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INVARIANCE OF THE AREA OF OVALOIDS

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In memory of Professor Sung Ki Kim

Abstract. Consider a two dimensional smooth convex body with a marked point on the boundary of it, sitting tangentially at the marked point over a base curve in $\mathbb{E}^2$, $\mathbb{H}^2$ or $\mathbb{S}^2$ and the image of this body by the reflection with respect to the tangent line of the base curve at the marked point. When we roll these two bodies simultaneously along the base curve, the trajectories of the marked point bound a closed region. We show that the area of the closed region is independent of the shape of the base curve if the base curve is not highly curved with respect to the boundary curve of the convex body.

1. Introduction

Suppose a two dimensional convex body $\mathcal{A}$ in the Euclidean plane $\mathbb{E}^2$ whose boundary $\partial\mathcal{A}$ is a smooth closed curve rolls without slipping along a curve, called the *base curve*, which is not necessarily a geodesic line. Let us call the curve traced by a marked point O on $\partial\mathcal{A}$ as $\mathcal{A}$ rolls along the base curve an *ovaloid.* There are two ovaloids, each of which is drawn on one side of the base curve. Let us call one of these ovaloid as the *left ovaloid* and the other *right ovaloid* over the base curve. If $\mathcal{A}$ is initially sitting tangentially at the marked point O over a base curve, then these two ovaloids, that is, the left ovaloid together with the right ovaloid make a closed curve. Let us call the region enclosed by these two ovaloids as the *ovaloidal region* over the base curve.

When the convex body is a round disk, it was shown in [1] and [2] that the area of the ovaloidal region is independent of the shape of the base curve if the base curve is not highly curved with respect to the boundary circle of the disk.

One can think of the ovaloidal region in the hyperbolic plane $\mathbb{H}^2$ of the curvature -1 or in the sphere $\mathbb{S}^2$ of the curvature 1. In this article, we show that the area of the ovaloidal region is independent of the shape of the base

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curve if the base curve is not highly curved with respect to the boundary curve $\partial\mathcal{A}$ of the convex body $\mathcal{A}$, which is not necessarily a disk, not only in $\mathbb{E}^2$ but in $\mathbb{H}^2$ or in $\mathbb{S}^2$.

Theorem 1.1. *Let $\alpha(s)$ be the arclength parametrization of $\partial\mathcal{A}$ and $\beta(s)$ the arclength parametrization of the base curve such that $\alpha(0) = O = \beta(0)$. Let $\kappa_\alpha(s)$ and $\kappa_\beta(s)$ be their signed curvature functions. If*

$$|\kappa_\beta(s)| \leq \kappa_\alpha(s),$$

then the area of the ovaloidal region over the base curve β in $\mathbb{E}^2$, $\mathbb{H}^2$ or in $\mathbb{S}^2$ is independent of the shape of the base curve β.

2. An observation

For the convex body $\mathcal{A}$ with $O \in \partial\mathcal{A}$ and for every point $P \in \partial\mathcal{A}$ consider the geodesic line segment $\overline{OP}$. Then, since $\mathcal{A}$ is convex, these geodesic line segments fill up without intersection the convex body $\mathcal{A}$:

$$\mathcal{A} = \bigcup_{P \in \partial\mathcal{A}} \overline{OP}.$$

Assume the convex body $\mathcal{A}$ lies tangent at O on the base curve. We roll a convex body $\mathcal{A}$ along a base curve to get an ovaloid, which is the trajectory of the marked point $O \in \partial\mathcal{A}$. While the convex body rolls along the base curve, the collection of line segments forming the convex body unfolds to another collection of line segments, one of whose end point lies on the ovaloid and the other on the base curve to form the region bounded by the ovaloid and the base curve.

Let $Q \in \partial\mathcal{A}$ be the point such that the arclength from O to Q along $\partial\mathcal{A}$ is q. Consider the directed geodesic line segment $\overrightarrow{QO}$. Then both the length $|\overrightarrow{QO}|$ and the angle formed by the line tangent to $\partial\mathcal{A}$ at Q and the geodesic line segment $\overrightarrow{QO}$ are functions of the arclength q. Now suppose when rolling $\mathcal{A}$, the point Q touches the base curve at Q' and at that moment, the point O is moved to the point O' of the ovaloid. Then the geodesic line segments $\overrightarrow{Q'O'}$ fill up the region bounded by the ovaloid and the base curve without intersection. Moreover we see that

- The arclength along the base curve from O to Q' is the same as the arclengh from O to Q along $\partial\mathcal{A}$.
- The length of the directed line segment $\overrightarrow{Q'O'}$ is the same as that of the directed line segment $\overrightarrow{QO}$:

$$|\overrightarrow{Q'O'}| = |\overrightarrow{QO}|.$$

- The angle formed by the line tangent to the base curve at Q' and the directed line segment $\overrightarrow{Q'O'}$ is the same as the angle formed by the line tangent to $\partial\mathcal{A}$ at Q and the directed line segment $\overrightarrow{QO}$.

