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CONVERGENCE OF THE EULER-MARUYAMA METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY G-BROWNIAN MOTION

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Abstract. In this paper, we deal with the Euler-Maruyama (EM) scheme for stochastic differential equations driven by G-Brownian motion (G-SDEs). Under the linear growth and the local Lipschitz conditions, the strong convergence as well as the rate of convergence of the EM numerical solution to the exact solution for G-SDEs are established.

1. Introduction

In the last decade, the fundamental theory of G-expectation, G-Brownian motion and its related stochastic calculus were established in Peng [18–20], due to its potential applications in uncertain problems, risk measures as well as the super-hedging in finance. Since then, a lot of works have been devoted to the study of G-expectation and G-Brownian motion, one can see Hu and Peng [9], Denis et al. [6], Li and Peng [14], Soner et al. [21], Song [22, 23] and the references therein. Under the G-framework, stochastic differential equations driven by G-Brownian motion (G-SDEs) were introduced in Peng [20]. The solvability of G-SDEs has been obtained in Peng [20] and Gao [7] under the Lipschitz assumptions on the coefficients. Since the global Lipschitz condition is somewhat restrict in applications, some non-Lipschitz conditions were introduced. For instance, Bai and Lin [1] proposed the existence and uniqueness of solutions for G-SDEs under the integral-Lipschitz conditions, Li et al. [12] showed the existence and uniqueness result under a locally Lipschitz condition and a Lyapunov-type condition on the coefficients.

It is known that the explicit solutions for SDEs or G-SDEs are often difficult to obtain, so as an alternative, the numerical solutions are considered naturally. For the classical framework, a lot of literature has been focused

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on the convergence of the EM schemes for SDEs under reasonable conditions, see for example, Kloeden and Platen [11] and Mao [16] for SDEs with global Lipschitz condition, Higham et al. [8] for SDEs with local Lipschitz condition and the linear growth condition. For SDEs without the linear growth condition, Hutzenthaler et al. [10] developed the tamed EM schemes, and Mao [17] developed the truncated EM schemes. Recently, Li, Mao, Yin [13] presented an approximation technique to study truncated EM schemes for SDEs in finite or infinite horizons. Wu et al. [25] showed the convergence of the numerical solutions for pantograph stochastic functional differential equations. On the other hand, the convergence rates of numerical approximations were studied extensively, see Higham et al. [8], Yuan and Mao [27], Bao and Yuan [4], Bao et al. [3], Bao and Huang [2] and the references therein.

Accordingly, so far as we know, there are very few works on the numerical solution for G-SDEs. Yang and Zhao [26] proposed the numerical simulations of the G-Brownian motion B and the related quadratic variation process $\langle B\rangle$. Ullah and Faizullah [24] showed the strong convergence of the EM approximate solutions for G-SDEs under the linear growth and the global Lipschitz assumptions on the coefficients. Li and Yang [15] proved the convergence of the stochastic theta solution and the asymptotical stability of the backward EM numerical simulation for neutral SDEs in the G-framework. Deng et al. [5] showed the stability equivalence between the G-SDE with delay and the corresponding Euler-Maruyama numerical method.

Motivated by the aforementioned works, we aim to investigate the EM scheme for G-SDEs. With the local Lipschitz and the linear growth assumptions, our first goal of this paper is to study the strong convergence of the EM numerical solution for G-SDEs, and then, it allows us to derive the rate of convergence. It should be pointed out that our results are obtained by a technique similar to that in Higham et al. [8] and Yuan and Mao [27], but the model is nontrivial due to the uncertainty of G-Brownian motion. Moreover, we hope that the current discussion will play a fundamental role for more in-depth study of the numerical solutions for G-SDEs.

The rest of the paper is organized as follows. In Section 2, we propose some notations and preliminaries. In Section 3, we discuss the strong convergence of the EM numerical solution for G-SDEs. In Section 4, the rate of convergence is provided.

