufficiently large. Indeed, since $\{f_n\}_{n\in\mathbb{Z}_+}$ converges uniformly to f, there exists $N_0 \in \mathbb{Z}_+$ such that for each $n \geq N_0$, $(f_n(x), f(x)) \in V_1$ for every $x \in X$. Then for every $n \geq N_0$ and every $i \in \mathbb{Z}_0$, $(f_n(x_i), f(x_i)) \in V_1$, $(f(x_i), x_{i+1}) \in V_1$ and so $(f_n(x_i), x_{i+1}) \in V_1 \circ V_1 \subset V$. This proves the claim.

The uniform shadowing property of $\{f_n\}_{n\in\mathbb{Z}_+}$ implies that for every $n \geq N_0$, there is $y_n \in X$ such that for every $i \in \mathbb{Z}_0$, $(f_n^i(y_n), x_i) \in U_1$. Let $\{y_n\}_{n\geq N_0}$ be the sequence of such points. Since X is compact and satisfies the first axiom of countability, we assume that $\{y_n\}_{n\geq N_0} \to y$ for some $y \in X$. Fix $i \in \mathbb{Z}_+$. Since $\{f_n\}_{n\in\mathbb{Z}_+}$ converges uniformly to f, $\{f_n^i\}_{n\in\mathbb{Z}_+}$ converges uniformly to f^i. So we can choose $N \in \mathbb{Z}_+$ with $N \geq N_0$ such that for each $n \geq N$, $(f_n^i(y_n), f^i(y_n)) \in U_1 = U_1^{-1}$. On the other hand, the continuity of f^i implies $f^i(y_n) \to f^i(y)$, then we can choose $n \in \mathbb{Z}_+$ such that $(f^i(y_n), f^i(y)) \in U_1 = U_1^{-1}$. Therefore, if we choose n sufficiently large, we obtain $(f^i(y), x_i) \in U_1 \circ U_1 \circ U_1 \subset U$. Thus f has the shadowing property and the proof is completed. □

Lemma 5.2. *Let $(X,\ \mathcal{U})$ be a compact uniform space, $f : X \to X$ be a continuous map and $\{f_n\}_{n\in\mathbb{Z}_+}$ be a sequence of continuous self-maps of X which converges uniformly to f. If $\{f_n\}_{n\in\mathbb{Z}_+}$ is uniformly expansive, then f is expansive.*

Proof. Let $W \in \widetilde{\mathcal{U}}$ be an expansive neighborhood for $\{f_n\}_{n\in\mathbb{Z}_+}$ and let $U \in \widetilde{\mathcal{U}}$ with $U \circ U \circ U \subset W$. Since $\{f_n\}_{n\in\mathbb{Z}_+}$ converges uniformly to f, there exists $N_0 \in \mathbb{Z}_+$ such that for every $n \geq N_0$, $(f_n(x), f(x)) \in U$ for each $x \in X$. Note that if $\{f_n\}_{n\in\mathbb{Z}_+}$ converges uniformly to f, then $\{f_n^i\}_{n\in\mathbb{Z}_+}$ converges uniformly to f^i for each $i \in \mathbb{Z}_0$. Taking $n = N_0$ sufficiently large, we obtain $(f_{N_0}^i(y), f^i(y)) \in U$ for each $y \in X$.

For each pair $x, y \in X$ and each $n \in \mathbb{Z}_0$, if $(f^n(x), f^n(y)) \in U$, then for every $n \in \mathbb{Z}_0$,

$$(f_{N_0}^n(x), f_{N_0}^n(y)) \in U \circ U \circ U^{-1} = U \circ U \circ U \subset W,$$

which shows that $x = y$ since f_{N_0} is expansive. Hence f is expansive. The proof is finished. □

Lemma 5.3. *Let $(X,\mathcal{U})$ be a compact uniform space with the first axiom of countability, $f : X \to X$ be an expansive continuous map with an expansivity neighborhood $W \in \widetilde{\mathcal{U}}$. Then, for each symmetric entourage $U \in \mathcal{U}$, there exists $N \geq 1$ such that for every $0 \leq n \leq N$, $(f^n(x), f^n(y)) \in W$ implies $(x, y) \in U$.*