3. Parametrizations of $\mathcal{A}$, left and right regions

3.1. Parametrization of $\mathcal{A}$

Let the length of $\partial\mathcal{A}$ be l and $\alpha(s)$ be the arclength parametrization of $\partial\mathcal{A}$ such that

$$\alpha(0) = \alpha(l) = O$$

and oriented so that $\kappa_\alpha(s) \geq 0$. Let $\gamma_s(u)$ be the unit-speed geodesic connecting the point $\alpha(s) = \gamma_s(0)$ and O whose length is $r(s)$. Let $\mathbf{t}_\alpha(s), \mathbf{n}_\alpha(s)$ be the unit tangent vector field and the unit normal vector field of $\alpha(s)$ and $\phi(s)$ be the oriented angle between $\mathbf{t}_\alpha(s)$ and $\frac{d\gamma_s}{du}(0)$:

$$\phi(s) = \angle\left(\mathbf{t}_\alpha(s), \frac{d\gamma_s}{du}(0)\right). \tag{1}$$

Then we have

$$\begin{aligned} \gamma_s(0) &= \alpha(s), \\ \frac{d\gamma_s}{du}(0) &= \cos\phi(s)\,\mathbf{t}_\alpha(s) + \sin\phi(s)\,\mathbf{n}_\alpha(s). \end{aligned}$$

We can parametrize $\mathcal{A}$ as

$$A(s,u) = \gamma_s(u), \quad 0 \leq s \leq l,\ 0 \leq u \leq r(s).$$

Note that

$$\frac{\partial A}{\partial u}(s,0) = \frac{d\gamma_s}{du}(0) = \cos\phi(s)\,\mathbf{t}_\alpha(s) + \sin\phi(s)\,\mathbf{n}_\alpha(s).$$

Note also that $A_s(s,u) = \frac{\partial A}{\partial s}(s,u)$ is the Jacobi field $J_s(u)$ along the geodesic $\gamma_s(u)$ whose initial data are

$$\begin{aligned} J_s(0) &= A_s(s,0) = \frac{d\gamma_s(0)}{ds} = \frac{d\alpha}{ds} = \mathbf{t}_\alpha(s), \\ \frac{DJ_s}{du}(0) &= \frac{d}{du}\bigg|_{u=0} A_s(s,u) = \frac{D}{\partial u}\frac{\partial A}{\partial s}(s,0) = \frac{D}{\partial s}\frac{\partial A}{\partial u}(s,0) \\ &= \frac{D}{ds}(\cos\phi(s)\,\mathbf{t}_\alpha(s) + \sin\phi(s)\,\mathbf{n}_\alpha(s)) \\ &= -\phi'(s)\sin\phi(s)\,\mathbf{t}_\alpha(s) + \cos\phi(s)\,\mathbf{t}'_\alpha(s) + \phi'(s)\cos\phi(s)\,\mathbf{n}_\alpha(s) \\ &\quad + \sin\phi(s)\,\mathbf{n}'_\alpha(s) \\ &= -\phi'(s)\sin\phi(s)\,\mathbf{t}_\alpha(s) + \kappa_\alpha(s)\cos\phi(s)\,\mathbf{n}_\alpha(s) + \phi'(s)\cos\phi(s)\,\mathbf{n}_\alpha(s) \\ &\quad - \kappa_\alpha(s)\sin\phi(s)\,\mathbf{t}_\alpha(s) \\ &= \big(\phi'(s) + \kappa_\alpha(s)\big)\big(-\sin\phi(s)\,\mathbf{t}_\alpha(s) + \cos\phi(s)\,\mathbf{n}_\alpha(s)\big) \end{aligned}$$

since

$$\mathbf{t}'_\alpha(s) = \kappa_\alpha(s)\mathbf{n}_\alpha(s), \quad \mathbf{n}'_\alpha(s) = -\kappa_\alpha(s)\mathbf{t}_\alpha(s).$$

If we decompose $J_s(0) = \mathbf{t}_\alpha(s)$ into tangential and normal components to $\frac{d\gamma_s}{du}(0)$, we have from (1) that the tangential component is

$$\cos\phi(s)\frac{d\gamma_s}{du}(0) = \cos\phi(s)\big(\cos\phi(s)\,\mathbf{t}_\alpha(s) + \sin\phi(s)\,\mathbf{n}_\alpha(s)\big)$$

and the normal component is

$$\begin{aligned} J_s(0) - \cos\phi(s)\frac{d\gamma_s}{du}(0) &= \mathbf{t}_\alpha(s) - \cos\phi(s)(\cos\phi(s)\,\mathbf{t}_\alpha(s) + \sin\phi(s)\,\mathbf{n}_\alpha(s)) \\ &= \sin\phi(s)\big(\sin\phi(s)\,\mathbf{t}_\alpha(s) - \cos\phi(s)\,\mathbf{n}_\alpha(s)\big). \end{aligned}$$

One can see that $\frac{DJ_s}{du}(0)$ is normal to $\frac{d\gamma_s}{du}(0)$.