2. Preliminaries

In this section, we propose some basic notations and results in the framework of G-expectation, the readers are referred to Peng [18], Denis et al. [6] and Gao [7] for more details. In the sequel, $\mathbb{R}^n$ represents the real n-dimensional space and for $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Let Ω be the space of all $\mathbb{R}^d$-valued continuous paths with $\omega_0 = 0$ equipped with the distance $\rho\left(\omega^1, \omega^2\right) :=$

$\sum_{N=1}^{\infty} 2^{-N}\left(\left(\max_{t\in[0,N]}\left|\omega_t^1-\omega_t^2\right|\right)\wedge 1\right)$ and let $\mathcal{B}(\Omega)$ be the Borel σ-algebra of Ω.

For $t\in[0,\infty)$, we list the following notations:

- $B_t(\omega):=\omega_t$ be the canonical process;
- $\Omega_t=\{\omega_{\cdot\wedge t}:\omega\in\Omega\}$, $\mathcal{F}_t:=\mathcal{B}(\Omega_t)$;
- $L^0(\Omega):$ the space of all $\mathcal{B}(\Omega)$-measurable real functions;
- $L^0(\Omega_t):$ the space of all $\mathcal{B}(\Omega_t)$-measurable real functions;
- $B_b(\Omega):$ all bounded elements in $L^0(\Omega)$; $B_b(\Omega_t):=B_b(\Omega)\cap L^0(\Omega_t)$;
- $C_b(\Omega):$ all continuous elements in $B_b(\Omega)$; $C_b(\Omega_t):=C_b(\Omega)\cap L^0(\Omega_t)$;
- $L_{ip}\left(\mathbb{R}^{d\times n}\right):$ the collection of all bounded and Lipschitz functions on $\mathbb{R}^{d\times n}$;
- $L_{ip}(\Omega):=\left\{\varphi(B_{t_1},\ldots,B_{t_n}):n\geq 1, 0\leq t_1<\cdots<t_n<\infty,\varphi\in L_{ip}(\mathbb{R}^{d\times n})\right\}$;
- $L_{ip}(\Omega_t):=L_{ip}(\Omega)\cap L^0(\Omega_t)$.

For each $p\geq 1$, we denote by $L_G^p(\Omega)$ the completion of $L_{ip}(\Omega)$ under the norm $\|\cdot\|_p:=\mathbb{E}\left[|\cdot|^p\right]^{\frac{1}{p}}$, where $\mathbb{E}(\cdot)$ denotes the related G-expectation on $(\Omega,L_{ip}(\Omega))$. Similarly, we can define $L_G^p(\Omega_T)$ for each $0\leq T<\infty$. In Denis et al. [6], they derived that there exists a weakly compact set $\mathcal{P}$ of probability measures defined on $(\Omega,\mathcal{B}(\Omega))$ such that

$$\mathbb{E}[Y]=\sup_{P\in\mathcal{P}} E_P[Y]\quad \text{for all } Y\in L_G^1(\Omega),$$

where E_P is the linear expectation with respect to probability measure P. For this $\mathcal{P}$, the associated capacity is defined by $\mathbb{C}(A):=\sup_{P\in\mathcal{P}}P(A)$, $A\in\mathcal{B}(\Omega)$.

Definition 1. A set $A\in\mathcal{B}(\Omega)$ is called polar if $\mathbb{C}(A)=0$. A property is said to hold quasi surely (q.s., in short) if it holds outside a polar set.

We define

$$M_G^{p,0}([0,T]):=\{\eta_t=\sum_{i=0}^{N-1}\xi_i\mathbf{1}_{[t_i,t_{i+1})}(t):\forall N\in\mathbb{N},0=t_0<\cdots<t_N=T,\\ \xi_i\in L_G^p(\Omega_{t_i})\}.$$

We denote by $M_G^p([0,T])$ the completion of $M_G^{p,0}([0,T])$ under the norm:

$$\|\eta\|_{M_p}:=\left(\mathbb{E}\left[\frac{1}{T}\int_0^T|\eta_t|^p\,dt\right]\right)^{1/p}.$$