Proof. We assume by contradiction that there exists a symmetric entourage $V \in \mathcal{U}$ such that for each $k \geq 1$, there exist $x_k, y_k \in X$ satisfying that for every $0 \leq n \leq k$, $(f^n(x_k), f^n(y_k)) \in W$ but $(x_k, y_k) \notin V$. Note that the family of open symmetric entourages is a base of the uniformity $\mathcal{U}$. We can pick an open symmetric entourage V_0 such that $V_0 \subset V$ and thus for each $k \geq 1$, $(x_k, y_k) \notin V_0$. Since X is compact and satisfies the first axiom of countability, there exists a subsequence $\{x_{k_l}\}_{l\in\mathbb{Z}_+}$ of $\{x_k\}_{k\in\mathbb{Z}_+}$ such that $\{x_{k_l}\}_{l\in\mathbb{Z}_+} \to x_0$ for some $x_0 \in X$ and there exists a subsequence $\{y_{k_l}\}_{l\in\mathbb{Z}_+}$ of $\{y_k\}_{k\in\mathbb{Z}_+}$ such that $\{y_{k_l}\}_{l\in\mathbb{Z}_+} \to y_0$ for some $y_0 \in X$. Then, $(x_0, y_0) \notin V_0$ and the continuity of f^n implies $(f^n(x_0), f^n(y_0)) \in W$ for each $n \in \mathbb{Z}_+$. Since f is expansive, $x_0 = y_0$ and so there is a contradiction and the proof is finished. □

Remark 5.4. In Lemma 2.5 of [30], the authors proved Lemma 5.3 for the case of homeomorphisms.

We say that f is *uniformly continuous* if for each symmetric entourage $U \in \mathcal{U}$, there exists a symmetric entourage $V \in \mathcal{U}$ such that $(x, y) \in V$ implies $(f(x), f(y)) \in U$, whenever $x, y \in X$.

Theorem 5.5. *Let $(X,\mathcal{U})$ be a compact uniform space with the first axiom of countability. Then every expansive continuous map of X with the shadowing property is topologically stable.*

Proof. Let $f : X \to X$ be an expansive continuous map with the shadowing property and $W \in \widetilde{\mathcal{U}}$ be an expansive neighborhood for f. Let $U \in \mathcal{U}$ be a symmetric entourage with $U \circ U \subset W$ and $U \circ U \circ U \subset W$. For $U \in \mathcal{U}$, there exists a symmetric entourage $V \in \mathcal{U}$ satisfying the definition of the shadowing property of f. Fix a continuous map $g : X \to X$ with $\Gamma(f, g) \subset V$. Clearly, every g-orbit $O_g(x)$ of $x \in X$ is a V-pseudo-orbit of f. Hence, it can be U-shadowed by some point in X, which implies that there exists $y \in X$ such that for every $n \in \mathbb{Z}_0$, $(f^n(y), g^n(x)) \in U$. If $z \in X$ is also U-shadowing the V-pseudo-orbit $O_g(x)$ of f for $x \in X$, then $(f^n(z), g^n(x)) \in U$ for every $n \in \mathbb{Z}_0$. Thus, for every $n \in \mathbb{Z}_0$, $(f^n(y), f^n(z)) \in U \circ U^{-1} = U \circ U \subset W$, which shows that $y = z$ from the expansivity of f. Therefore for every $x \in X$, the V-pseudo-orbit $O_g(x)$ of f can be U-shadowed by a unique point in X.

For each $x \in X$, define a map $h : X \to X$ by $h(x) = y_x$, where $y_x \in X$ is the unique U-shadowing point of the V-pseudo-orbit $O_g(x)$ of f. Then for every $n \in \mathbb{Z}_0$, $(f^n(h(x)), g^n(x)) \in U$. Pick $n = 0$, then we obtain that $(h(x), x) \in U$ and hence $\Gamma(h, I) \subset U$. Since for each $x \in X$ and every $n \in$

$\mathbb{Z}_0$, $(f^{n+1}(h(x)), g^{n+1}(x)) \in U$ and $(f^n(h(g(x))), g^n(g(x))) \in U$, by the above argument, we have $f(h(x)) = h(g(x))$ for each $x \in X$, i.e., $f \circ h = h \circ g$.