Let $T_\alpha(s,u)$, $N_\alpha(s,u)$ be the parallel translates of $\mathbf{t}_\alpha(s)$, $\mathbf{n}_\alpha(s)$ along the geodesic $\gamma_s(u)$. Then we have the followings:

(a) For $\mathcal{A} \subset \mathbb{E}^2$,

$$\begin{aligned} A_s(s,u) &= J_s(u) \\ &= \cos\phi(s)\big(\cos\phi(s)\,T_\alpha(s,u) + \sin\phi(s)\,N_\alpha(s,u)\big) \\ &\quad + \sin\phi(s)\big(\sin\phi(s)\,T_\alpha(s) - \cos\phi(s)\,N_\alpha(s)\big) \\ &\quad + \big(\phi'(s) + \kappa_\alpha(s)\big)\big(-\sin\phi(s)\,T_\alpha(s,u) + \cos\phi(s)\,N_\alpha(s,u)\big)u \\ &= \big(1 - (\phi'(s) + \kappa_\alpha(s))u\sin\phi(s)\big)\,T_\alpha(s,u) \\ &\quad + (\phi'(s) + \kappa_\alpha(s))u\cos\phi(s)\,N_\alpha(s,u), \\ A_u(s,u) &= \cos\phi(s)\,T_\alpha(s,u) + \sin\phi(s)\,N_\alpha(s,u). \end{aligned}$$

(b) For $\mathcal{A} \subset \mathbb{H}^2$,

$$\begin{aligned} A_s(s,u) &= J_s(u) \\ &= \cos\phi(s)(\cos\phi(s)\,T_\alpha(s,u) + \sin\phi(s)\,N_\alpha(s,u)) \\ &\quad + \sin\phi(s)(\sin\phi(s)\,T_\alpha(s,u) - \cos\phi(s)\,N_\alpha(s,u))\cosh u \\ &\quad + (\phi'(s) + \kappa_\alpha(s))(-\sin\phi(s)\,T_\alpha(s,u) + \cos\phi(s)\,N_\alpha(s,u))\sinh u \\ &= \big(\cos^2\phi(s) + \cosh u\sin^2\phi(s) - (\phi'(s) + \kappa_\alpha(s))\sinh u\sin\phi(s)\big)\,T_\alpha(s,u) \\ &\quad + \big((1-\cosh u)\sin\phi(s) + (\phi'(s) + \kappa_\alpha(s))\sinh u\big)\cos\phi(s)\,N_\alpha(s,u), \\ A_u(s,u) &= \cos\phi(s)\,T_\alpha(s,u) + \sin\phi(s)\,N_\alpha(s,u). \end{aligned}$$

(c) For $\mathcal{A} \subset \mathbb{S}^2$,

$$\begin{aligned} A_s &= \cos\phi(s)(\cos\phi(s)\,T_\alpha(s,u) + \sin\phi(s)\,N_\alpha(s,u)) \\ &\quad + \sin\phi(s)(\sin\phi(s)\,T_\alpha(s,u) - \cos\phi(s)\,N_\alpha(s,u))\cos u \\ &\quad + (\phi'(s) + \kappa_\alpha(s))(-\sin\phi(s)\,T_\alpha(s,u) + \cos\phi(s)\,N_\alpha(s,u))\sin u \\ &= \big(\cos^2\phi(s) + \cos u\sin^2\phi(s) - (\phi'(s) + \kappa_\alpha(s))\sin u\sin\phi(s)\big)\,T_\alpha(s,u) \\ &\quad + \big((1-\cos u)\sin\phi(s) + (\phi'(s) + \kappa_\alpha(s))\sin u\big)\cos\phi(s)\,N_\alpha(s,u), \\ A_u &= \cos\phi(s)\,T_\alpha(s,u) + \sin\phi(s)\,N_\alpha(s,u). \end{aligned}$$

3.2. Parametrization of the left region

Let $\mathbf{t}_\beta(s), \mathbf{n}_\beta(s)$ be the unit tangent vector field and the unit normal vector field of the base curve $\beta(s)$. Our observation gives the following parametrizations $L(s,t)$ of the left region:

$$L(s,u) = \gamma_s^L(u), \quad 0 \le s \le l,\ 0 \le u \le r(s),$$

where $\gamma^L s(u)$ is the geodesic with

$$\gamma_s^L(0) = \beta(s), \quad \frac{d\gamma_s^L}{du}(0) = \cos\phi(s)\,\mathbf{t}_\beta(s) + \sin\phi(s)\,\mathbf{n}_\beta(s).$$

Let $\kappa_\beta(s)$ be the curvature function of the base curve $\beta(s)$ and let $T_\beta(s,u)$, $N_\beta(s,u)$ be the parallel translates of $\mathbf{t}_\beta(s)$, $\mathbf{n}_\beta(s)$ along the geodesic $\gamma_s^L(u)$. The same computations as in §3.1 give the followings:
(a) In $\mathbb{E}^2$,

$$\begin{aligned} L_s(s,u) &= \big(1 - (\phi'(s) + \kappa_\beta(s))u\sin\phi(s)\big)\,T_\beta(s,u) \\ &\qquad + (\phi'(s) + \kappa_\beta(s))u\cos\phi(s)\,N_\beta(s,u), \\ L_u(s,u) &= \cos\phi(s)\,T_\beta(s,u) + \sin\phi(s)\,N_\beta(s,u). \end{aligned}$$

(b) In $\mathbb{H}^2$,

$$\begin{aligned} L_s(s,u) &= (\cos^2\phi(s) + \cosh u\sin^2\phi(s) - (\phi'(s) + \kappa_\beta(s))\sinh u\sin\phi(s))\,T_\beta(s,u) \\ &\qquad + \big((1-\cosh u)\sin\phi(s) + (\phi'(s) + \kappa_\beta(s))\sinh u\big)\cos\phi(s)\,N_\beta(s,u), \\ L_u(s,u) &= \cos\phi(s)\,T_\beta(s,u) + \sin\phi(s)\,N_\beta(s,u). \end{aligned}$$

