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ON SIMILARITY AND REDUCING SUBSPACES OF A CLASS OF OPERATOR ON THE DIRICHLET SPACE

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Abstract. Let Y_p be the multiplication operator M_p plus the Volterra operator V_p induced by $p(z)$, where p is a polynomial. Under a mild condition, we prove that Y_p acting on the Dirichlet space $\mathfrak{D}$ is similar to multiplication operator M_p acting on a subspace $S(\mathbb{D})$ of $\mathfrak{D}$. Furthermore, it shows that T_{z^n} $(n \geq 2)$ has exactly 2^n reducing subspaces on $\mathfrak{D}$.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane $\mathbb{C}$, and $H(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$. The Dirichlet space $\mathfrak{D}$ is defined by

$$\mathfrak{D} = \{f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty,\ f(0) = 0\},$$

where $dA(z) = \frac{1}{\pi} dxdy = \frac{r}{\pi} drd\theta$ is the normalized Lebesgue area measure on $\mathbb{D}$. $\mathfrak{D}$ is a reproducing kernel Hilbert space with the norm

$$\|f\|_{\mathfrak{D}}^2 = \int_{\mathbb{D}} |f'(z)|^2 dA(z).$$

Let $f, g \in \mathfrak{D}$, and $f(z) = \sum_{k=1}^{\infty} a_k z^k$, $g(z) = \sum_{k=1}^{\infty} b_k z^k$, then the inner product of f and g is defined by

$$\langle f, g\rangle = \sum_{k=1}^{\infty} k a_k \bar{b}_k.$$

Finding the invariant subspace and reducing subspace of an operator is a fundamental problem in the field of operator theory and operator algebra, and this issue has attracted lots of attention. Since Beurling (see [4,6]) characterized the lattice of the shift operator acting on the Hardy space in the middle of last century, numerous scholars have studied the invariant subspace and reducing

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subspace problem of certain class of operators on the Hardy space and Bergman space (see [1,3], [5,7–11], [13,15], [16], [18–20]). In [2], Aleman characterized the boundedness and compactness of integral operator $T_g f(z) = \int_0^z g'(w)f(w)dw$ between Hardy space H^p and H^q for $p, q > 0$, where the symbol g is an analytic function in $\mathbb{D}$. In [3], using Beurling's theorem, Aleman and Korenblum studied the complex Volterra operator (see [17]) in the Hardy space $H^2(\mathbb{D})$ defined by $(Vf)(z) = \int_0^z f(w)dw$. Then they characterized the lattice of closed invariant subspaces of operator V. Čučković and Paudyal (see [7]) characterized the lattice of closed invariant subspaces of the shift plus complex Volterra operator on the Hardy space. Ball (see [5]) and Nordgren (see [16]) studied the problem of determining the reducing subspaces for an analytic Toeplitz operator on the Hardy space. In [19], Stessin and Zhu gave a complete description of the weighted unilateral shift operator of finite multiplicity on some Hilbert spaces type I and type II. In 2011, Douglas and Kim in [8] studied the reducing subspaces for an analytic multiplication operator M_{z^n} on the Bergman space $A^2_\alpha(A_r)$ of the annulus A_r.

Based on the above works, now we consider the Dirichlet space setting. Let Y_p be the multiplication operator M_p plus the Volterra operator V_p induced by $p(z)$, where p is a polynomial. Because the Dirichlet space involves the derivative of a function, under a mild condition, we prove that Y_p acting on the Dirichlet space $\mathfrak{D}$ is similar to multiplication operator M_p acting on a subspace $S(\mathbb{D})$ of $\mathfrak{D}$. Furthermore, using the techniques in operator theory, we show that T_{z^n} $(n \geq 2)$ has exactly 2^n reducing subspaces on $\mathfrak{D}$.

The main theorem of this paper is as follows.

Theorem 1.1. *Let V_p be the Volterra operator induced by $p(z)$ on $\mathfrak{D}$. Then the following conclusions hold*:

(i) $\operatorname{ran}(V_p) = S(\mathbb{D})$;

(ii) $V_p : \mathfrak{D} \to S(\mathbb{D})$ *is a bounded isomorphism induced by $p(z)$ and*

$$V_p^{-1} = \frac{1}{p'(z)} D;$$

(iii) Y_p *is similar to* M_p.

