Bull. Korean Math. Soc. **61** (2024), No. 4, pp. 1087–1105
https://doi.org/10.4134/BKMS.b230648
pISSN: 1015-8634 / eISSN: 2234-3016

A NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF INVARIANT GIBBS MEASURES

Minkyu Kim

Abstract. In this paper, we study a relation between the existence of invariant Gibbs measures and the balanced property of subshifts. We show that a subshift X has an invariant Gibbs measure for $f \in C(X, \mathbb{R})$ if and only if it is balanced with respect to f.

1. Introduction

Let X be a subshift and let f be a real-valued continuous function from X to $\mathbb{R}$. Then a measure μ on X is called a Gibbs measure for f if there exist $c \geq 1$ and $P \in \mathbb{R}$ such that for every $n \in \mathbb{N}$, $w \in \mathcal{B}_n(X)$, and $x \in [w]$

$$c^{-1} \leq \frac{\mu([w]) \exp(nP)}{\exp S_n f(x)} \leq c,$$

where $S_n f(x) = \sum_{i=0}^{n-1} f\left(\sigma^i(x)\right)$ and $[w]$ is a cylinder.

It is well known that if X is a mixing shift of finite type and f is a Hölder continuous function from X to $\mathbb{R}$, then there exists a unique invariant Gibbs measure μ for f. This invariant Gibbs measure is a unique equilibrium state (Definition 2.6) for f and gives rise to a Bernoulli shift [6]. For the case where X is a subshift with the specification property (Definition 2.2) and f is a function in the Bowen class $\mathrm{Bow}(X)$ (Definition 3.3), there exists a unique invariant Gibbs measure for f. Moreover, it is a unique equilibrium state for f [5]. Walters showed that if X is a mixing shift of finite type and f is a Bowen function, then a unique invariant Gibbs measure μ for f gives rise to a Bernoulli shift [13].

Let $\{f_n\}$ be a sequence of functions from X to $\mathbb{R}$. We call the sequence an almost additive sequence if there exists $c > 0$ such that for every $m, n \in \mathbb{N}$ and $x \in X$ we have

$$-c + f_m(x) + f_n(\sigma^m(x)) \leq f_{m+n}(x) \leq c + f_m(x) + f_n(\sigma^m(x)).$$

Received November 6, 2023; Revised December 19, 2023; Accepted December 21, 2023.
2020 *Mathematics Subject Classification.* 37B10.
Key words and phrases. Symbolic dynamics, subshifts, Gibbs measures, balanced property.

The sequence has a bounded variation if

$$\sup_{n\in\mathbb{N}}\{|f_n(x)-f_n(y)| : x_0x_1\cdots x_{n-1}=y_0y_1\cdots y_{n-1}\}<\infty.$$

It is known that if $\{f_n\}$ is an almost additive sequence with bounded variation on a mixing shift of finite type X, then there exists a unique equilibrium state μ with the Gibbs property, i.e., for μ there exist $c\geq 1$ and $P\in\mathbb{R}$ such that for every $n\in\mathbb{N}$, $w\in\mathcal{B}_n(X)$, and $x\in[w]$

$$c^{-1}\leq\frac{\mu([w])\exp(nP)}{\exp f_n(x)}\leq c.$$

Moreover, μ is mixing [2, 3].

There are literature on the Gibbs measures on shifts of finite type over an infinite alphabet [10, 11].

In Section 2, we give basic definitions and notions on symbolic dynamics. For more details, we refer the reader to [9].

In Section 3, we discuss the existence of the invariant Gibbs measures. Existence of invariant Gibbs measures do not give much information outside the specification property. Beyond the specification property, it has been shown that a subshift X has a Gibbs measure for $f=0$ if and only if it has the right balanced property with respect to $f=0$ (Theorem 3.14 in [1]). The constructed measure in the proof of the theorem, however, need not be invariant (Theorem 3.7 and Theorem 3.14). Since one is more interested in invariant measures on dynamical systems, we investigate a necessary and sufficient condition for the existence of invariant Gibbs measures for $f=0$. We define boundedly supermultiplicative property (Definition 3.4) and balanced property (Definition 3.5) for real-valued continuous functions on subshifts. We show that a subshift X has a measure of maximal entropy with the Gibbs property if and only if it has the balanced property with respect to $f=0$ (Corollary 3.15), and extend this result for real-valued continuous functions on X (Theorem 3.14).

2. Preliminary

We recall some definitions which are relevant in Section 3. A finite set with the discrete topology, denoted by $\mathcal{A}$, is called an *alphabet* and an element of $\mathcal{A}$ is called a *symbol* or *letter*. The collection of bi-infinite sequences of symbols from $\mathcal{A}$ is called the *full $\mathcal{A}$-shift* and denoted by $\mathcal{A}^{\mathbb{Z}}$, i.e., $\mathcal{A}^{\mathbb{Z}}=\{x=(\cdots x_{-1}x_0x_1\cdots)\mid x_i\in\mathcal{A},\ i\in\mathbb{Z}\}$. It is a compact and metrizable space with respect to the product topology. Together with the *shift map*, denoted by σ and defined by $\sigma(x)_i=x_{i+1}$, it is regarded as a topological dynamical system. A σ-invariant closed subset X of $\mathcal{A}^{\mathbb{Z}}$ is called a *subshift*.

A *block* or *word* is a finite sequence of symbols in $\mathcal{A}$ and the *length* of a word u, denoted by $|u|$, is the number of symbols it contains. For example, the length $|u|$ of $u=aabab$ is 5. The unique word of length 0 is the *empty word* denoted by ϵ. For $n\in\mathbb{N}\cup\{0\}$, an n-word is a word of length n. For a word u, a word u' is a *factor* or *subword* of u, denoted by $u' \prec u$, if there exist words v and w such that $u = vu'w$, a *prefix* if there exists a word w such that $u = u'w$, and a *suffix* if there exists a word v such that $u = vu'$. For a word u and $n \in \mathbb{N}$, denote n-times concatenation of u by u^n. For $x \in \mathcal{A}^{\mathbb{Z}}$ and $i \leq j \in \mathbb{Z}$, we denote the word of coordinates in x from i to j by

$$x_{[i,j]} = x_i x_{i+1} \cdots x_j.$$

If X is a subshift, we denote by $\mathcal{B}_n(X)$ the set of all words of length n appearing in points in X and $\mathcal{B}(X) = \bigcup_{n \geq 0} \mathcal{B}_n(X)$ is called the *language* of X. For $u \in \mathcal{B}_n(X)$, the *cylinder* u, denoted by $[u]$, is $\{x \in X \mid x_{[0,n)} = x_0 x_1 \cdots x_{n-1} = u\}$. It is open and closed in X.

Definition 2.1. Let X be a subshift, let u be a word in $\mathcal{B}(X)$, and let n be a nonnegative integer. A word v is called an *n-follower* of u if $|v| = n$ and $uv \in \mathcal{B}(X)$ and the *n-follower set* $F_n(u)$ of u is the set of all n-followers of u. A word w is called an *n-predecessor* of u if $|w| = n$ and $wu \in \mathcal{B}(X)$ and the *n-predecessor set* $P_n(u)$ of u is the set of all n-predecessors of u.

A subshift X is *irreducible* if for all $u, v \in \mathcal{B}(X)$, there exists a word w such that $uwv \in \mathcal{B}(X)$.

Definition 2.2. Let X be a subshift. We say that X has the *specification property* [4] if there exists a nonnegative integer N such that for all $u, v \in \mathcal{B}(X)$, there exists a word w of length N with $uwv \in \mathcal{B}(X)$. We say that X has the *almost specification property* [8] if there exists a nonnegative integer N such that for all $u, v \in \mathcal{B}(X)$, there exists a word w of length less than or equal to N with $uwv \in \mathcal{B}(X)$.