In this paper we consider the following n-dimensional stochastic differential equation in the G-framework

$$dx(t)=f(x(t))dt+g(x(t))d\langle B\rangle_t+h(x(t))dB_t,\ t\in[0,T] \tag{1}$$

with given initial condition $x(0)=x_0\in\mathbb{R}^n$, where B_t is a one-dimensional G-Brownian motion under the G-expectation space $(\Omega,L_{ip}(\Omega),\mathbb{E}(\cdot))$ with $G(a):=\frac{1}{2}\mathbb{E}[aB_1^2]=\frac{1}{2}\left(\bar{\sigma}^2a^+-\underline{\sigma}^2a^-\right)$ for $a\in\mathbb{R}$, where $\bar{\sigma}^2=\mathbb{E}[B_1^2]$, $\underline{\sigma}^2=-\mathbb{E}[-B_1^2]$, $0\leq\underline{\sigma}\leq\bar{\sigma}<\infty$. The quadratic variation process of G-Brownian motion B_t is denoted by $\langle B\rangle_t$. A process $x(t) \in M_G^2([0,T];\mathbb{R}^n)$ with t-continuous path and satisfying the G-SDE (1) is said to be its solution.

We impose the following conditions on the coefficients f, g and h:

(**A1**) Assume that $f, g, h : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the local Lipschitz condition: for each $R = 1, 2, \ldots$, there exist positive constants $L_1(R)$, $L_2(R)$ and $L_3(R)$ such that

$$|f(x) - f(y)| \leq L_1(R)|x - y|, \ |g(x) - g(y)| \leq L_2(R)|x - y|$$

and

$$|h(x) - h(y)| \leq L_3(R)|x - y|$$

for all $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \leq R$.

(**A2**) Assume that f, g, h satisfy the linear growth condition: there is a constant $u > 0$ such that

$$|f(x)| \vee |g(x)| \vee |h(x)| \leq u(1 + |x|)$$

for all $x \in \mathbb{R}^n$.

Remark 2.1. Under assumptions (**A1**) and (**A2**), the G-SDE (1) has a unique solution (see [12]).

The following Burkholder-Davis-Gundy type inequalities in the G-framework are borrowed from [7].

Lemma 2.2. *For each* $p \geq 1$, $\eta \in M_G^p([0,T])$ *and* $0 \leq t \leq T$, *we have*

$$\mathbb{E}\left[\sup_{0\leq r\leq t}\left|\int_0^r \eta_s d\langle B\rangle_s\right|^p\right] \leq \bar{\sigma}^{2p} t^{p-1} \int_0^t \mathbb{E}\left[|\eta_s|^p\right] ds.$$

Lemma 2.3. *Let* $p \geq 2$, $\eta \in M_G^p([0,T])$ *and* $0 \leq t \leq T$. *Then,*

$$\mathbb{E}\left[\sup_{0\leq r\leq t}\left|\int_0^r \eta_s dB_s\right|^p\right] \leq C(p)\bar{\sigma}^p t^{\frac{p}{2}-1} \int_0^t \mathbb{E}\left[|\eta_s|^p\right] ds,$$

where $C(p)$ *is a positive constant independent of* η.

3. Strong convergence

In this section, we prove that the strong convergence of the EM approximate solution for the G-SDE (1) under the local Lipschitz and the linear growth conditions.

We now set up the EM approximate solution for the G-SDE (1). Given a stepsize $0 < \Delta < 1$, let $t_k = k\Delta$ for $k \geq 0$. Then the discrete EM approximate solution $y_k\,(\approx x(t_k))$ for the G-SDE (1) is defined by

$$y_{k+1} = y_k + f\left(y_k\right)\Delta + g\left(y_k\right)\Delta\langle B\rangle_k + h\left(y_k\right)\Delta B_k, \ k \geq 0, \ y_0 = x_0,$$

where $\Delta\langle B\rangle_k = \Delta\langle B\rangle_{t_{k+1}} - \Delta\langle B\rangle_{t_k}$, $\Delta B_k = B_{t_{k+1}} - B_{t_k}$. We extend the discrete solution to the continuous one by

$$y(t) = y_0 + \int_0^t f(\bar{y}(s))ds + \int_0^t g(\bar{y}(s))d\langle B\rangle_s + \int_0^t h(\bar{y}(s))dB_s, \ t \in [0,T],$$

where $\bar{y}(t)$ is defined by $\bar{y}(t) := y_k$ for $t \in [t_k, t_{k+1})$, it is obvious that $y(t_k) = \bar{y}(t_k) = y_k$.