Theorem 1.2. *For $n \geq 2$, let T_{z^n} be an operator $(T_{z^n} f)(z) = M_{z^n} f(z) + (V_{z^n} f)(z)$ defined on $\mathfrak{D}$. Then T_{z^n} has exactly 2^n reducing subspaces and n minimal reducing subspaces.*

2. The similarity of the operator Y_p

Characterizing the similarity of certain operators can help us understand the structure of the operators better, therefore many scholars have studied the similarity of operators (see [11, 12], [14]). Let $\mathcal{H}$ be a Hilbert space, $A, B \in L(\mathcal{H})$, if there exists a bounded linear invertible operator T such that $T^{-1}AT = B$, then we call A is similar to B.

Let $p(z) = \sum_{k=0}^{n} d_k z^k$, and the coefficients $d_k \in \mathbb{C}$ satisfying

$$|d_1| > \sum_{k=2}^{n} k\,|d_k|\,.$$

By direct computations, we know this condition guarantees that $p'(z)$ does not have zero points in $\overline{\mathbb{D}}$.

Define the Volterra operator V_p induced by $p(z)$ on $\mathfrak{D}$ as

$$(V_p f)(z) = \int_0^z p'(w)f(w)dw.$$

Let M_p be the multiplication operator on $\mathfrak{D}$ such that $M_p f(z) = p(z)f(z)$ for $f(z) \in \mathfrak{D}$. Define the operator Y_p as follows:

$$\begin{aligned}(Y_p f)(z) &= M_p f(z) + (V_p f)(z)\\ &= p(z)f(z) + \int_0^z p'(w)f(w)dw,\ f(z) \in \mathfrak{D}.\end{aligned}$$

In order to characterize the similarity of Y_p, we need to introduce the space $S(\mathbb{D})$ defined as

$$S(\mathbb{D}) = \{h \in H(\mathbb{D}) : h^{(l)}\,(0) = 0\,(l = 0, 1),\ Dh \in \mathfrak{D}\},$$

where D is the differential operator. Set

$$p(z) = \sum_{k=0}^{n} d_k z^k,\ |d_1| > \sum_{k=2}^{n} k\,|d_k|\,. \tag{1}$$

Since p' dose not vanish on the closed unit disk $\overline{\mathbb{D}}$, we then have the following lemma.

Lemma 2.1. *For $h \in S(\mathbb{D})$, $p(z)$ is a polynomial described as* (1)*. Then $\frac{Dh}{p'(z)} \in \mathfrak{D}$.*

Define the norm of $S(\mathbb{D})$ as follows:

$$\|h\|^2_{S(\mathbb{D})} = \|h\|^2_{\mathfrak{D}} + \|Dh\|^2_{\mathfrak{D}}. \tag{2}$$

For any two functions $h_1, h_2 \in S(\mathbb{D})$, the corresponding inner product is given by

$$\langle h_1, h_2\rangle_{S(\mathbb{D})} := \langle h_1, h_2\rangle_{\mathfrak{D}} + \langle Dh_1, Dh_2\rangle_{\mathfrak{D}}.$$

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1. (i) On one hand, for all $f \in \mathfrak{D}$, let

$$h(z) = (V_p f)(z) = \int_0^z p'(w)f(w)dw.$$

Then

$$\int_{\mathbb{D}} |(Dh(z))'|^2 dA(z) = \int_{\mathbb{D}} |(p'(z)f(z))'|^2 dA(z)$$
$$\leq C_1 \|f(z)\|_{\mathfrak{D}}^2,$$

where C_1 is a positive constant. Moreover, $h(0) = 0$ and $h'(0) = p'(z)f(z)|_{z=0} = 0$. Hence $h(z) \in S(\mathbb{D})$.

On the other hand, for every $h \in S(\mathbb{D})$, there exists $f(z) \in \mathfrak{D}$ such that $V_p(f) = h$. In fact, let $f(z) = \frac{Dh}{p'(z)}$, then

$$V_p\left(\frac{Dh}{p'(z)}\right)(z) = \int_0^z p'(w)\frac{Dh}{p'(w)}(w)dw = h(z) - h(0) = h(z).$$

(ii) Firstly, V_p is bounded from $\mathfrak{D}$ to $S(\mathbb{D})$. Let $f \in \mathfrak{D}$, by the equality (2),

$$\|V_p f\|_{S(\mathbb{D})}^2 = \|V_p f\|_{\mathfrak{D}}^2 + \|D(V_p f)\|_{\mathfrak{D}}^2$$
$$\leq 2\|D(V_p f)\|_{\mathfrak{D}}^2$$
$$\leq 2C_1\|f(z)\|_{\mathfrak{D}}^2.$$

Hence V_p is linear bounded.