We call N in Definition 2.2 the gap. It is obvious that the specification property implies the almost specification property and the almost specification property implies the irreducibility.

Definition 2.3. Let X be an irreducible subshift. A word u is *synchronizing* for X if $vuw \in \mathcal{B}(X)$ whenever vu and uw are in $\mathcal{B}(X)$. A subshift X is called a *synchronized system* if it has a synchronizing word.

In the topological dynamics, the topological entropy is an important notion. The number $|\mathcal{B}_n(X)|$ of n-words appearing in a subshift X gives us a way to define the entropy of X which is equal to the topological entropy of (X, σ).

Definition 2.4. Let X be a subshift. The *entropy* of X is defined by

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log |\mathcal{B}_n(X)|.$$

Here and throughout log is the logarithm to the base e. Let $C(X, \mathbb{R})$ be the set of all real-valued continuous functions on X. For $f \in C(X, \mathbb{R})$, the following is a generalized notion of the topological entropy.

Definition 2.5. Let X be a subshift and let $f \in C(X, \mathbb{R})$. The *topological pressure* of f is defined by

$$P(f) = \lim_{n\to\infty} \frac{1}{n} \log \left[\sum_{w\in\mathcal{B}_n(X)} \exp\left(\sup_{x\in[w]} S_n f(x) \right) \right],$$

where $S_n f(x) = \sum_{i=0}^{n-1} f\left(\sigma^i(x)\right)$.

Since $[w]$ is a compact subset of X for $w \in \mathcal{B}(X)$, we can use maximum instead of supremum in Definition 2.5. The limit exists, since we have the subadditivity that for $m, n \in \mathbb{N}$

$$\begin{aligned} &\sum_{w\in\mathcal{B}_{m+n}(X)} \exp\left(\sup_{x\in[w]} S_{m+n} f(x) \right) \\ &\le \left[\sum_{w\in\mathcal{B}_m(X)} \exp\left(\sup_{x\in[w]} S_m f(x) \right) \right] \left[\sum_{w\in\mathcal{B}_n(X)} \exp\left(\sup_{x\in[w]} S_n f(x) \right) \right]. \end{aligned}$$

In fact

$$\begin{aligned} P(f) &= \lim_{n\to\infty} \frac{1}{n} \log \left[\sum_{w\in\mathcal{B}_n(X)} \exp\left(\sup_{x\in[w]} S_n f(x) \right) \right] \\ &= \inf_{n\in\mathbb{N}} \frac{1}{n} \log \left[\sum_{w\in\mathcal{B}_n(X)} \exp\left(\sup_{x\in[w]} S_n f(x) \right) \right]. \end{aligned}$$

If $f = 0$, then the topological pressure of f is exactly same as the entropy of X. The topological pressure of f and the measure-theoretic entropies of invariant measures on (X, σ) have a relation called the variational principle [12]: If f is a real-valued continuous function on X, then

$$P(f) = \sup \left\{ h_\mu(\sigma) + \int f d\mu \,\middle|\, \mu \in M_\sigma(X) \right\},$$

where $h_\mu(\sigma)$ is the measure-theoretic entropy of (X, σ, μ) and $M_\sigma(X)$ is the set of σ-invariant Borel probability measures on X. Because measures on X which attain the supremum are special, we give these measures a name.

Definition 2.6. Let X be a subshift and let $f \in C(X, \mathbb{R})$. A measure $\mu \in M_\sigma(X)$ is called an *equilibrium state* for f if $P(f) = h_\mu(\sigma) + \int f d\mu$. In particular, if $f = 0$, then we call μ a *measure of maximal entropy*.

3. A relation between Gibbs measures and the balanced property

Throughout the set of all Borel probability measures on a subshift X is denoted by $M(X)$ and the set of all real-valued continuous functions on a subshift X is denoted by $C(X, \mathbb{R})$. If $w \in \mathcal{B}(X)$, then we let x_w denote an arbitrary point in $[w]$.

Definition 3.1. A measure $\mu \in M(X)$ is called a *Gibbs measure* for $f \in C(X, \mathbb{R})$ if there exist $c \geq 1$ and $P \in \mathbb{R}$ such that for each $n \in \mathbb{N}$, $w \in \mathcal{B}_n(X)$, and $x \in [w]$

$$c^{-1} \leq \frac{\mu([w]) \exp(nP)}{\exp S_n f(x)} \leq c. \tag{1}$$

Remark 3.2. If μ satisfies (1) for $f \in C(X, \mathbb{R})$, then P must be equal to the topological pressure $P(f)$ of f. If μ is an invariant Gibbs measure for f, then μ is an equilibrium state for f, i.e., $P(f) = h_\mu(\sigma) + \int f d\mu$ [6].

Definition 3.3. A function $f \in C(X, \mathbb{R})$ is called a *Bowen function* if there exists $M \geq 0$ such that, for every $n \in \mathbb{N}$, we have

$$|S_n f(x) - S_n f(y)| \leq M$$

whenever $x, y \in X$ and $x_0 x_1 \cdots x_{n-1} = y_0 y_1 \cdots y_{n-1}$. We call the set of all Bowen functions the *Bowen class* denoted by $\mathrm{Bow}(X)$.

If μ is a Gibbs measure for $f \in C(X, \mathbb{R})$, then f must be a Bowen function. The following are extension of notions in [1].

Definition 3.4. Let X be a subshift and let $f \in C(X, \mathbb{R})$. Then X is *boundedly supermultiplicative with respect to* f (BSM(f) for short) if there exists $c \geq 1$ such that for each $n, m \in \mathbb{N}$

$$c^{-1} \leq \frac{\left(\sum_{u \in \mathcal{B}_m(X)} \exp S_m f(x_u)\right)\left(\sum_{v \in \mathcal{B}_n(X)} \exp S_n f(x_v)\right)}{\sum_{w \in \mathcal{B}_{m+n}(X)} \exp S_{m+n} f(x_w)} \leq c$$

for each choice of $x_u \in [u]$, $x_v \in [v]$, and $x_w \in [w]$.

Definition 3.5. Let X be a subshift and let $f \in C(X, \mathbb{R})$. We say that X is *right balanced with respect to* f if there exists $c \geq 1$ such that for each $m, n \in \mathbb{N}$ and $u \in \mathcal{B}_m(X)$

$$c^{-1} \leq \frac{\sum_{v \in F_n(u)} \exp S_{m+n} f(x_{uv})}{\exp S_m f(x_u) \sum_{w \in \mathcal{B}_n(X)} \exp S_n f(x_w)} \leq c$$

for each choice of $x_u \in [u]$, $x_{uv} \in [uv]$, and $x_w \in [w]$. We say that X is *left balanced with respect to* f if there exists $c \geq 1$ such that for each $m, n \in \mathbb{N}$ and $u \in \mathcal{B}_m(X)$

$$c^{-1} \leq \frac{\sum_{v \in P_n(u)} \exp S_{m+n} f(x_{vu})}{\exp S_m f(x_u) \sum_{w \in \mathcal{B}_n(X)} \exp S_n f(x_w)} \leq c$$

for each choice of $x_u \in [u]$, $x_{vu} \in [vu]$, and $x_w \in [w]$. We say that X is *balanced with respect to* f if it is left and right balanced with respect to f. We say that X is *one-sided balanced with respect to* f if it is left or right balanced with respect to f.

We note that if $f = 0$, then the right balanced with respect to f is same as the balanced property defined in [1].

We construct a concrete example of X and f, where X is BSM(f).