(c) In $\mathbb{S}^2$,

$$\begin{aligned} L_s(s,u) &= (\cos^2\phi(s) + \cos u\sin^2\phi(s) - (\phi'(s) + \kappa_\beta(s))\sin u\sin\phi(s))\,T_\beta(s,u) \\ &\qquad + \big((1-\cos u)\sin\phi(s) + (\phi'(s) + \kappa_\beta(s))\sin u\big)\cos\phi(s)\,N_\beta(s,u), \\ L_u(s,u) &= \cos\phi(s)\,T_\beta(s,u) + \sin\phi(s)\,N_\beta(s,u). \end{aligned}$$

3.3. Parametrization of the right region

Our observation also gives the following parametrizations $R(s,t)$ of the right region:

$$R(s,u) = \gamma_s^R(u), \quad 0 \le s \le l,\ 0 \le u \le r(s),$$

where $\gamma_s^R(u)$ is the geodesic with

$$\begin{aligned} \gamma_s^R(0) = \beta(s), \quad \frac{d\gamma_s^R}{du}(0) &= \cos(-\phi(s))\,\mathbf{t}_\beta(s) + \sin(-\phi(s))\,\mathbf{n}_\beta(s) \\ &= \cos\phi(s)\,\mathbf{t}_\beta(s) - \sin\phi(s)\,\mathbf{n}_\beta(s). \end{aligned}$$

Then the same computations as in §3.1 give the followings:
(a) In $\mathbb{E}^2$,

$$\begin{aligned} R_s(s,u) &= \big(1 + (-\phi'(s) + \kappa_\beta(s))u\sin\phi(s)\big)\,T_\beta(s,u) \\ &\qquad + (-\phi'(s) + \kappa_\beta(s))u\cos\phi(s)\,N_\beta(s,u), \end{aligned}$$

$$R_u(s,u) = \cos\phi(s)\, T_\beta(s,u) - \sin\phi(s)\, N_\beta(s,u).$$

(b) In $\mathbb{H}^2$,

$$\begin{aligned} R_s(s,u) =& (\cos^2\phi(s) + \cosh u \sin^2\phi(s) + (-\phi'(s) + \kappa_\beta(s)) \sinh u \sin\phi(s))\, T_\beta(s,u) \\ &+ \big((-1+\cosh u)\sin\phi(s) + (-\phi'(s)+\kappa_\beta(s))\sinh u\big)\cos\phi(s) N_\beta(s,u), \\ R_u(s,u) =& \cos\phi(s)\, T_\beta(s,u) - \sin\phi(s)\, N_\beta(s,u). \end{aligned}$$

(c) In $\mathbb{S}^2$,

$$\begin{aligned} R_s(s,u) =& (\cos^2\phi(s) + \cos u \sin^2\phi(s) + (-\phi'(s) + \kappa_\beta(s)) \sin\phi(s)) \sin u\, T_\beta(s,u) \\ &+ \big((-1+\cos u)\sin\phi(s) + (-\phi'(s)+\kappa_\beta(s))\sin u\big)\cos\phi(s) N_\beta(s,u), \\ R_u(s,u) =& \cos\phi(s)\, T_\beta(s,u) - \sin\phi(s)\, N_\beta(s,u). \end{aligned}$$

4. Proof of the theorem

4.1. Proof for $\mathbb{E}^2$

The area element of the parametrization $A(s,u)$ of $\mathcal{A} \subset \mathbb{E}^2$ is computed as

$$\big(\langle A_s, A_s\rangle\langle A_u, A_u\rangle - \langle A_s, A_s\rangle^2\big)^{1/2} = |\sin\phi(s) - (\phi'(s) + \kappa_\alpha(s))u|,$$

that of the parametrization $L(s,u)$ of the left region as

$$\big(\langle L_s, L_s\rangle\langle L_u, L_u\rangle - \langle L_s, L_s\rangle^2\big)^{1/2} = |\sin\phi(s) - (\phi'(s) + \kappa_\beta(s))u|$$

and that of the parametrization $R(s,u)$ of the right region as

$$\big(\langle R_s, R_s\rangle\langle R_u, R_u\rangle - \langle R_s, R_s\rangle^2\big)^{1/2} = |\sin\phi(s) - (\phi'(s) - \kappa_\beta(s))u|.$$

Hence the area of the ovaloidal region is

$$\int_0^l \int_0^{r(s)} \big(|\sin\phi(s) - (\phi'(s) + \kappa_\beta(s))u| + |\sin\phi(s) - (\phi'(s) - \kappa_\beta(s))u|\big)\, du\, ds$$

and the area of $\mathcal{A}$ is

$$\int_0^l \int_0^{r(s)} |\sin\phi(s) - (\phi'(s) + \kappa_\alpha(s))u|\, du\, ds.$$