Next we show that $h : X \to X$ is continuous. Given a symmetric entourage $T \in \mathcal{U}$, by Lemma 5.3, there exists $N \geq 1$ such that for every $0 \leq n \leq N$, $(f^n(a), f^n(b)) \in W$ implies $(a, b) \in T$. Since X is compact, g is uniformly continuous on X and so there is a symmetric entourage $H \in \mathcal{U}$ such that $(x, y) \in H$ implies $(g^n(x), g^n(y)) \in U$ for every $0 \leq n \leq N$. Then if $(x, y) \in H$, we have that for every $0 \leq n \leq N$, then

$$(f^n(h(x)), f^n(h(y))) = (h(g^n(x)), h(g^n(y))) \in U \circ U \circ U^{-1} = U \circ U \circ U \subset W.$$

Thus by Lemma 5.3, $(h(x), h(y)) \in T$. This ends the proof. □

Remark 5.6. See Theorem 3.2 of [30] for the case of homeomorphisms of Theorem 5.5.

Theorem 5.7. *Let $(X,\mathcal{U})$ be a compact uniform space with the first axiom of countability, $f : X \to X$ be a continuous map and $\{f_n\}_{n\in\mathbb{Z}_+}$ be a sequence of continuous self-maps of X which converges uniformly to f. If $\{f_n\}_{n\in\mathbb{Z}_+}$ is uniformly expansive and has the uniform shadowing property, then f is topologically stable.*

Proof. Suppose that $\{f_n\}_{n\in\mathbb{Z}_+}$ is uniformly expansive and has the uniform shadowing property. By Lemma 5.1, f has the shadowing property and by Lemma 5.2, f is expansive, and so by Theorem 5.5, f is topologically stable. This completes the result. □

6. Topologically stable point and shadowable point of continuous maps

In this section, we will introduce several notions of dynamical systems from the viewpoint of open covers. At first, we introduce the notion of shadowable points of continuous maps of compact topological spaces. For convenience, we always assume that (X, f) is a dynamical system which means that X is a compact topological space and $f : X \to X$ is a continuous map, and all the involved finite open covers of X are non-trivial, namely, if $\mathcal{U}$ is a finite open cover of X, then $\mathcal{U} \neq \{X\}$.

Let $\mathcal{U}$ be a finite open cover of X. We say that a sequence $\{x_n\}_{n\in\mathbb{Z}_0}$ of points in X is a *$\mathcal{U}$-pseudo orbit* of f if for every $n \in \mathbb{Z}_0$, there exists $U_{n+1} \in \mathcal{U}$ with $x_{n+1} \in U_{n+1}$ such that $f(x_n) \in U_{n+1}$. We say that a sequence $\xi = \{x_n\}_{n\in\mathbb{Z}_0}$ of X is *$\mathcal{U}$-shadowed* by $z \in X$ if for every $n \in \mathbb{Z}_0$, there exists $U_n \in \mathcal{U}$ with $x_n \in U_n$ such that $f^n(z), x_n \in \overline{U_n}$.

We say that (X, f) has the *shadowing property* if for each finite open cover $\mathcal{V}$ of X, there exists a finite open cover $\mathcal{W}$ of X such that every $\mathcal{W}$-pseudo-orbit of f can be $\mathcal{V}$-shadowed by some point in X. Given a subset $B \subset X$, we say that the sequence $\{x_n\}_{n\in\mathbb{Z}_0}$ of X is *through* B if $x_0 \in B$. We say that (X, f) has the *shadowing property through a subset* $B \subset X$ if for each finite open cover $\mathcal{V}$ of X, there exists a finite open cover $\mathcal{W}$ of X such that every $\mathcal{W}$-pseudo-orbit of f through B can be $\mathcal{V}$-shadowed by some point in X. In addition, a homeomorphism of X has the shadowing property through $B \subset X$ can be similarly defined.

Definition 14. Let $\mathcal{U}_0$ be a finite open cover of X. A point $x \in X$ is called a $\mathcal{U}_0$-shadowable point of f if there exists a finite open cover $\mathcal{W}$ of X such that every $\mathcal{W}$-pseudo-orbit of f through $\{x\}$ can be $\mathcal{U}_0$-shadowed by some point in X. Denoted by $Sh_+(f,\mathcal{U}_0)$ the set of $\mathcal{U}_0$-shadowable points of f. We say that $x \in X$ is a shadowable point of f if for each finite open cover $\mathcal{U}$ of X, $x \in Sh_+(f,\mathcal{U})$.