Example 3.6. Let $X = \{0,1\}^{\mathbb{Z}}$ be the full 2-shift. Define a continuous real-valued function f by

$$f(x) = \begin{cases} 0 & \text{if } x_i = 0 \text{ for all } i \geq 0, \\ \frac{1}{2^j} & \text{otherwise,} \end{cases}$$

where j is the minimum of $\{i \,|\, x_i = 1,\ i \geq 0\}$. If $x_{[0,n)} = y_{[0,n)}$, then we have

$$\left| f\big(\sigma^i(x)\big) - f\big(\sigma^i(y)\big) \right| \leq \frac{1}{2^{n-i}}$$

for $i = 0, \dots, n-1$. For each $u \in \mathcal{B}(X)$, take $\tilde{x}_u \in [u]$. Then we have

$$\left| S_{|u|} f(\tilde{x}_u) - S_{|u|} f(x_u) \right| \leq \sum_{i=0}^{|u|-1} \frac{1}{2^{|u|-i}} \leq 1.$$

It follows that

$$\begin{aligned} &\frac{\left(\sum_{u\in\mathcal{B}_m(X)} \exp S_m f(x_u)\right)\left(\sum_{v\in\mathcal{B}_n(X)} \exp S_n f(x_v)\right)}{\sum_{w\in\mathcal{B}_{m+n}(X)} \exp S_{m+n} f(x_w)} \\ \leq\ & \exp(3) \frac{\left(\sum_{u\in\mathcal{B}_m(X)} \exp S_m f(\tilde{x}_u)\right)\left(\sum_{v\in\mathcal{B}_n(X)} \exp S_n f(\tilde{x}_v)\right)}{\sum_{w\in\mathcal{B}_{m+n}(X)} \exp S_{m+n} f(\tilde{x}_w)} \\ =\ & \exp(3) \frac{\left(\sum_{u\in\mathcal{B}_m(X)} \exp S_m f(\tilde{x}_u)\right)\left(\sum_{v\in\mathcal{B}_n(X)} \exp S_n f(\tilde{x}_v)\right)}{\sum_{p\in\mathcal{B}_m(X)} \sum_{q\in\mathcal{B}_n(X)} \exp S_{m+n} f(\tilde{x}_{pq})} \\ \leq\ & \exp(5) \frac{\left(\sum_{u\in\mathcal{B}_m(X)} \exp S_m f(\tilde{x}_u)\right)\left(\sum_{v\in\mathcal{B}_n(X)} \exp S_n f(\tilde{x}_v)\right)}{\sum_{p\in\mathcal{B}_m(X)} \sum_{q\in\mathcal{B}_n(X)} \exp[S_m f(\tilde{x}_p) + S_n f(\tilde{x}_q)]} \\ =\ & \exp(5). \end{aligned}$$

The second inequality follows from

$$S_{m+n} f\left(\tilde{x}_{pq}\right) = S_m f\left(\tilde{x}_{pq}\right) + S_n f\left(\sigma^m(\tilde{x}_{pq})\right) \geq S_m f\left(\tilde{x}_p\right) + S_n f\left(\tilde{x}_q\right) - 2.$$

From similar calculations, we obtain

$$\exp(-5) \leq \frac{\left(\sum_{u\in\mathcal{B}_m(X)} \exp S_m f(x_u)\right)\left(\sum_{v\in\mathcal{B}_n(X)} \exp S_n f(x_v)\right)}{\sum_{w\in\mathcal{B}_{m+n}(X)} \exp S_{m+n} f(x_w)}.$$

Therefore X is BSM(f).

We note that the full $\mathcal{A}$-shift is BSM(f) for any $f \in$Bow(X).

It is not hard to see that the one-sided balanced property with respect to f implies that f is a Bowen function as follows. Suppose that a subshift X is right balanced with respect to $f \in C(X,\mathbb{R})$ and $u \in \mathcal{B}(X)$. Let x and y be points in $[u]$. For each $w \in \mathcal{B}(X)$, take a point $\hat{x}_w \in [w]$. Since X is right balanced with respect to f, there exists $c \geq 1$ such that for each $n \in \mathbb{N}$

$$c^{-1} \leq \frac{\sum_{v\in F_n(u)} \exp S_{m+n} f(\hat{x}_{uv})}{\exp S_m f(x) \sum_{w\in\mathcal{B}_n(X)} \exp S_n f(\hat{x}_w)} \leq c$$

and

$$c^{-1} \leq \frac{\sum_{v\in F_n(u)} \exp S_{m+n} f(\hat{x}_{uv})}{\exp S_m f(y) \sum_{w\in\mathcal{B}_n(X)} \exp S_n f(\hat{x}_w)} \leq c,$$

where m is the length of u. It follows that

$$c^{-2} \leq \frac{\exp S_m f(x)}{\exp S_m f(y)} \leq c^2$$

or equivalently,

$$|S_m f(x) - S_m f(y)| \leq 2\log c.$$

That is, f is a Bowen function. For the case where X is left balanced with respect to f, we get the same result by similar calculations.

We also note that if f is in $\mathrm{Bow}(X)$, then there exists c such that for each $m, n \in \mathbb{N}$ and $u \in \mathcal{B}_m(X)$

$$\frac{\sum_{v\in F_n(u)} \exp S_{m+n} f(x_{uv})}{\exp S_m f(x_u) \sum_{w\in\mathcal{B}_n(X)} \exp S_n f(x_w)} \leq c$$

and

$$\frac{\sum_{v\in P_n(u)} \exp S_{m+n} f(x_{vu})}{\exp S_m f(x_u) \sum_{w\in\mathcal{B}_n(X)} \exp S_n f(x_w)} \leq c$$

for each choice of $x_u \in [u]$, $x_{uv} \in [uv]$, $x_{vu} \in [vu]$, and $x_w \in [w]$ (see the proof of Proposition 3.9).

Theorem 3.7. *Let X be a subshift and let $f \in C(X,\mathbb{R})$. The right balanced property with respect to f and the left balanced property with respect to f are not equivalent.*

Proof. Let $Y = \{a,b\}^{\mathbb{Z}}$ and let $S = \{s \mid s = 2^n \text{ for } n \in \mathbb{N}\}$. Then define a sequence $\{a_s\}_{s\in S}$ by $a_s = \log_2 s$ and $C = \{1wb^{s-a_s} \mid w \in \mathcal{B}_{a_s}(Y) \text{ for } s \in S\}$. Let X be the closure of the collection of all bi-infinite concatenations of elements in C and let Z be the closure of the collection of all bi-infinite concatenations of elements in $\{10^s \mid s \in S\}$. Then X and Z are subshifts and X contains Y because $\{a_s\}_{s\in S}$ is unbounded. It is well known that the entropy of Z is equal to $\log\lambda$, where λ is a unique positive solution of $\sum_{s\in S} x^{-(s+1)} = 1$ [9]. Since Z is BSM(0) [7], there exists $d > 0$ such that

$$\lambda^n \leq |\mathcal{B}_n(Z)| \leq d\lambda^n$$

for $n \in \mathbb{N}$ [1]. Define a block map Φ from $\mathcal{B}_1(X)$ to $\{0,1\}$ by

$$\Phi(w) = \begin{cases} 1 & \text{if } w = 1, \\ 0 & \text{if } w \neq 1, \end{cases}$$

and a block map Ψ from $\mathcal{B}(X)$ to $\mathcal{B}(Z)$ by

$$\Psi(w_1 w_2 \cdots w_n) = \Phi(w_1)\Phi(w_2)\cdots\Phi(w_n).$$

For each $s \in S$, $\left|\Psi^{-1}(10^s 1)\right| = |\mathcal{B}_{a_s}(Y)| = 2^{a_s}$. Since $\{a_s\}_{s\in S}$ is unbounded, we obtain $\left|\Psi^{-1}(10^n)\right| = |\mathcal{B}_n(Y)| = 2^n$ for $n \in \mathbb{N}$. For $n \in \mathbb{N}$, an element of $\Psi^{-1}(0^n 1)$ is a suffix of length $n+1$ of $u1$, where $u \in C$. Since $\{s - a_s\}_{s\in S}$ is an unbounded increasing sequence, $u_{[a_s+2,s+1]}$ is a suffix of $v_{[a_{s'}+2,s'+1]}$ whenever $u, v \in C$, $|u| = s+1$, $|v| = s'+1$, $s, s' \in S$, and $s \leq s'$. Hence we have $\left|\Psi^{-1}(0^n 1)\right| \leq 2^{n\alpha}$, where $\alpha = \sup_{s\in S} \frac{a_s}{s}$. Since 1 is a synchronizing word for X, we have