Now, suppose it holds that

$$\sin\phi(s) - (\phi'(s) + \kappa_\alpha(s))u \geq 0 \tag{2}$$

in the parameter region

$$0 \leq s \leq l,\ 0 \leq u \leq r(s).$$

Furthermore, if $|\kappa_\beta(s)| \leq \kappa_\alpha(s)$, we will have

$$\begin{aligned} \sin\phi(s) - (\phi'(s) + \kappa_\beta(s))u \geq \sin\phi(s) - (\phi'(s) + \kappa_\alpha(s))u \geq 0, \\ \sin\phi(s) - (\phi'(s) - \kappa_\beta(s))u \geq \sin\phi(s) - (\phi'(s) + \kappa_\alpha(s))u \geq 0, \end{aligned}$$

then the area of the ovaloidal region will be

$$\int_0^l \int_0^{r(s)} \big(|\sin\phi(s)-(\phi'(s)+\kappa_\beta(s))u|+|\sin\phi(s)-(\phi'(s)-\kappa_\beta(s))u|\big)\,du\,ds$$
$$= \int_0^l \int_0^{r(s)} \big(\sin\phi(s)-(\phi'(s)+\kappa_\beta(s))u+\sin\phi(s)-(\phi'(s)-\kappa_\beta(s))u\big)\,du\,ds$$
$$= 2\int_0^l \int_0^{r(s)} \big(\sin\phi(s)-\phi'(s)u\big)\,du\,ds$$

which shows that the area of the ovaloidal region is independent of the shape of the base curve β.

In order to prove (2), take a polar coordinate (r,θ) centered at O and write

$$\alpha(s) = r(s)\big(\cos\theta(s),\sin\theta(s)\big).$$

Then we have

$$\frac{d\gamma_s}{du}(0) = -(\cos\theta(s),\sin\theta(s)), \tag{3}$$
$$\mathbf{t}_\alpha(s) = r'(s)\big(\cos\theta(s),\sin\theta(s)\big) + r(s)\theta'(s)\big(-\sin\theta(s),\cos\theta(s)\big) \tag{4}$$

and since $\alpha(s)$ is of unit speed, we also have

$$r'(s)^2 + r^2(s)\theta'(s)^2 = 1. \tag{5}$$

Take a constant vector field $\mathbf{e}(s) = (\alpha(s);1,0)$ along $\alpha(s)$ parallel to the x axis and let the oriented angle between $\mathbf{e}(s)$ and $\mathbf{t}_\alpha(s)$ be $\psi(s)$:

$$\psi(s) = \angle(\mathbf{e}(s),\mathbf{t}_\alpha(s)),$$

then we have

$$\psi(s) = \theta(s) - \phi(s) + \pi$$

which gives

$$\kappa_\alpha(s) = \psi'(s) = \theta'(s) - \phi'(s). \tag{6}$$

Note that $\theta'(s) \geq 0$ since $\alpha(s)$ is a convex curve. We have from (1) and (3), (4)

$$\cos\phi(s) = \langle \mathbf{t}_\alpha(s), \frac{d\gamma_s}{du}(0)\rangle = -r'(s)$$

and then from (5)

$$\sin\phi(s) = r(s)\theta'(s),$$

which, together with (6), gives

$$\sin\phi(s) - (\phi'(s)+\kappa_\alpha(s))u = \big(r(s)-u\big)\theta'(s) \geq 0$$

since $r(s) \geq u$ and $\theta'(s) \geq 0$. This completes the proof of (2) and the Theorem for $\mathbb{E}^2$.

4.2. Proof for $\mathbb{H}^2$

The area element of the parametrization $A(s,u)$ of $\mathcal{A} \subset \mathbb{H}^2$ is computed as

$$\big(\langle A_s, A_s\rangle\langle A_u, A_u\rangle - \langle A_s, A_s\rangle^2\big)^{1/2} = |\cosh u \sin\phi(s) - (\phi'(s) + \kappa_\alpha(s))\sinh u|,$$

that of the parametrization $L(s,u)$ of the left region as

$$\big(\langle L_s, L_s\rangle\langle L_u, L_u\rangle - \langle L_s, L_s\rangle^2\big)^{1/2} = |\cosh u \sin\phi(s) - (\phi'(s) + \kappa_\beta(s))\sinh u|$$

and that of the parametrization $R(s,u)$ of the right region as

$$\big(\langle R_s, R_s\rangle\langle R_u, R_u\rangle - \langle R_s, R_s\rangle^2\big)^{1/2} = |\cosh \sin\phi(s) - (\phi'(s) - \kappa_\beta(s))\sinh u|.$$

Hence the area of the ovaloidal region is

$$\int_0^l \int_0^{r(s)} \big(|\cosh u \sin\phi(s) - (\phi'(s)+\kappa_\beta(s))\sinh u| + |\cosh u \sin\phi(s) - (\phi'(s) - \kappa_\beta(s))\sinh u|\big)\, du\, ds$$

and the area of $\mathcal{A}$ is

$$\int_0^l \int_0^{r(s)} |\cosh u \sin\phi(s) - (\phi'(s) + \kappa_\alpha(s))\sinh u|\, du\, ds.$$