The set of shadowable points of f is denoted by $Sh_+(f)$. It is easy to see that f has the shadowing property if and only if $Sh_+(f) = X$. The shadowable points of a homeomorphism of a compact topological space can be similarly defined. We denote by $Sh(f)$ the set of shadowable points of a homeomorphism f of a compact topological space X.

Definition 15. (see [11]) Let $\mathcal{U}$ be a finite open cover of X. We say that f is $\mathcal{U}$-expansive if for every $n \in \mathbb{Z}_0$, there exists $U_n \in \mathcal{U}$ such that if $f^n(x)$, $f^n(y) \in \overline{U_n}$ whenever x, $y \in X$, then $x = y$.

Next we introduce the notion of topologically stable points of dynamical systems.

Given a continuous map $g : X \to X$ and a finite open cover $\mathcal{U}$ of X, we say that g is *$\mathcal{U}$-close* to f if for each $x \in X$, there exists $U \in \mathcal{U}$ such that $f(x), g(x) \in U$ (see [11]).

Definition 16. (see [11]) We say that f is topologically stable if for each finite open cover $\mathcal{U}$ of X, there exists a finite open cover $\mathcal{V}$ of X such that for every continuous map $g : X \to X$ that is $\mathcal{V}$-close to f, there is a continuous map $h : X \to X$ satisfying

(1) for each $x \in X$, there exists $U_x \in \mathcal{U}$ with $x \in U_x$ such that $h(x) \in \overline{U_x}$;
(2) $f \circ h = h \circ g$.

Definition 17. We say that $x \in X$ is a topologically stable point of f if for each finite open cover $\mathcal{U}$ of X, there exists a finite open cover $\mathcal{V}$ of X such that for every continuous map $g : X \to X$ that is $\mathcal{V}$-close to f, there is a continuous map $h : \overline{O_g(x)} \to X$ satisfying

(1) for each $y \in \overline{O_g(x)}$, there exists $U_y \in \mathcal{U}$ with $y \in U_y$ such that $h(y) \in \overline{U_y}$;
(2) $f \circ h = h \circ g$.

The set of topologically stable points of f in the sense of open covers is denoted by $T^*(f)$. We note that a necessary condition for a continuous map $f : X \to X$ of a compact topological space X to be topologically stable is that every point of X is topologically stable. But we do not know whether the converse holds.

Lemma 6.1. *Let X be a compact topological space with the first axiom of countability, $\mathcal{W}$ be a finite open cover of X and $f : X \to X$ be a $\mathcal{W}$-expansive continuous map of X. Then, given a finite open cover $\mathcal{U}$ of X, there exists $N \geq 1$ such that for every $0 \leq n \leq N$, $f^n(x), f^n(y) \in \overline{W_n}$ for some $W_n \in \mathcal{W}$ implies $x, y \in U$ for some $U \in \mathcal{U}$.*

Proof. We assume by contradiction that there exists a finite open cover $\mathcal{U}$ of X such that for each $k \geq 1$, there exist x_k, y_k satisfying for every $0 \leq n \leq k$, $f^n(x_k), f^n(y_k) \in \overline{W_{n,k}}$ for some $W_{n,k} \in \mathcal{W}$ but $x_k, y_k \notin U$ for every $U \in \mathcal{U}$. Since X is compact and satisfies the first axiom of countability, there are sequences $\{x_{k_l}\}_{l\in\mathbb{Z}_0} \to x_0 \in X$ and $\{y_{k_l}\}_{l\in\mathbb{Z}_0} \to y_0 \in X$, respectively. Fix $n \in \mathbb{Z}_0$, because $\mathcal{W}$ is a finite open cover of X, we can choose the same open set $W_n \in \mathcal{W}$ such that $f^n(x_{k_l}), f^n(y_{k_l}) \in \overline{W_n}$ for infinitely many $l \in \mathbb{N}$. The continuity of f^n implies $f^n(x_0), f^n(y_0) \in \overline{W_n}$. Since f is $\mathcal{W}$-expansive, we obtain $x_0 = y_0$.