$$\begin{aligned}\left|\Phi^{-1}\left(0^n 10^{s_1} 10^{s_2} 1 \cdots 10^{s_k} 10^m\right)\right| &= \left|\Phi^{-1}\left(0^n 1\right)\right| \left(\prod_{i=1}^{k} \left|\Phi^{-1}\left(10^{s_i} 1\right)\right|\right) \left|\Phi^{-1}\left(10^m\right)\right| \\ &\leq 2^m 2^{(n+s_1+\cdots+s_k)\alpha} \\ &\leq 2^m 2^{l\alpha},\end{aligned}$$

where l is the length of $0^n 10^{s_1} 10^{s_2} 1 \cdots 10^{s_k}$.

Let $\mathcal{B}_n(Z,k) = \{w = w_1 \cdots w_n \in \mathcal{B}_n(Z) \mid w_k = 1 \text{ and } w_i \neq 1 \text{ for } i > k\}$. Then, for $n \in \mathbb{N}$

$$\begin{aligned}|\mathcal{B}_n(X)| &\leq \sum_{k=0}^{n} \left(|\mathcal{B}_n(Z,k)| 2^{n-k} 2^{k\alpha}\right) \\ &\leq \sum_{k=0}^{n} d\lambda^k 2^{n-k} 2^{k\alpha} \\ &= \sum_{k=0}^{n} d 2^{k\frac{\log\lambda}{\log 2}} 2^{n-k} 2^{k\alpha} \\ &= d2^n \sum_{k=0}^{n} 2^{\left(\frac{\log\lambda}{\log 2}+\alpha-1\right)k} \\ &\leq d2^n \sum_{k=0}^{\infty} 2^{\left(\frac{\log\lambda}{\log 2}+\alpha-1\right)k}.\end{aligned}$$

Since $\sum_{s\in S}\left(\sqrt{2}\right)^{-(s+1)} < 1$ and $\alpha = \frac{1}{2}$, we have $\lambda < \sqrt{2}$ and $\frac{\log\lambda}{\log 2} + \alpha - 1 < 0$. It follows that there exists $d' \geq 1$ such that, for $n \in \mathbb{N}$, $|\mathcal{B}_n(X)| \leq d' 2^n$. Since $\{a_s\}_{s\in S}$ is unbounded, we have $\mathcal{B}_n(Y) \subset F_n(u)$ for any $u \in \mathcal{B}(X)$ and $n \in \mathbb{N}$. Therefore we obtain

$$\frac{|F_n(u)|}{|\mathcal{B}_n(X)|} \geq \frac{|\mathcal{B}_n(Y)|}{d' 2^n} = \frac{1}{d'}$$

i.e., X is right balanced with respect to $f = 0$.

Suppose that X is left balanced with respect to $f = 0$. There exists $0 < c \leq 1$ such that, for $u \in \mathcal{B}(X)$ and $n \in \mathbb{N}$, the number $|P_n(u)|$ of words v of length n followed by u is greater than or equal to $c\,|\mathcal{B}_n(X)|$. The set $\mathcal{B}_n(X)$ of all words of length n in X is disjoint union $\bigcup_{k=0}^{n} \mathcal{B}_n(X,k)$, where $\mathcal{B}_n(X,k)$ is the set of all elements of $\mathcal{B}_n(X)$ such that the k-th coordinate is 1 and the j-th coordinate is not 1 for all $j > k$. Since 1 is synchronizing, we have

$$\begin{aligned}|\mathcal{B}_n(X,k)| &= |P_{k-1}(1)|\,|F_{n-k}(1)| \\ &\geq \frac{1}{d'}|P_{k-1}(1)|\,|\mathcal{B}_{n-k}(X)| \quad \text{(by the right balanced property of } X) \\ &\geq \frac{c}{d'}|\mathcal{B}_{k-1}(X)|\,|\mathcal{B}_{n-k}(X)| \quad \text{(by the left balanced property of } X) \\ &\geq \frac{c}{d'}2^{n-1} \quad \text{(since } X \text{ contains } Y \text{ and } h(Y) = \log 2).\end{aligned}$$

Let $c' = \frac{c}{2d'}$. Then

$$|\mathcal{B}_n(X)| = \sum_{k=0}^{n} |\mathcal{B}_n(X,k)| \geq \sum_{k=0}^{n} c' 2^n = c'(n+1)2^n.$$

This is a contradiction because $|\mathcal{B}_n(X)| \leq d' 2^n$ for $n \in \mathbb{N}$. Therefore X is not left balanced with respect to $f = 0$. □

There exists a nonzero function f such that the right balanced property and the left balanced property are not equivalent.

Example 3.8. Let X and Y be the subshifts in the above proof. Define a real-valued continuous function f on X by

$$f(x) = \sum_{i \in \mathbb{N} \cup \{0\}} \delta_i(x),$$

where $\delta_i(x) = \begin{cases} 0 & \text{if } x_i = 1, \\ \frac{1}{2^i} & \text{otherwise.} \end{cases}$ For $u \in \mathcal{B}(X)$, let $m = |u|$. We have

$$\left|f\big(\sigma^i(x)\big) - f\big(\sigma^i(y)\big)\right| \leq \frac{1}{2^{m-1-i}}$$

whenever $x, y \in [u]$ and $i = 0, 1, \ldots, m-1$. Thus $|S_m f(x) - S_m f(y)| \leq 2$, that is, f is in $\mathrm{Bow}(X)$. It follows that for any $n \in \mathbb{N}$, we have

$$\frac{\sum_{v \in F_n(u)} \exp S_{m+n} f(x_{uv})}{\exp S_m f(x_u) \sum_{v \in \mathcal{B}_n(X)} \exp S_n f(x_v)} \geq \exp(-2) \frac{\sum_{v \in F_n(u)} \exp S_n f(\sigma^m x_{uv})}{\sum_{v \in \mathcal{B}_n(X)} \exp S_n f(x_v)}.$$

Since $\{a_s\}_{s \in S}$ is unbounded, $F_n(u)$ contains $\mathcal{B}_n(Y)$. Thus we get

$$\frac{\exp(-2) \sum_{v \in F_n(u)} \exp S_n f(\sigma^m x_{uv})}{\sum_{v \in \mathcal{B}_n(X)} \exp S_n f(x_v)} \geq \frac{\exp(-2) \sum_{v \in \mathcal{B}_n(Y)} \exp S_n f(\sigma^m x_{uv})}{\sum_{v \in \mathcal{B}_n(X)} \exp S_n f(x_v)}.$$