Now, suppose it holds that

$$\cosh u \sin\phi(s) - (\phi'(s) + \kappa_\alpha(s))\sinh u \geq 0 \tag{7}$$

in the parameter region

$$0 \leq s \leq l,\ \ 0 \leq u \leq r(s).$$

Furthermore, if $|\kappa_\beta(s)| \leq \kappa_\alpha(s)$, we will have

$$\cosh u \sin\phi(s) - (\phi'(s)+\kappa_\beta(s))\sinh u \geq \cosh u \sin\phi(s) - (\phi'(s)+\kappa_\alpha(s))\sinh u \geq 0,$$
$$\cosh u \sin\phi(s) - (\phi'(s)-\kappa_\beta(s))\sinh u \geq \cosh u \sin\phi(s) - (\phi'(s)+\kappa_\alpha(s))\sinh u \geq 0,$$

then the area of the ovaloidal region will be

$$\begin{aligned}
&\int_0^l \int_0^{r(s)} \big(|\cosh u \sin\phi(s) - (\phi'(s)+\kappa_\beta(s))\sinh u| + |\cosh u \sin\phi(s) - (\phi'(s)-\kappa_\beta(s))\sinh u|\big)\, du\, ds \\
= &\int_0^l \int_0^{r(s)} \big(\cosh u \sin\phi(s) - (\phi'(s)+\kappa_\beta(s))\sinh u + \cosh u \sin\phi(s) - (\phi'(s)-\kappa_\beta(s))\sinh u\big)\, du\, ds \\
= 2&\int_0^l \int_0^{r(s)} \big(\cosh u \sin\phi(s) - \phi'(s)\sinh u\big)\, du\, ds
\end{aligned}$$

which shows that the area of the ovaloidal region in $\mathbb{H}^2$ is independent of the shape of the base curve β.

In order to prove (7), take the geodesic polar coordinate (ρ, θ) centered at O. Let us write for brevity

$$\partial_\rho := \frac{\partial}{\partial\rho}, \quad \partial_\theta := \frac{\partial}{\partial\theta}.$$

We have

$$\langle\partial_\rho, \partial_\rho\rangle = 1, \quad \langle\partial_\rho, \partial_\theta\rangle = \langle\partial_\theta, \partial_\rho\rangle = 0, \quad \langle\partial_\theta, \partial_\theta\rangle = \sinh^2\rho$$

and

$$\nabla_{\partial_\rho}\partial_\rho = 0, \quad \nabla_{\partial_\rho}\partial_\theta = \nabla_{\partial_\theta}\partial_\rho = \frac{\cosh\rho}{\sinh\rho}\partial_\theta, \quad \nabla_{\partial_\theta}\partial_\theta = -\cosh\rho\sinh\rho.$$

Let $\alpha(s) = \big(\rho(s), \theta(s)\big)$. Then we have

$$\mathbf{t}_\alpha(s) = \alpha'(s) = \rho'(s)\partial_\rho + \theta'(s)\partial_\theta,$$

and since α is of unit speed, we also have

$$\rho'(s)^2 + \theta'(s)^2\sinh^2\rho(s) = 1.$$

Let $\mathbf{e}(s)$ be a parallel vector field along $\alpha(s)$ with $\langle \mathbf{e}(s), \mathbf{e}(s)\rangle = 1$ and let

$$\begin{aligned}\epsilon_1(s) &= \angle\big(\mathbf{e}(s), \partial_\rho(s)\big),\\ \epsilon_2(s) &= \angle\big(\partial_\rho(s), \mathbf{t}_\alpha(s)\big).\end{aligned}$$

Then we have

$$\epsilon_2(s) + \phi(s) = \pi, \quad \phi'(s) = -\epsilon_2'(s)$$

and

$$\cos\epsilon_2(s) = \langle \partial_\rho(s), \mathbf{t}_\alpha(s)\rangle = \rho'(s).$$

Since $\angle\big(\mathbf{e}(s), \mathbf{t}_\alpha(s)\big) = \epsilon_1(s) + \epsilon_2(s)$, we have

$$\kappa_\alpha(s) = \frac{d}{ds}\angle\big(\mathbf{e}(s), \mathbf{t}_\alpha(s)\big) = \epsilon_1'(s) + \epsilon_2'(s)$$

and hence

$$\phi'(s) + \kappa_\alpha(s) = \epsilon_1'(s).$$

Since $\epsilon_1(s) = \angle\big(\mathbf{e}(s), \partial_\rho(s)\big)$, we have

$$\begin{aligned}\cos\epsilon_1(s) &= \langle \mathbf{e}(s), \partial_\rho\rangle,\\ \angle\big(\mathbf{e}(s), \partial_\theta(s)\big) &= \epsilon_1(s) + \frac{\pi}{2},\\ \langle \mathbf{e}(s), \partial_\theta(s)\rangle &= \sinh\rho(s)\cos(\epsilon_1(s) + \pi/2) = -\sinh\rho(s)\sin\epsilon_1(s).\end{aligned}$$