Next, we show V_p is one to one. Suppose that $f_1, f_2 \in \mathfrak{D}$ satisfying

$$\int_0^z p'(w)f_1(w)dw = \int_0^z p'(w)f_2(w)dw,\ z \in \mathbb{D}.$$

Taking the derivative with respect to w gives $f_1(z) = f_2(z)$. For $h \in S(\mathbb{D})$, from the definition of V_p, we have

$$V_p\left(\frac{1}{p'(z)}Dh\right)(z) = h(z), \tag{3}$$

where $\frac{1}{p'(z)}Dh(z) = f(z) \in \mathfrak{D}$. So V_p is a bijection from $\mathfrak{D}$ onto $S(\mathbb{D})$. And $V_p^{-1} = \frac{1}{p'}D$, as desired.

(iii) For $f \in \mathfrak{D}$, write $h(z) = (V_p f)(z)$. Note that

$$(Y_p f)(z) = (M_p f)(z) + (V_p f)(z)$$
$$= p(z)\frac{h'(z)}{p'(z)} + h(z).$$

Applying V_p acts on the equality above, we have

$$(V_p Y_p f)(z) = \int_0^z p'(w)\left[p(w)\frac{h'(w)}{p'(w)} + h(w)\right]dw$$
$$= \int_0^z D(p(w)h(w))dw$$
$$= p(z)h(z)$$
$$= M_p(V_p f)(z).$$

Hence $V_p Y_p = M_p V_p$. So Y_p is similar to M_p. The proof is complete. □

3. The reducing subspaces of T_{z^n}

Let M be a closed nontrivial subspace of Hilbert space $\mathcal{H}$, and T be a linear bounded operator on $\mathcal{H}$. M is called an invariant subspace for T if $TM \subset M$. If M and $M^{\perp}$ are both invariant subspaces for T, then M is said to be a reducing subspace for T. If M does not contain any reducing subspace $N \neq \{0\}$, then M is called a minimal reducing subspace for T. All bounded linear operators that can commute with T is called the commutator of T, and we denote it by $\mathcal{A}'(T)$, i.e., $\mathcal{A}'(T) := \{S : ST = TS \text{ and } S \in L(\mathcal{H})\}$. An operator P is a projection if P is idempotent and self-adjoint. The reducing subspace for T is determined by the projection operator in the commutant algebra of T (see [8]). Suppose that P_M is a projection onto M, then M is a reducing subspace for T if and only if $P_M T = T P_M$.

In the following, we will consider the operator T_{z^n} $(n \geq 2)$ and characterize its reducing subspace. T_{z^n} is defined as follows:

$$\begin{aligned}(T_{z^n} f)(z) &= M_{z^n} f(z) + (V_{z^n} f)(z) \\ &= z^n f(z) + \int_0^z n w^{n-1} f(w) dw,\ f \in \mathfrak{D}.\end{aligned}$$

Let $\{e_k(z) = \frac{z^k}{\sqrt{k}} = \gamma_k z^k\}_{k=1}^{\infty}$ be the orthonormal basis of the Dirichlet space $\mathfrak{D}$. In this paper, $\text{Span}\{v_i, i \in I\}$ always means the closed linear span of $\{v_i, i \in I\}$. We have the following lemma.

Lemma 3.1. *Let $H_j = \text{Span}\{e_{(k-1)n+j} : k \geq 1\}$ $(j = 1, 2, \ldots, n)$. Then H_j is a reducing subspace of T_{z^n}.*

Proof. Since

$$\begin{aligned}&\langle e_{(k-1)n+j}, e_{(m-1)n+j}\rangle \\ &= \int_{\mathbb{D}} \frac{(k-1)n+j}{\sqrt{(k-1)n+j}} z^{(k-1)n+j-1} \frac{(m-1)n+j}{\sqrt{(m-1)n+j}} \overline{z}^{(m-1)n+j-1} dA(z),\end{aligned}$$

it is easy to see that

$$\langle e_{(k-1)n+j}, e_{(m-1)n+j}\rangle = \begin{cases} 1, & k = m, \\ 0, & k \neq m. \end{cases}$$