If $v \in \mathcal{B}_n(X) \setminus \mathcal{B}_n(Y)$ and $w \in \mathcal{B}_n(Y)$, then the following holds:

$$\sup\{\exp S_n f(x) \mid x \in [v]\} \leq \sup\{\exp S_n f(x) \mid x \in [w]\}$$

because there exists $y \in Y \subset X$ such that $y_0 y_1 \cdots y_{n-1} = w$. Since X is right balanced with respect to $f = 0$, $h(X) = \log 2$, and $|\mathcal{B}_n(Y)| = 2^n$ for all $n \in \mathbb{N}$, there exists $d \geq 1$ such that $|\mathcal{B}_n(X)| \leq d\,|\mathcal{B}_n(Y)|$ for all $n \in \mathbb{N}$ [1]. Hence

$$\begin{aligned}
&\exp(-2)\frac{\sum_{v\in\mathcal{B}_n(Y)} \exp S_n f(\sigma^m x_{uv})}{\sum_{v\in\mathcal{B}_n(X)} \exp S_n f(x_v)} \\
&= \exp(-2)\frac{\sum_{v\in\mathcal{B}_n(Y)} \exp S_n f(\sigma^m x_{uv})}{\sum_{v\in\mathcal{B}_n(Y)} \exp S_n f(x_v) + \sum_{v\in\mathcal{B}_n(X)\setminus\mathcal{B}_n(Y)} \exp S_n f(x_v)} \\
&\geq \exp(-2)\frac{\sum_{v\in\mathcal{B}_n(Y)}(\sup\{\exp S_n f(x) \mid x \in [v]\}\exp(-2))}{d\sum_{v\in\mathcal{B}_n(Y)} \sup\{\exp S_n f(x) \mid x \in [v]\}} \\
&= \frac{\exp(-4)}{d}.
\end{aligned}$$

Since f is in $\mathrm{Bow}(X)$, we already know that there exists c such that

$$\frac{\sum_{v\in F_n(u)} \exp S_{m+n} f(x_{uv})}{\exp S_m f(x_u) \sum_{v\in\mathcal{B}_n(X)} \exp S_n f(x_v)} \leq c.$$

Therefore X is right balanced with respect to f.

Now, we show that X is not left balanced with respect to f. Let n be a positive integer and let $u = b^{1+2^n}1$. Then every $(2^{n+1} - 2^n - 1)$-predecessor of u is of the form $vb^{2^{n+1}-2^n-1-(n+1)}$, where $v \in \mathcal{B}_{n+1}(Y)$. Since $|\mathcal{B}_{n+1}(Y)| = 2^{n+1}$, the number of $(2^{n+1} - 2^n - 1)$-predecessors of u is equal to 2^{n+1}. Let $i = 2^{n+1} - 2^n - 1$. For each $o \in \mathcal{B}(X)$, take a point $\hat{x}_o \in [o]$ such that $S_{|o|} f(\hat{x}_o) = \sup_{x\in[o]} S_{|o|} f(x)$. Then we have

$$\begin{aligned}
&\frac{\sum_{v\in P_i(u)} \exp S_{|u|+i} f(\hat{x}_{vu})}{\exp S_{|u|} f(\hat{x}_u) \sum_{w\in\mathcal{B}_i(X)} \exp S_i f(\hat{x}_w)} \\
&\leq \frac{\exp(2)\sum_{v\in\mathcal{B}_{n+1}(Y)} \exp S_i f(\hat{x}_{vb^{i-(n+1)}u})}{\sum_{w\in\mathcal{B}_i(Y)} \exp S_i f(\hat{x}_w)} \\
&\leq \frac{\exp(2)\sum_{v\in\mathcal{B}_{n+1}(Y)} \exp S_i f(\hat{x}_{vb^{i-(n+1)}})}{\sum_{w\in\mathcal{B}_i(Y)} \exp S_i f(\hat{x}_w)}.
\end{aligned}$$

For $w \in \mathcal{B}_i(Y)$, there exists $x \in Y \subset X$ such that $x_0 x_1 \cdots x_{i-1} = w$, so $S_i f(\hat{x}_w) = 2i$. It follows that

$$\begin{aligned}
\exp(2)\frac{\sum_{v\in\mathcal{B}_{n+1}(Y)} \exp S_i f(\hat{x}_{vb^{i-(n+1)}})}{\sum_{w\in\mathcal{B}_i(Y)} \exp S_i f(\hat{x}_w)} &= \exp(2)\frac{|\mathcal{B}_{n+1}(Y)|\exp(2i)}{|\mathcal{B}_i(Y)|\exp(2i)} \\
&= \frac{\exp(2)}{2^{i-(n+1)}}.
\end{aligned}$$

As n goes to ∞, $\frac{\exp(2)}{2^{2^{n+1}-2^n-1-(n+1)}}$ converges to 0. Therefore X is not left balanced with respect to f.

However if a subshift X has the almost specification property, then the left balanced property with respect to f and the right balanced property with respect to f are equivalent. Moreover, if a subshift X is almost specified, then it is balanced with respect to every function $f \in \mathrm{Bow}(X)$.

Proposition 3.9. *Let X be an almost specified subshift and let $f \in C(X, \mathbb{R})$. Then the following are equivalent*:

1) *X has the balanced property with respect to f.*
2) *X has the one-sided balanced with respect to f.*
3) *f is a Bowen function.*

Proof. We already know that 1) implies 2) and 2) implies 3).

Now, let's show that 3) implies 1). Suppose that f is a Bowen function on X. Thus there exists M such that for $u \in \mathcal{B}(X)$, $\exp(S_n f(y) - M) \leq \exp S_n f(x) \leq \exp(S_n f(y) + M)$ whenever $x, y \in [u]$ and $n = |u|$. Since X is almost specified, there exists N such that whenever u, $v \in \mathcal{B}(X)$, there exists w such that $|w| \leq N$ and $uwv \in \mathcal{B}(X)$. Let $m, n \in \mathbb{N}$ and let $u \in \mathcal{B}_m(X)$. For each $v \in F_n(u)$, $(x_u)_{[0,m)} = (x_{uv})_{[0,m)} = u$ and $(x_v)_{[0,n)} = \sigma^m(x_{uv})_{[0,n)} = v$. Hence we obtain

$$\begin{aligned} &\exp S_m f(x_u) \sum_{w \in \mathcal{B}_n(X)} \exp S_n f(x_w) \\ \geq\ &\exp S_m f(x_u) \sum_{v \in F_n(u)} \exp\left(S_n f(x_v) - M\right) \\ \geq\ &\sum_{v \in F_n(u)} \exp\left(S_{m+n} f(x_{uv}) - 3M\right). \end{aligned}$$

It follows that

$$\frac{\sum_{v \in F_n(u)} \exp S_{m+n} f(x_{uv})}{\exp S_m f(x_u) \sum_{w \in \mathcal{B}_n(X)} \exp S_n f(x_w)} \leq \exp(3M).$$

Since X is almost specified, for each $w \in \mathcal{B}(X)$ there exists $k \leq N$ such that $F_{|\omega|+k}(u)$ contains a word which ends with w. It follows that

$$\begin{aligned} &\exp S_m f(x_u) \sum_{w \in \mathcal{B}_n(X)} \exp S_n f(x_w) \\ \leq\ &\sum_{k=0}^{N} \left(\exp(k\|f\|_\infty) \sum_{v \in F_{n+k}(u)} \exp\left(S_{m+n+k} f\left(x_{uv}\right) + 2M\right) \right), \end{aligned}$$

where $\|f\|_\infty = \sup_{x\in X} f(x)$. If $v \in F_n(u)$, then the number of words in $F_{n+k}(u)$ which start with v is at most $|\mathcal{B}_k(X)|$. Thus we have

$$\begin{aligned} &\sum_{k=0}^{N}\left(\exp(k\|f\|_\infty)\sum_{v\in F_{n+k}(u)}\exp\left(S_{m+n+k}f(x_{uv})+2M\right)\right)\\ &\le \sum_{k=0}^{N}\left(\exp(k\|f\|_\infty)|\mathcal{B}_k(X)|\sum_{v\in F_n(u)}\exp(S_{m+n}f(x_{uv})+3M+k\|f\|_\infty)\right)\\ &= \left(\sum_{k=0}^{N}\exp(2k\|f\|_\infty+3M)|\mathcal{B}_k(X)|\right)\sum_{v\in F_n(u)}\exp S_{m+n}f(x_{uv}). \end{aligned}$$

Consequently, we get

$$\begin{aligned} &\left(\sum_{k=0}^{N}\left[|\mathcal{B}_k\left(X\right)|\exp\left(2k\left\|f\right\|_\infty+3M\right)\right]\right)^{-1}\\ &\le \frac{\sum_{v\in F_n(u)}\exp S_{m+n}f(x_{uv})}{\exp S_m f(x_u)\sum_{w\in\mathcal{B}_n(X)}\exp S_n f(x_w)}. \end{aligned}$$

Therefore X is right balanced with respect to f. Similarly, X is left balanced with respect to f. □

Example 3.10. Let $X = \{0,1\}^{\mathbb{Z}}$. Then it is balanced with respect to f defined in Example 3.6 because X is almost specified and f is a Bowen function.