Differentiating the equation $\cos\epsilon_1(s) = \langle \mathbf{e}(s), \partial_\rho\rangle$, we then have

$$\begin{aligned}-\epsilon_1'(s)\sin\epsilon_1(s) &= \frac{d}{ds}\langle \mathbf{e}(s), \partial_\rho\rangle = \langle \mathbf{e}(s), \nabla_{\mathbf{t}_\alpha(s)}\partial_\rho\rangle\\ &= \langle \mathbf{e}(s), \rho'(s)\nabla_{\partial_\rho}\partial_\rho + \theta'(s)\nabla_{\partial_\theta}\partial_\rho\rangle\\ &= \theta'(s)\,\frac{\cosh\rho(s)}{\sinh\rho(s)}\langle \mathbf{e}(s), \partial_\theta\rangle = -\theta'(s)\,\frac{\cosh\rho(s)}{\sinh\rho(s)}\sinh\rho(s)\sin\epsilon_1(s)\\ &= -\theta'(s)\,\cosh\rho(s)\sin\epsilon_1(s),\end{aligned}$$

which gives

$$\epsilon_1'(s) = \theta'(s)\cosh\rho(s)$$

and we have

$$\phi'(s) + \kappa_\alpha(s) = \theta'(s)\cosh\rho(s). \tag{8}$$

On the other hand, since $\phi(s) = \pi - \epsilon_2(s)$, we have

$$\sin\phi(s) = \sin\epsilon_2(s) = \sqrt{1-\cos^2\epsilon_2(s)} = \sqrt{1-\rho'(s)^2} = \theta'(s)\sinh\rho(s). \tag{9}$$

Now we have from(8) and (9)

$$\begin{aligned} &\cosh u\sin\phi(s) - (\phi'(s)+\kappa_\alpha(s))\sinh u \\ &= \theta'(s)(\cosh u\sinh\rho(s) - \sinh u\cosh\rho(s)) \\ &= \theta'(s)\sinh(\rho(s)-u) \geq 0 \end{aligned}$$

since $\rho(s) \geq u$, which completes the proof of (7) and the theorem for $\mathbb{H}^2$.

4.3. Proof for $\mathbb{S}^2$

The area element of the parametrization $A(s,u)$ of $\mathcal{A} \subset \mathbb{S}^2$ is computed as

$$\big(\langle A_s, A_s\rangle\langle A_u, A_u\rangle - \langle A_s, A_s\rangle^2\big)^{1/2} = |\cos u\sin\phi(s) - (\phi'(s)+\kappa_\alpha(s))\sin u|,$$

that of the parametrization $L(s,u)$ of the left region as

$$\big(\langle L_s, L_s\rangle\langle L_u, L_u\rangle - \langle L_s, L_s\rangle^2\big)^{1/2} = |\cos u\sin\phi(s) - (\phi'(s)+\kappa_\beta(s))\sin u|$$

and that of the parametrization $R(s,u)$ of the right region as

$$\big(\langle R_s, R_s\rangle\langle R_u, R_u\rangle - \langle R_s, R_s\rangle^2\big)^{1/2} = |\cos\sin\phi(s) - (\phi'(s)-\kappa_\beta(s))\sin u|.$$

Hence the area of the ovaloidal region is

$$\int_0^l\int_0^{r(s)} \big(|\cos u\sin\phi(s) - (\phi'(s)+\kappa_\beta(s))\sin u| + |\cos u\sin\phi(s) - (\phi'(s)-\kappa_\beta(s))\sin u|\big)\,du\,ds$$

and the area of $\mathcal{A}$ is

$$\int_0^l\int_0^{r(s)} |\cos u\sin\phi(s) - (\phi'(s)+\kappa_\alpha(s))\sin u|\,du\,ds.$$

Now, suppose it holds that

$$\cos u\sin\phi(s) - (\phi'(s)+\kappa_\alpha(s))\sin u \geq 0 \tag{10}$$

in the parameter region

$$0 \leq s \leq l,\ 0 \leq u \leq r(s).$$

Furthermore, if $|\kappa_\beta(s)| \leq \kappa_\alpha(s)$, we will have

$$\cos u\sin\phi(s) - (\phi'(s)+\kappa_\beta(s))\sin u \geq \cos u\sin\phi(s) - (\phi'(s)+\kappa_\alpha(s))\sin u \geq 0,$$
$$\cos u\sin\phi(s) - (\phi'(s)-\kappa_\beta(s))\sin u \geq \cos u\sin\phi(s) - (\phi'(s)+\kappa_\alpha(s))\sin u \geq 0,$$

then the area of the ovaloidal region will be

$$\begin{aligned} &\int_0^l\int_0^{r(s)} \big(|\cos u\sin\phi(s) - (\phi'(s)+\kappa_\beta(s))\sin u| + |\cos u\sin\phi(s) - (\phi'(s)-\kappa_\beta(s))\sin u|\big)\,du\,ds \\ =& \int_0^l\int_0^{r(s)} \big(\cos u\sin\phi(s) - (\phi'(s)+\kappa_\beta(s))\sin u + \cos u\sin\phi(s) - (\phi'(s)-\kappa_\beta(s))\sin u\big)\,du\,ds \end{aligned}$$

$$= 2\int_0^l \int_0^{r(s)} \big(\cos u \sin\phi(s) - \phi'(s)\sin u\big)\, du\, ds$$

which shows that the area of the ovaloidal region in $\mathbb{S}^2$ is independent of the shape of the base curve β.