In addition, since $\mathcal{U}$ is a finite open cover of X, there is $\widetilde{U} \in \mathcal{U}$ such that $x_0 = y_0 \in \widetilde{U}$. Note that $\{x_{k_l}\} \to x_0$ and $\{y_{k_l}\} \to y_0$, it follows that there is $N \in \mathbb{Z}_+$ such that when $l \geq N$, $x_{k_l}, y_{k_l} \in \widetilde{U}$, and so there is a contradiction and the proof is finished. ☐

We say that $f : X \to X$ is *uniformly continuous* if for each finite open cover $\mathcal{U}$ of X, there is a finite open cover $\mathcal{V}$ of X that relates only to $\mathcal{U}$ such that $x, y \in V$ for some $V \in \mathcal{V}$ implies $f(x), f(y) \in U$ for some $U \in \mathcal{U}$.

Theorem 6.2. *Let X be a disconnected compact topological space with the first axiom of countability and $\mathcal{W}$ be a finite open cover of X. Then every shadowable point of a $\mathcal{W}$-expansive continuous map of X is topologically stable.*

Proof. Let $f : X \to X$ be a $\mathcal{W}$-expansive continuous map (since X is disconnected, there exists a finite open cover of X of which the elements are pairwise disjoint, without loss of generality, we assume the elements of $\mathcal{W}$ are pairwise disjoint). Suppose that $x \in Sh_+(f)$. For each finite open cover $\widetilde{\mathcal{U}}$ of X, clearly, $\mathcal{U} =: \widetilde{\mathcal{U}} \cap \mathcal{W} = \{\widetilde{U} \cap W : \widetilde{U} \in \widetilde{\mathcal{U}},\ W \in \mathcal{W}\}$ is also a finite open cover of X. For this $\mathcal{U}$, there exists a finite open cover $\mathcal{V}$ of X satisfying the definition of the shadowableness of x. Fix a continuous map $g : X \to X$ that is $\mathcal{V}$-close to f. Obviously, the g-orbit $O_g(x)$ of $x \in X$ is a $\mathcal{V}$-pseudo-orbit of f. Hence, it can be $\mathcal{U}$-shadowed by some point in X, which implies that there exists $y \in X$ such that for every $n \in \mathbb{Z}_0$, there is $U_{g^n(x)} \in \mathcal{U}$ with $g^n(x) \in U_{g^n(x)}$ such that $f^n(y), g^n(x) \in \overline{U_{g^n(x)}}$. If $z \in X$ is also $\mathcal{U}$-shadowing the $\mathcal{V}$-pseudo-orbit $O_g(x)$ of f, then there is $U'_{g^n(x)} \in \mathcal{U}$ with $g^n(x) \in U'_{g^n(x)}$ such that $f^n(z), g^n(x) \in \overline{U'_{g^n(x)}}$ for every $n \in \mathbb{Z}_0$. Then it follows from the choice of $\mathcal{U}$ that there is $W_n \in \mathcal{W}$ such that $U_{g^n(x)}, U'_{g^n(x)} \subset W_n$. Thus, for every $n \in \mathbb{Z}_0$, $f^n(y), f^n(z) \in \overline{W_n}$ for some $W_n \in \mathcal{W}$ which shows that $y = z$ from the $\mathcal{W}$-expansiveness of f. Therefore, for every $x \in X$, the $\mathcal{V}$-pseudo-orbit $O_g(x)$ of f can be $\mathcal{U}$-shadowed by a unique point in X.

Let $y_x \in X$ be the unique $\mathcal{U}$-shadowing point of the $\mathcal{V}$-pseudo-orbit $O_g(x)$ of f, then for every $n \in \mathbb{Z}_0$, there exists $U_{g^n(x)} \in \mathcal{U}$ with $g^n(x) \in U_{g^n(x)}$ such that $f^n(y_x), g^n(x) \in \overline{U_{g^n(x)}}$. Then, it is not hard to conclude that for each $n \in \mathbb{Z}_0$ and every $k \in \mathbb{Z}_0$, $f^n(f^k(y_x)), g^n(g^k(x)) \in \overline{U_{g^{n+k}(x)}}$ for some $U_{g^{n+k}(x)} \in \mathcal{U}$ with $g^{n+k}(x) \in U_{g^{n+k}(x)}$. Therefore, $f^k(y_x) \in X$ is the unique $\mathcal{U}$-shadowing point of the $\mathcal{V}$-pseudo-orbit $O_{g^k(x)}$ of f.