Define g on X by

$$g(x) = \begin{cases} 0 & \text{if } x_i = 0 \text{ for all } i \ge 0,\\ \frac{1}{j+1} & \text{otherwise,} \end{cases}$$

where $j = \min\{i \,|\, x_i = 1,\ i \ge 0\}$. For each $k \in \mathbb{N}$, choose $x(k) \in X$ such that $x(k)_i = 0$ for $i < k$ and $x(k)_i = 1$ for $i \ge k$. Let y be the point in X such that $y_i = 0$ for all $i \in \mathbb{Z}$. Then $x(k), y \in \left[0^k\right]$ and $|S_k g(x(k)) - S_k g(y)| = \sum_{l=0}^{k-1}\frac{1}{l+2}$. Since $\sum_{l=0}^{\infty}\frac{1}{l+2}$ diverges to ∞, given g is not a Bowen function on X. By Proposition 3.9, X is not one-sided balanced with respect to g. That is, it is neither left nor right balanced with respect to g.

It is known that a subshift X has a Gibbs measure for $f = 0 \in C(X,\mathbb{R})$ if and only if it has the right balanced property with respect to $f = 0$ (Theorem 3.14 in [1]). However, we can not guarantee that the constructed Gibbs measure in the proof of the theorem is invariant (Theorem 3.7 and Theorem 3.14). So we investigate a necessary and sufficient condition for the existence of invariant Gibbs measures. To obtain a necessary and sufficient condition for the existence of invariant Gibbs measures, we need the following lemmas. The lemmas are extension of results in [1].

Lemma 3.11. *Let X be a subshift and $f \in C(X, \mathbb{R})$. Then X is BSM(f) if and only if there exist $c \geq 1$ and $P \in \mathbb{R}$ such that for each $n \in \mathbb{N}$*

$$c^{-1} \leq \frac{\exp nP}{\sum_{u \in \mathcal{B}_n(X)} \exp S_n f(x_u)} \leq c$$

for each choice of $x_u \in [u]$.

If such P exists, then P must be equal to the topological pressure $P(f)$ of f.

Proof. First we show if part. Observe that

$$c^{-2} \leq \frac{\exp (m+n)P}{\left(\sum_{u \in \mathcal{B}_m(X)} \exp S_m f(x_u)\right)\left(\sum_{v \in \mathcal{B}_n(X)} \exp S_n f(x_v)\right)} \leq c^2.$$

Since $c^{-1} \exp (m+n)P \leq \sum_{w \in \mathcal{B}_{m+n}(X)} \exp S_{m+n} f(x_w) \leq c \exp (m+n)P$, we get

$$c^{-2} \leq \frac{c \sum_{w \in \mathcal{B}_{m+n}(X)} \exp S_{m+n} f(x_w)}{\left(\sum_{u \in \mathcal{B}_m(X)} \exp S_m f(x_u)\right)\left(\sum_{v \in \mathcal{B}_n(X)} \exp S_n f(x_v)\right)}$$

and

$$\frac{c^{-1} \sum_{w \in \mathcal{B}_{m+n}(X)} \exp S_{m+n} f(x_w)}{\left(\sum_{u \in \mathcal{B}_m(X)} \exp S_m f(x_u)\right)\left(\sum_{v \in \mathcal{B}_n(X)} \exp S_n f(x_v)\right)} \leq c^2.$$

Then we have

$$c^{-3} \leq \frac{\sum_{w \in \mathcal{B}_{m+n}(X)} \exp S_{m+n} f(x_w)}{\left(\sum_{u \in \mathcal{B}_m(X)} \exp S_m f(x_u)\right)\left(\sum_{v \in \mathcal{B}_n(X)} \exp S_n f(x_v)\right)} \leq c^3.$$

Therefore X is BSM(f).

Let m and n be two positive integers. Then take $\hat{x}_u \in [u]$ for each $u \in \mathcal{B}(X)$. Then there exists $d \geq 1$ such that

$$d^{-1} \leq \frac{\left(\sum_{u \in \mathcal{B}_n(X)} \exp S_n f(x_u)\right)\left(\sum_{v \in \mathcal{B}_m(X)} \exp S_m f(\hat{x}_v)\right)}{\sum_{w \in \mathcal{B}_{m+n}(X)} \exp S_{m+n} f(\hat{x}_w)} \leq d \tag{2}$$

and

$$d^{-1} \leq \frac{\left(\sum_{u \in \mathcal{B}_n(X)} \exp S_n f(y_u)\right)\left(\sum_{v \in \mathcal{B}_m(X)} \exp S_m f(\hat{x}_v)\right)}{\sum_{w \in \mathcal{B}_{m+n}(X)} \exp S_{m+n} f(\hat{x}_w)} \leq d. \tag{3}$$

By (2) and (3), we have

$$d^{-2} \leq \frac{\sum_{u \in \mathcal{B}_n(X)} \exp S_n f(x_u)}{\sum_{u \in \mathcal{B}_n(X)} \exp S_n f(y_u)} \leq d^2. \tag{4}$$

Let $p_n = \sum_{u\in\mathcal{B}_n(X)} \exp\left(\sup_{x\in[u]} S_n f(x_u)\right)$ for each $n \in \mathbb{N}$ and let P be the topological pressure of f. Since $p_{m+n} \le p_m p_n$ for $m, n \in \mathbb{N}$, P is equal to $\inf_{n\in\mathbb{N}} \frac{1}{n}\log p_n$. By (4)

$$P \le \frac{1}{n}\log p_n \le \frac{1}{n}\log\left(d^2 \sum_{u\in\mathcal{B}_n(X)} \exp S_n f(x_u)\right).$$

It follows that

$$\exp nP \le d^2 \sum_{u\in\mathcal{B}_n(X)} \exp S_n f(x_u).$$

To obtain a lower bound of $\frac{\exp nP}{\sum_{u\in\mathcal{B}_n(X)} \exp S_n f(x_u)}$, consider $p'_n = d^{-1}p_n$. Since X is BSM(f), we have $p'_{m+n} \ge p'_m p'_n$. Thus

$$P = \lim_{n\to\infty} \frac{1}{n}\log p'_n = \sup_{n\in\mathbb{N}} \frac{1}{n}\log p'_n.$$

It follows that

$$\frac{1}{n}\log\left(d^{-1} \sum_{u\in\mathcal{B}_n(X)} \exp S_n f(x_u)\right) \le \frac{1}{n}\log p'_n \le P.$$

Therefore we obtain

$$d^{-1} \sum_{u\in\mathcal{B}_n(X)} \exp S_n f(x_u) \le \exp nP.$$

□

Lemma 3.12. *Let X be a subshift and $f \in C(X,\mathbb{R})$. If μ is a Gibbs measure for f, then X is BSM(f).*