Let $\{e_{(k-1)n+j}\}_{k=1}^{\infty}$, $\{e_{(k-1)n+s}\}_{k=1}^{\infty}$ $(1 \leq j \neq s \leq n)$ be the orthonormal basis of H_j and H_s, respectively. For each $f_j \in H_j$, we have $\langle f_j, e_{(k-1)n+s}\rangle = 0$. So $H_j \perp H_s$ for all $1 \leq j \neq s \leq n$. For each $f \in \mathfrak{D}$,

$$f = \sum_{k=1}^{\infty} a_{1k} e_{(k-1)n+1} + \sum_{k=1}^{\infty} a_{2k} e_{(k-1)n+2} + \cdots + \sum_{k=1}^{\infty} a_{nk} e_{(k-1)n+n}.$$

Suppose that $f = 0$, we then have $\langle \sum_{k=1}^{\infty} \sum_{j=1}^{n} a_{jk} e_{nk+j}, e_l \rangle = 0$ $(l = 1, 2, \dots)$. Therefore, $a_{jk} = 0$ $(j = 1, 2, \dots, n; k = 1, 2, \dots)$, i.e.,

$$0 = \underbrace{0 \bigoplus 0 \bigoplus \cdots \bigoplus 0}_{n}.$$

Hence

$$\mathfrak{D} = H_1 \bigoplus H_2 \bigoplus \cdots \bigoplus H_n.$$

Since

$$\begin{aligned} T_{z^n} e_{(k-1)n+j}(z) &= M_{z^n} e_{(k-1)n+j}(z) + V_{z^n} e_{(k-1)n+j}(z) \\ &= z^n \frac{z^{(k-1)n+j}}{\gamma_{(k-1)n+j}} + \int_0^z n w^{n-1} \frac{w^{(k-1)n+j}}{\gamma_{(k-1)n+j}} dw \\ &= \left(1 + \frac{n}{kn+j}\right) \frac{\gamma_{kn+j}}{\gamma_{(k-1)n+j}} e_{kn+j}, \end{aligned} \tag{4}$$

we have $T_{z^n} H_j \subset H_j$, and H_j is an invariant subspace of T_{z^n}. Moreover,

$$\begin{aligned} &T_{z^n} \left(H_1 \bigoplus H_2 \bigoplus \cdots \bigoplus H_{j-1} \bigoplus H_{j+1} \bigoplus \cdots \bigoplus H_n\right) \\ &\subset H_1 \bigoplus H_2 \bigoplus \cdots \bigoplus H_{j-1} \bigoplus H_{j+1} \bigoplus \cdots \bigoplus H_n, \end{aligned}$$

so H_j is a reducing subspace of T_{z^n}. □

From the proof of Lemma 3.1, we have the following lemma.

Lemma 3.2. *For $k \geq 0$, we have*

$$T_{z^n} e_{kn+j} = \delta_{k,j} e_{(k+1)n+j}, \ j = 1, 2, \dots, n,$$

where

$$\delta_{k,j} = \left(1 + \frac{n}{(k+1)n+j}\right) \frac{\sqrt{(k+1)n+j}}{\sqrt{kn+j}}, \ j = 1, 2, \dots, n.$$

Next we will show that T_{z^n} is unitarily equivalent to a weighted shift with matrix weights. For this purpose, we briefly introduce weighted shifts.

Let E be a complex Hilbert space. Let $l^2(E)$ be the E-valued l^2 space such that

$$l^2(E) = \left\{ y = (y_0, y_1, \dots) : \|y\|^2 = \sum_{i=0}^{\infty} \|y_i\|^2 < \infty \right\}.$$

Let $e_i = (0, \dots, 0, 1, 0, \dots)$, where the 1 is in the i-th place. Then we write

$$y = (y_0, y_1, \dots) = \sum_{i=0}^{\infty} y_i e_i.$$

We identify E as a subspace of $l^2(E)$ by mapping y to ye_0 for $y \in E$. By an abuse of notation, we just write y instead of ye_0 for $y \in E$. Let $\Phi =$

$\{\Phi_i, i \geq 0\} \subset B(E)$ be a sequence of invertible operators. The weighted shift S_Φ with weight Φ is an operator on $l^2(E)$ defined by

$$S_\Phi y e_i = [\Phi_i y]\, e_{i+1},\ i \geq 0,\ y \in E.$$

It follows that $S_\Phi^* y = 0$ for $y \in E$ and

$$S_\Phi^* y e_{i+1} = [\Phi_i^* y]\, e_i,\ i \geq 0,\ y \in E.$$

Thus $\ker(S_\Phi^*) = E$. Furthermore, S_Φ is a bounded operator if and only if $\sup_{i\geq 0} \|\Phi_i\| < \infty$ and $\|S_\Phi\| = \sup_{i\geq 0} \|\Phi_i\|$.