Define a map $h : O_g(x) \to X$ by $h(g^k(x)) = f^k(y_x)$ for each $y = g^k(x) \in O_g(x)$. By the above argument, it is obvious that the map h is well defined. Then for every $n \in \mathbb{Z}_0$ and each $k \in \mathbb{Z}_0$, there exists $\overline{U_{g^n(g^k(x))}} \in \mathcal{U}$ with $g^n(g^k(x)) \in U_{g^n(g^k(x)}$ such that $f^n(h(g^k(x))), g^n(g^k(x) \in \overline{U_{g^n(g^k(x))}}$. In particular, pick $n = 0$, we obtain that there exists $U_y \in \mathcal{U}$ with $y \in U_y$ such that $h(y), y \in \overline{U_y}$. Since

$$\begin{aligned}(f \circ h)(g^k(x)) &= f(f^k(y_x)) = f^{k+1}(y_x) \\ &= h(g^{k+1}(x)) = h(g(g^k(x))) = (h \circ g)(g^k(x))\end{aligned}$$

for every $k \in \mathbb{Z}_0$, we obtain $f \circ h = h \circ g$.

Next we show that $h : O_g(x) \to X$ is continuous. Given a finite open cover $\mathcal{T}$ of X, by Lemma 6.1, there exists $N \geq 1$ such that for every $0 \leq n \leq N$, $f^n(a), f^n(b) \in \overline{W_n}$ for some $W_n \in \mathcal{W}$ implies $a, b \in T$ for some $T \in \mathcal{T}$. Since X is compact, g is uniformly continuous on X and so there is a finite open cover $\mathcal{H}$ of X such that $x, y \in H$ for some $H \in \mathcal{H}$ implies $g^n(x), g^n(y) \in \overline{\widetilde{W}_n}$ for some $\widetilde{W}_n \in \mathcal{W}$ and every $0 \leq n \leq N$. Now take $c, d \in O_g(x)$ with $c, d \in \overline{H}$ for some $H \in \mathcal{H}$, we have that for every $0 \leq n \leq N$,

$$\begin{aligned}\{f^n(h(c)), f^n(h(d))\} &= \{h(g^n(c)), h(g^n(d))\} \\ &\subset \{h(g^n(c)), g^n(c)\} \cup \{g^n(c), g^n(d)\} \cup \{h(g^n(d)), g^n(d)\} \\ &\subset \overline{U_{g^n(c)}} \cup \overline{\widetilde{W}_n} \cup \overline{U_{g^n(d)}}.\end{aligned}$$

Then by the choice of $\mathcal{U}$ and the disjointness of the elements of $\mathcal{W}$, it is not hard to conclude that for every $0 \leq n \leq N$, $\{f^n(h(c)), f^n(h(d))\} \subset \overline{W_n^{'}}$ for some $W_n^{'} \in \mathcal{W}$. Thus by Lemma 6.1, $h(c), h(d) \in T^{'}$ for some $T^{'} \in \mathcal{T}$ and so h is continuous. Then we can extend continuously h to $\overline{O_g(x)}$ to obtain a continuous map, still denote it by $h : \overline{O_g(x)} \to X$. Clearly, by the above argument, we get that for each $y \in \overline{O_g(x)}$, there exists $U_y \in \mathcal{U} \subset \widetilde{\mathcal{U}}$ with $y \in U_y$ such that $h(y) \in \overline{U_y}$ and $f \circ h = h \circ g$. Thus $x \in T^*(f)$ and the proof is completed. □

Remark 6.3. In Theorem 4.6 of [30], the authors obtained that every shadowable point of an expansive homeomorphism of a compact uniform space is topologically stable. Clearly, Theorem 6.2 and Theorem 4.6 of [30] are independent.

We say that f has the *finite shadowing property* through $B \subset X$ if for each finite open cover $\mathcal{U}$ of X, there exists a finite open cover $\mathcal{V}$ of X such that every finite $\mathcal{V}$-pseudo-orbit of f through B can be $\mathcal{U}$-shadowed by some point in X. In addition, a homeomorphism of X has the finite shadowing property through a subset B of X can be similarly defined.