Proof. Since μ is a Gibbs measure for f, there exists $c \ge 1$ such that whenever $n \in \mathbb{N}$ and $u \in \mathcal{B}_n(X)$, we have

$$c^{-1} \le \frac{\mu([u]) \exp nP}{\exp S_n f(x_u)} \le c.$$

It is equivalent to

$$c^{-1} \exp S_n f(x_u) \le \mu([u]) \exp nP \le c \exp S_n f(x_u).$$

By summing over all $u \in \mathcal{B}_n(X)$, we obtain

$$c^{-1} \sum_{u\in\mathcal{B}_n(X)} \exp S_n f(x_u) \le \exp nP \le c \sum_{u\in\mathcal{B}_n(X)} \exp S_n f(x_u).$$

By Lemma 3.11, X is BSM(f). □

Lemma 3.13. *Let X be a subshift and $f \in C(X,\mathbb{R})$. If X is one-sided balanced with respect to f, then X is BSM(f).*

Proof. Suppose X is right balanced with respect to f. Then there exists $c > 0$ such that for each $m, n \in \mathbb{N}$ and $u \in \mathcal{B}_m(X)$

$$c^{-1} \leq \frac{\sum_{v \in F_n(u)} \exp S_{m+n} f(x_{uv})}{\exp S_m f(x_u) \sum_{w \in \mathcal{B}_n(X)} \exp S_n f(x_w)} \leq c$$

for each choice of $x_u \in [u]$, $x_{uv} \in [uv]$, and $x_w \in [w]$. Hence we have

$$\begin{aligned} & c^{-1} \sum_{u \in \mathcal{B}_m(X)} \left[\exp S_m f(x_u) \sum_{v \in \mathcal{B}_n(X)} \exp S_n f(x_v) \right] \\ \leq & \sum_{u \in \mathcal{B}_m(X)} \left[\sum_{v \in F_n(u)} \exp S_{m+n} f(x_{uv}) \right] \\ \leq & \; c \sum_{u \in \mathcal{B}_m(X)} \left[\exp S_m f(x_u) \sum_{v \in \mathcal{B}_n(X)} \exp S_n f(x_v) \right]. \end{aligned}$$

It follows that

$$c^{-1} \leq \frac{\left(\sum_{u \in \mathcal{B}_m(X)} \exp S_m f(x_u) \right) \left(\sum_{v \in \mathcal{B}_n(X)} \exp S_n f(x_v) \right)}{\sum_{u \in \mathcal{B}_m(X)} \left[\sum_{v \in F_n(u)} \exp S_{m+n} f(x_{uv}) \right]} \leq c.$$

Since every word $w \in \mathcal{B}_{m+n}(X)$ has a unique decomposition uv such that $w = uv$, $u \in \mathcal{B}_m(X)$, and $v \in \mathcal{B}_n(X)$, we have

$$\sum_{u \in \mathcal{B}_m(X)} \left[\sum_{v \in F_n(u)} \exp S_{m+n} f(x_{uv}) \right] = \sum_{w \in \mathcal{B}_{m+n}(X)} \exp S_{m+n} f(x_w).$$

Hence we get

$$c^{-1} \leq \frac{\left(\sum_{u \in \mathcal{B}_m(X)} \exp S_m f(x_u) \right) \left(\sum_{v \in \mathcal{B}_n(X)} \exp S_n f(x_v) \right)}{\sum_{w \in \mathcal{B}_{m+n}(X)} \exp S_{m+n} f(x_w)} \leq c.$$

For the case where X is left balanced with respect to f, we obtain the same result by similar calculations. □

Theorem 3.14. *A subshift X has an invariant Gibbs measure μ for $f \in C(X, \mathbb{R})$ if and only if it is balanced with respect to f.*

Proof. First we show the only if part. Let c be the constant in Definition 3.1 and let u be an m-word of X. Then, for $n \in \mathbb{N}$

$$\frac{c^{-1} \exp S_m f(x_u)}{\exp mP} \leq \mu([u]) = \sum_{v \in F_n(u)} \mu([uv]) \leq \frac{c \sum_{v \in F_n(u)} \exp S_{m+n} f(x_{uv})}{\exp (m+n)P}$$

and

$$\frac{c^{-1}\sum_{v\in F_n(u)}\exp S_{m+n}f(x_{uv})}{\exp{(m+n)P}} \le \sum_{v\in F_n(u)} \mu\left([uv]\right) = \mu\left([u]\right) \le \frac{c\exp S_m f(x_u)}{\exp mP}.$$

It follows from the fact that the exponential function is positive that

$$c^{-2} \le \frac{\sum_{v\in F_n(u)}\exp S_{m+n}f(x_{uv})}{\exp S_m f(x_u)\exp nP}$$

and

$$\frac{\sum_{v\in F_n(u)}\exp S_{m+n}f(x_{uv})}{\exp S_m f(x_u)\exp nP} \le c^2.$$

By Lemma 3.11, and Lemma 3.12, there exists $d \ge 1$ such that

$$c^{-2} \le \frac{\sum_{v\in F_n(u)}\exp S_{m+n}f(x_{uv})}{\exp S_m f(x_u)\exp nP} \le \frac{\sum_{v\in F_n(u)}\exp S_{m+n}f(x_{uv})}{d^{-1}\exp S_m f(x_u)\sum_{w\in\mathcal{B}_n(X)}\exp S_n f(x_w)}$$

and

$$\frac{\sum_{v\in F_n(u)}\exp S_{m+n}f(x_{uv})}{d\exp S_m f(x_u)\sum_{w\in\mathcal{B}_n(X)}\exp S_n f(x_w)} \le \frac{\sum_{v\in F_n(u)}\exp S_{m+n}f(x_{uv})}{\exp S_m f(x_u)\exp nP} \le c^2.$$

Thus X is right balanced with respect to f. Since μ is invariant, we have

$$\sum_{v\in P_n(u)} \mu([vu]) = \mu([u])$$

for $m, n \in \mathbb{N}$ and $u \in \mathcal{B}_m(X)$. By similar calculations, we can show that X is left balanced with respect to f. Therefore X is balanced with respect to f.

Conversely, suppose that X is balanced with respect to f. Then it is BSM(f) by Lemma 3.13. Let c be a constant satisfying Definition 3.4, Definition 3.5, and Lemma 3.11. For each $n \in \mathbb{N}$ and $u \in \mathcal{B}_n(X)$, fix a point $\hat{x}_u \in [u]$ and define a measure

$$\nu_n = \frac{1}{\sum_{u\in\mathcal{B}_n(X)}\exp S_n f(\hat{x}_u)} \sum_{u\in\mathcal{B}_n(X)} \exp S_n f(\hat{x}_u)\delta_{\hat{x}_u},$$

where $\delta_{\hat{x}_u}$ is the point measure concentrated at the point $\hat{x}_u$.

Let $m \in \mathbb{N}$ and let $u \in \mathcal{B}_m(X)$. If $n \ge m$, $0 \le k \le n-m$, and $l = n-m-k$, then we note that

$$\begin{aligned}\nu_n\left(\sigma^{-k}([u])\right) &= \sum_{\substack{vuw\in\mathcal{B}_n(X)\\ |v|=k}} \nu_n(\hat{x}_{vuw}) \\ &= \frac{\sum_{v\in P_k(u)}\sum_{w\in F_l(vu)}\exp S_n f(\hat{x}_{vuw})}{\sum_{t\in\mathcal{B}_n(X)}\exp S_n f(\hat{x}_t)}.\end{aligned}$$

We claim that

$$c^{-5}\frac{\exp S_m f(x_u)}{\exp mP} \le \nu_n\left(\sigma^{-k}([u])\right) \le c^5\frac{\exp S_m f(x_u)}{\exp mP} \tag{5}$$

for each choice of $x_u \in [u]$. Since X is right balanced with respect to f, we have

$$(6)\qquad \sum_{v\in P_k(u)} \sum_{w\in F_l(vu)} \exp S_n f(\hat{x}_{vuw}) \\ \leq \sum_{v\in P_k(u)} \left(c\exp S_{m+k} f(x_{vu}) \sum_{s\in\mathcal{B}_l(X)} \exp S_l f(\hat{x}_s) \right).$$