In order to prove (10), take the geodesic polar coordinate (ρ, θ) centered at O. Let us write for brevity

$$\partial_\rho := \frac{\partial}{\partial\rho}, \quad \partial_\theta := \frac{\partial}{\partial\theta}.$$

We have

$$\langle\partial_\rho, \partial_\rho\rangle = 1, \quad \langle\partial_\rho, \partial_\theta\rangle = \langle\partial_\theta, \partial_\rho\rangle = 0, \quad \langle\partial_\theta, \partial_\theta\rangle = \sin^2\rho$$

and

$$\nabla_{\partial_\rho}\partial_\rho = 0, \quad \nabla_{\partial_\rho}\partial_\theta = \nabla_{\partial_\theta}\partial_\rho = \frac{\cos\rho}{\sin\rho}\partial_\theta, \quad \nabla_{\partial_\theta}\partial_\theta = -\cos\rho\sin\rho.$$

Let $\alpha(s) = \big(\rho(s), \theta(s)\big)$. Then we have

$$\mathbf{t}_\alpha(s) = \alpha'(s) = \rho'(s)\partial_\rho + \theta'(s)\partial_\theta,$$

and since α is of unit speed, we also have

$$\rho'(s)^2 + \theta'(s)^2\sin^2\rho(s) = 1.$$

Let $\mathbf{e}(s)$ be a parallel vector field along $\alpha(s)$ with $\langle\mathbf{e}(s), \mathbf{e}(s)\rangle = 1$ and let

$$\begin{aligned}\epsilon_1(s) &= \angle\big(\mathbf{e}(s), \partial_\rho(s)\big),\\ \epsilon_2(s) &= \angle\big(\partial_\rho(s), \mathbf{t}_\alpha(s)\big).\end{aligned}$$

Then we have

$$\epsilon_2(s) + \phi(s) = \pi, \quad \phi'(s) = -\epsilon_2'(s)$$

and

$$\cos\epsilon_2(s) = \langle\big(\partial_\rho(s), \mathbf{t}_\alpha(s)\big)\rangle = \rho'(s).$$

Since $\angle\big(\mathbf{e}(s), \mathbf{t}_\alpha(s)\big) = \epsilon_1(s) + \epsilon_2(s)$, we have

$$\kappa_\alpha(s) = \frac{d}{ds}\angle\big(\mathbf{e}(s), \mathbf{t}_\alpha(s)\big) = \epsilon_1'(s) + \epsilon_2'(s)$$

and hence

$$\phi'(s) + \kappa_\alpha(s) = \epsilon_1'(s).$$

Since $\epsilon_1(s) = \angle\big(\mathbf{e}(s), \partial_\rho(s)\big)$, we have

$$\begin{aligned}\cos\epsilon_1(s) &= \langle\mathbf{e}(s), \partial_\rho\rangle,\\ \angle\big(\mathbf{e}(s), \partial_\theta(s)\big) &= \epsilon_1(s) + \frac{\pi}{2},\\ \langle\mathbf{e}(s), \partial_\theta(s)\rangle &= \sin\rho(s)\cos(\epsilon_1(s) + \pi/2) = -\sin\rho(s)\sin\epsilon_1(s).\end{aligned}$$

Differentiating the equation $\cos\epsilon_1(s) = \langle\mathbf{e}(s), \partial_\rho\rangle$, we then have

$$-\epsilon_1'(s)\sin\epsilon_1(s) = \frac{d}{ds}\langle\mathbf{e}(s), \partial_\rho\rangle = \langle\mathbf{e}(s), \nabla_{\mathbf{t}_\alpha(s)}\partial_\rho\rangle$$

$$
\begin{aligned}
&= \langle \mathbf{e}(s), \rho'(s)\nabla_{\partial_\rho}\partial_\rho + \theta'(s)\nabla_{\partial_\theta}\partial_\rho\rangle \\
&= \theta'(s)\,\frac{\cos\rho(s)}{\sin\rho(s)}\langle \mathbf{e}(s), \partial_\theta\rangle \\
&= -\theta'(s)\,\frac{\cos\rho(s)}{\sin\rho(s)}\sin\rho(s)\sin\epsilon_1(s) \\
&= -\theta'(s)\,\cos\rho(s)\sin\epsilon_1(s),
\end{aligned}
$$

which gives

$$\epsilon_1'(s) = \theta'(s)\cos\rho(s)$$

and we have

$$\phi'(s) + \kappa_\alpha(s) = \theta'(s)\cos\rho(s). \tag{11}$$

On the other hand, since $\phi(s) = \pi - \epsilon_2(s)$, we have

$$\sin\phi(s) = \sin\epsilon_2(s) = \sqrt{1-\cos^2\epsilon_2(s)} = \sqrt{1-\rho'(s)^2} = \theta'(s)\sin\rho(s). \tag{12}$$

Now we have from(11) and (12)

$$
\begin{aligned}
&\cos u\sin\phi(s) - (\phi'(s) + \kappa_\alpha(s))\sin u \\
&= \theta'(s)(\cos u\sinh\rho(s) - \sin u\cosh\rho(s)) \\
&= \theta'(s)\sin(\rho(s) - u) \geq 0
\end{aligned}
$$

since $\rho(s) \geq u$, which completes the proof of (10) and the theorem for $\mathbb{S}^2$.

5. A remark

The curvature condition in the theorem guarantees that a convex body and the base curve do not intersect near the initial tangent point. However, we are curious if there is an assurance that, during the rolling of the convex body, the convex body and the base curve will not intersect at locations away from the tangent point.

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