Proposition 6.4. *Let X be a compact topological space with the first axiom of countability, $f : X \to X$ be a homeomorphism of X. Then f has the shadowing property through $Sh(f)$ if and only if f has the finite shadowing property through $Sh(f)$.*

Proof. Suppose that f has the shadowing property through $Sh(f)$. Fix a finite open cover $\mathcal{U}$ of X and choose a finite open cover $\mathcal{V}$ of X satisfying the definition of the shadowing property through $Sh(f)$. Let

$$\Gamma = \{x_{-n}, \dots, x_{-1}, x_0, x_1, \dots, x_m\}$$

be a finite $\mathcal{V}$-pseudo-orbit of f with $x_0 \in Sh(f)$. Then the sequence $\widetilde{\Gamma} = \{y_k : k \in \mathbb{Z}\}$ given by

$$y_k = \begin{cases} f^{k+n}(x_{-n}), & \text{if} \quad k < -n; \\ x_k, & \text{if} \quad -n \le k \le m; \\ f^{k-m}(x_m), & \text{if} \quad k > m. \end{cases} \tag{6}$$

is a $\mathcal{V}$-pseudo-orbit of f with $y_0 = x_0 \in Sh(f)$. Since f has the shadowing property through $Sh(f)$, there is a point $z \in X$ with the property that for every $k \in \mathbb{Z}$, there exists $U_k \in \mathcal{U}$ with $y_k \in U_k$ such that $f^k(z), y_k \in \overline{U_k}$. Thus, every finite $\mathcal{V}$-pseudo-orbit of f through $Sh(f)$ can be $\mathcal{U}$-shadowed by some point $z \in X$.

Conversely, given a finite open cover $\mathcal{U}$ of X and choose a finite open cover $\mathcal{V}$ of X such that every finite $\mathcal{V}$-pseudo-orbit of f through $Sh(f)$ can be $\mathcal{U}$-shadowed by some point in X. Assume that $\xi = \{x_n\}_{n\in\mathbb{Z}}$ is a $\mathcal{V}$-pseudo-orbit of f through $Sh(f)$. Fix $m \in \mathbb{Z}_+$ and let $\widetilde{x}_n = x_{n-m}$. Then take a finite sequence $\widehat{\Gamma} = \{x_{-m}, x_{-m+1}, \dots, x_0, x_1, \dots, x_m\}$, clearly, it is a $\mathcal{V}$-pseudo-orbit of f through $Sh(f)$. Since f has the finite shadowing property through $Sh(f)$, there is a point $z_m \in X$ such that for each $0 \le n \le 2m$, there exists $\widetilde{U}_{n,m} \in \mathcal{U}$ with $\widetilde{x}_n \in \widetilde{U}_{n,m}$ such that $f^n(z_m), \widetilde{x}_n \in \overline{\widetilde{U}_{n,m}}$. Setting $w_m = f^m(z_m)$, then we get that for each $-m \le n \le m$, there exists $U_{n,m} \in \mathcal{U}$ with $x_n \in U_{n,m}$ such that $f^n(w_m), x_n \in \overline{U_{n,m}}$. Since X is compact and satisfies the first axiom of countability, we assume that $w_{m_l} \to w$ for some $w \in X$. Fix $-m \le n \le m$, because $\mathcal{U}$ is a finite open cover of X, we can choose the same open set $U_n \in \mathcal{U}$ such that $f^n(w_{m_l}), x_n \in \overline{U_n}$ for infinitely many $l \in \mathbb{Z}_+$. The continuity of f^n implies $f^n(w), x_n \in \overline{U_n}$. Thus the $\mathcal{V}$-pseudo-orbit $\xi = \{x_n\}_{n\in\mathbb{Z}}$ can be $\mathcal{U}$-shadowed by $w \in X$. Hence f has the shadowing property through $Sh(f)$. This proof is completed. □

Definition 18. (see [18, 21]) Let (X, f) be a dynamical system. We say that f has the almost shadowing property if $Sh_+(f)$ is dense in X.