Since X is left balanced with respect to f, we have

$$(7)\qquad \sum_{v\in P_k(u)} \left(c\exp S_{m+k} f(x_{vu}) \sum_{s\in\mathcal{B}_l(X)} \exp S_l f(\hat{x}_s) \right) \\ \leq c^2 \left(\exp S_m f(x_u) \sum_{p\in\mathcal{B}_k(X)} \exp S_k f(\hat{x}_p) \right) \left(\sum_{s\in\mathcal{B}_l(X)} \exp S_l f(\hat{x}_s) \right).$$

By Lemma 3.11 and Lemma 3.13, we have

$$(8)\qquad \sum_{t\in\mathcal{B}_n(X)} \exp S_n f(x_t) \\ \geq c^{-1} \left(\sum_{o\in\mathcal{B}_m(X)} \exp S_m f(x_o) \right) \left(\sum_{p\in\mathcal{B}_{n-m}(X)} \exp S_{n-m} f(\hat{x}_p) \right) \\ \geq c^{-2} \left(\sum_{o\in\mathcal{B}_m(X)} \exp S_m f(x_o) \right) \left(\sum_{p\in\mathcal{B}_k(X)} \exp S_k f(\hat{x}_p) \right) \left(\sum_{s\in\mathcal{B}_l(X)} \exp S_l f(\hat{x}_s) \right) \\ \geq c^{-3} \exp mP \left(\sum_{p\in\mathcal{B}_k(X)} \exp S_k f(\hat{x}_p) \right) \left(\sum_{s\in\mathcal{B}_l(X)} \exp S_l f(\hat{x}_s) \right).$$

By (6), (7), and (8), we have

$$\nu_n\left(\sigma^{-k}([u])\right) \leq \frac{c^2 \left(\exp S_m f(x_u) \sum_{p\in\mathcal{B}_k(X)} \exp S_k f(\hat{x}_p)\right) \left(\sum_{s\in\mathcal{B}_l(X)} \exp S_l f(\hat{x}_s)\right)}{c^{-3} \exp mP \left(\sum_{p\in\mathcal{B}_k(X)} \exp S_k f(\hat{x}_p)\right) \left(\sum_{s\in\mathcal{B}_l(X)} \exp S_l f(\hat{x}_s)\right)} \\ = c^5 \frac{\exp S_m f(x_u)}{\exp mP}.$$

By similar calculations, we have

$$c^{-5} \frac{\exp S_m f(x_u)}{\exp mP} \leq \nu_n\left(\sigma^{-k}([u])\right).$$

Thus the claim holds.

Now define new measures $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} \nu_n \circ \sigma^{-i}$ for $n \in \mathbb{N}$. Then there exists a subsequence $\{\mu_{n_j}\}$ of the sequence $\{\mu_n\}$ which converges to an invariant measure. Let μ be the limit of $\{\mu_{n_j}\}$. By (5), for each $u \in \mathcal{B}_m(X)$ and $n \geq m$,

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-m} c^{-5}\frac{\exp S_m f(x_u)}{\exp mP} \leq \mu_n([u]) &= \frac{1}{n}\sum_{i=0}^{n-1}\nu_n\left(\sigma^{-i}([u])\right)\\ &\leq \frac{1}{n}\sum_{i=0}^{n-m} c^5\frac{\exp S_m f(x_u)}{\exp mP} + \frac{1}{n}\sum_{i=n-m+1}^{n-1} 1.\end{aligned}$$

Since $\lim_{j\to\infty}\mu_{n_j}([u]) = \mu([u])$, we obtain

$$\begin{aligned}c^{-5}\frac{\exp S_m f(x_u)}{\exp mP} &= \lim_{j\to\infty}\left(\frac{(n_j-m+1)c^{-5}}{n_j}\frac{\exp S_m f(x_u)}{\exp mP}\right)\\ &\leq \lim_{j\to\infty}\mu_{n_j}([u]) = \mu([u])\\ &\leq \lim_{j\to\infty}\left(\frac{(n_j-m+1)c^5}{n_j}\frac{\exp S_m f(x_u)}{\exp mP} + \frac{m-1}{n_j}\right)\\ &= c^5\frac{\exp S_m f(x_u)}{\exp mP}.\end{aligned}$$

□

If $f = 0$, then we have the following.

Corollary 3.15. *A subshift X is balanced with respect to $f = 0$ if and only if it has a measure of maximal entropy with the Gibbs property.*

By Theorem 3.7 and Theorem 3.14 together with the existence of a Gibbs measure (Theorem 3.14 in [1]), the following holds.

Corollary 3.16. *There exist a subshift X and $f \in C(X, \mathbb{R})$ such that none of the Gibbs measures for f is invariant.*

Acknowledgement. This work is a part of the author's dissertation. The author would like to thank Uijin Jung for encouragement, guidance and support. I would also like to thank Kyewon Koh Park and the referee for the careful reading of my manuscript and many valuable comments which have improved the paper. This research was supported by the Ministry of Education of the Republic of Korea and the National Research Foundation of Korea (NRF-2018R1D1A1B07050803).

References

[1] S. Baker and A. E. Ghenciu, *Dynamical properties of S-gap shifts and other shift spaces*, J. Math. Anal. Appl. **430** (2015), no. 2, 633–647. https://doi.org/10.1016/j.jmaa.2015.04.092

[2] L. Barreira, *A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems*, Ergodic Theory Dynam. Systems **16** (1996), no. 5, 871–927. https://doi.org/10.1017/S0143385700010117

[3] L. Barreira, *Nonadditive thermodynamic formalism: equilibrium and Gibbs measures*, Discrete Contin. Dyn. Syst. **16** (2006), no. 2, 279–305. https://doi.org/10.3934/dcds.2006.16.279

[4] A. Bertrand, *Specification, synchronisation, average length*, in Coding theory and applications (Cachan, 1986), 86–95, Lecture Notes in Comput. Sci., 311, Springer, Berlin, 1988. https://doi.org/10.1007/3-540-19368-5_9

[5] R. Bowen, *Some systems with unique equilibrium states*, Math. Systems Theory **8** (1974/75), no. 3, 193–202. https://doi.org/10.1007/BF01762666

[6] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Springer-Verlag, Berlin Heidelberg, 1975.

[7] V. Climenhaga and D. J. Thompson, *Intrinsic ergodicity beyond specification*: *β-shifts, S-gap shifts, and their factors*, Israel J. Math. **192** (2012), no. 2, 785–817. https://doi.org/10.1007/s11856-012-0052-x

[8] U. Jung, *On the existence of open and bi-continuing codes*, Trans. Amer. Math. Soc. **363** (2011), no. 3, 1399–1417. https://doi.org/10.1090/S0002-9947-2010-05035-4

[9] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge Univ. Press, Cambridge, 1995. https://doi.org/10.1017/CBO9780511626302

[10] O. Sarig, *Existence of Gibbs measures for countable Markov shifts*, Proc. Amer. Math. Soc. **131** (2003), no. 6, 1751–1758. https://doi.org/10.1090/S0002-9939-03-06927-2

[11] O. Sarig, *Thermodynamic formalism for countable Markov shifts*, in Hyperbolic dynamics, fluctuations and large deviations, 81–117, Proc. Sympos. Pure Math., 89, Amer. Math. Soc., Providence, RI, 2015. https://doi.org/10.1090/pspum/089/01485

[12] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, New York, 1982.

[13] P. Walters, *Regularity conditions and Bernoulli properties of equilibrium states and g-measures*, J. London Math. Soc. (2) **71** (2005), no. 2, 379–396. https://doi.org/10.1112/S0024610704006076

Minkyu Kim
Department of Mathematics
Ajou University
Suwon 16499, Korea
Email address: angellucifer@ajou.ac.kr