Let

$$W_k = \Phi_k \cdots \Phi_1 \Phi_0,\ k \geq 0,\ W_{-1} = I_E.$$

Note that $\Phi_k = W_k W_{k-1}^{-1}$.

The reducing subspaces of S_Φ as above is described by the following theorem from [9].

Theorem 3.3 ([9]). *A closed subspace X is a reducing subspace of S_Φ if and only if*

$$X = \operatorname{Span}\left\{S_\Phi^k x,\ k \geq 0,\ x \in E_0\right\}, \tag{5}$$

where $E_0 \subseteq E$ is an invariant subspace of the sequence of operators

$$\Omega = \left\{W_{k-2}^{-1} \Phi_{k-1}^* \Phi_{k-1} W_{k-2},\ k \geq 1\right\}.$$

Furthermore, X is a minimal reducing subspace of S_Φ if and only if E_0 is a minimal invariant subspace of Ω.

The following lemma from [9] will also be useful for us.

Lemma 3.4. *Let Ω be a set of invertible diagonal matrices on $\mathbb{C}^N$ with respect to an orthonormal basis $\{e_1, \ldots, e_N\}$. The following two statements are equivalent*:

(i) *For any $i \neq j$, there is $A \in \Omega$ such that $Ae_i = \lambda_i e_i$, $Ae_j = \lambda_j e_j$ with $\lambda_i \neq \lambda_j$.*

(ii) *There are exactly N minimal invariant subspaces of Ω. Namely,*

$$\operatorname{Span}\{e_i\} \quad \textit{for } i = 1, \ldots, N.$$

Lemma 3.5. *T_{z^n} is unitarily equivalent to S_Φ on $l^2(E)$, where Φ_k is a diagonal matrix with diagonal $\{\delta_{k,j},\ j = 1, 2, \ldots, n\}$.*

Proof. For $f \in \mathfrak{D}$, we can write

$$f = \sum_{i=1}^{\infty} f_i e_i = \sum_{k=0}^{\infty} \sum_{j=1}^{n} f_{kn+j} e_{kn+j}.$$

Thus we can view $\mathfrak{D}$ as $l^2(E)$ with $\dim E = n$. By Lemma 3.2,

$$T_{z^n} \begin{bmatrix} e_{1+nk} \\ \vdots \\ e_{n+nk} \end{bmatrix} = \begin{bmatrix} \delta_{k,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_{k,n} \end{bmatrix} \begin{bmatrix} e_{1+n(k+1)} \\ \vdots \\ e_{n+n(k+1)} \end{bmatrix}.$$

Thus T_{z^n} is unitarily equivalent to S_Φ. □

We now are in the position to prove Theorem 1.2.

Proof of Theorem 1.2. Lemma 3.5 shows that T_{z^n} is a weighted shift with weight $\{\Phi_k,\, k \geq 0\}$ being the diagonal matrix with diagonal $\{\delta_{k,j},\, j = 1, 2, \ldots, n\}$. Then by Theorem 3.3, the minimal reducing subspaces are determined by the common minimal invariant subspace of Ω. Since Φ_k is a diagonal matrix for each $k \geq 0$, we have

$$\Omega = \left\{\Phi_k^2 : k \geq 0\right\}.$$

Note that Φ_0 is a diagonal matrix with diagonals

$$\left(1 + \frac{n}{n+j}\right)\frac{\sqrt{n+j}}{\sqrt{j}},\ j = 1, 2, \ldots, n.$$

It is easy to see that the above n diagonals are distinct. By Lemma 3.4, Ω has n minimal invariant subspaces. By Theorem 3.3, T_{z^n} has n minimal reducing subspaces. Namely,

$$\begin{aligned} M_j &= \mathrm{Span}\{e_{j+nk} : k \geq 0\} \\ &= \mathrm{Span}\{z^{j+nk} : k \geq 0\}\ (j = 1, 2, \ldots, n) \end{aligned}$$

are the n minimal reducing subspaces. Furthermore, T_{z^n} has exactly 2^n reducing subspaces from direct sums of several minimal reducing subspaces. The proof is complete. □

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