Two dynamical systems (X, f) and (Y, g) are called *topologically conjugated* if there is a homeomorphism $h : X \to Y$ such that $g \circ h = h \circ f$, where the homeomorphism h is called a *topological conjugacy* between f and g.

Proposition 6.5. *If (X, f) and (Y, g) are topological conjugated, then*

$$h(Sh_+(f)) = Sh_+(g),$$

where h is the topological conjugacy between f and g.

Proof. Let $\mathcal{U}$ be a finite open cover of Y and $y \in h(Sh_+(f))$. Then there exists $x \in Sh_+(f)$ such that $h(x) = y$. Since X, Y are compact, h is uniformly continuous and so there exists a finite open cover $\mathcal{V}$ of X such that $a, b \in V$ for some $V \in \mathcal{V}$ implies $h(a), h(b) \in U$ for some $U \in \mathcal{U}$. Since $x \in X$ is a shadowable point of f, for the finite open cover $\mathcal{V}$, there exists a finite open cover $\mathcal{W}$ of X such that every $\mathcal{W}$-pseudo-orbit $\xi = \{x_n\}_{n\in\mathbb{Z}_0}$ of f with $x_0 = x$ can be $\mathcal{V}$-shadowed by some point in X.

Let $h(\mathcal{W}) =: \{h(W) : W \in \mathcal{W}\}$, then $h(\mathcal{W})$ is a finite open cover of Y. We claim that every $h(\mathcal{W})$-pseudo-orbit $\zeta = \{y_n\}_{n\in\mathbb{Z}_0}$ of g with $y_0 = y$ can be $\mathcal{U}$-shadowed by some point in Y. Put $x_n = h^{-1}(y_n)$ for each $n \in \mathbb{Z}_0$. Since for every $n \in \mathbb{Z}_0$, there exists $h(W) \in h(\mathcal{W})$ such that $g(y_n), y_{n+1} \in h(W)$, we have that for each $n \in \mathbb{Z}_0$,

$$\begin{aligned}\{f(x_n), x_{n+1}\} &= \{f(h^{-1}(y_n)), h^{-1}(y_{n+1})\}\\ &= \{h^{-1}(g(y_n)), h^{-1}(y_{n+1})\} \subset W\end{aligned}$$

for some $W \in \mathcal{W}$ and $x_0 = h^{-1}(y_0) = h^{-1}(y) = x$. Therefore, $\xi = \{x_n\}_{n\in\mathbb{Z}_0}$ is a $\mathcal{W}$-pseudo-orbit of f with $x_0 = x$ and so there is a point $z \in X$ such that for every $n \in \mathbb{Z}_0$, there exists $V_n \in \mathcal{V}$ with $x_n \in V_n$ such that $f^n(z), x_n \in \overline{V_n}$. By the uniform continuity of h, we obtain $h(x_n) \in U_n$ for some $U_n \in \mathcal{U}$ and $h(f^n(z)) \in h(\overline{V_n}) = \overline{h(V_n)} \subset \overline{U_n}$. Thus for each $n \in \mathbb{Z}_0$, there exists $U_n \in \mathcal{U}$ with $h(x_n) = y_n \in U_n$ such that $\{h(f^n(z)), y_n\} = \{g^n(h(z)), y_n\} \subset \overline{U_n}$ for some $U_n \in \mathcal{U}$. So the $h(\mathcal{W})$-pseudo-orbit $\zeta = \{y_n\}_{n\in\mathbb{Z}_0}$ of g with $y_0 = y$ can be $\mathcal{U}$-shadowed by a point $h(z) \in Y$. This proves the claim. Hence $y \in Sh_+(g)$ and so $h(Sh_+(f)) \subset Sh_+(g)$.

By a similar argument for h^{-1}, we can prove the converse. Hence,

$$h(Sh_+(f)) = Sh_+(g)$$

and the proof is completed. □

Proposition 6.6. *If (X, f) and (Y, g) are topological conjugated, then f has the almost shadowing property if and only if g has the almost shadowing property.*

Proof. Suppose that f has the almost shadowing property. Then $Sh_+(f)$ is a dense subset of X. By Proposition 6.5, we see that $Sh_+(g)$ is a dense subset of Y. Hence g has the almost shadowing property. Similarly, we can prove that the converse is also true. This completes the proof. □

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