Lemma 3.1. *Under assumptions* (**A1**) *and* (**A2**), *for any given* $p \geq 2$, *there is a constant* $K(p) := K(p, T, y_0, u, \bar{\sigma})$ *such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T} |x(t)|^p\right] \leq K(p).$$

Proof. By the Hölder inequality, we have

$$|y(t)|^p \leq 4^{p-1}\left[|x_0|^p + T^{p-1}\int_0^t |f(\bar{y}(s))|^p ds + \left|\int_0^t g(\bar{y}(s))d\langle B\rangle_s\right|^p + \left|\int_0^t h(\bar{y}(s))dB_s\right|^p\right].$$

Taking G-expectation on both sides yields that

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T}|y(t)|^p\right] \leq 4^{p-1}\Bigg[&|x_0|^p + T^{p-1}\int_0^T \mathbb{E}|f(\bar{y}(s))|^p ds \\ &+ \mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t g(\bar{y}(s))d\langle B\rangle_s\right|^p\right) \\ &+ \mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t h(\bar{y}(s))dB_s\right|^p\right)\Bigg].\end{aligned} \tag{2}$$

By Lemmas 2.2 and 2.3, we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t g(\bar{y}(s))d\langle B\rangle_s\right|^p\right] \leq \bar{\sigma}^{2p}T^{p-1}\int_0^T \mathbb{E}|g(\bar{y}(s))|^p ds$$

and

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t h(\bar{y}(s))dB_s\right|^p\right] &\leq \bar{\sigma}^p C(p)\mathbb{E}\left[\int_0^T |h(\bar{y}(s))|^2 ds\right]^{p/2} \\ &\leq \bar{\sigma}^p C(p) T^{p/2-1}\int_0^T \mathbb{E}|h(\bar{y}(s))|^p ds.\end{aligned}$$

Substituting this into (2), using the linear growth condition, we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|y(t)|^p\right] \leq 4^{p-1}\left[|y_0|^p + \Gamma_1\int_0^T (1+\mathbb{E}|\bar{y}(s)|^p)ds\right],$$

where $\Gamma_1 := 2^{p-1}u^p(T^{p-1} + \bar{\sigma}^{2p}T^{p-1} + \bar{\sigma}^p C(p)T^{p/2-1})$. Applying the Gronwall inequality we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|y(t)|^p\right] \leq K(p).$$

Similarly, we can show that

$$\mathbb{E}\left[\sup_{0\le t\le T}|x(t)|^p\right]\le K(p).$$

The proof is complete. □

Theorem 3.2. *Let assumptions* (**A1**) *and* (**A2**) *hold. Then, the EM approximate solution for the G-SDE* (1) *satisfies*

$$\lim_{\Delta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}|x(t)-y(t)|^2\right]=0.$$

Proof. First, we define

$$\tau_R:=\inf\{t\ge 0:|y(t)|\ge R\},\ \rho_R:=\inf\{t\ge 0:|x(t)|\ge R\},\ \theta_R:=\tau_R\wedge\rho_R$$

and

$$e(t):=x(t)-y(t).$$

Note that the Young inequality: for $r^{-1}+q^{-1}=1$

$$ab\le\frac{\delta}{r}a^r+\frac{1}{q\delta^{q/r}}b^q\quad\forall\ a,b,\delta>0.$$

We thus have for any $\delta>0$ and $p>2$

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right]&=\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\mathbf{1}_{\{\tau_R>T,\ \rho_R>T\}}\right]\\&\quad+\mathbb{E}\left[\sup_{0\le i\le T}|e(t)|^2\mathbf{1}_{\{\tau_R\le T\text{ or }\rho_R\le T\}}\right]\\&\le\mathbb{E}\left[\sup_{0\le t\le T}\left|e\left(t\wedge\theta_R\right)\right|^2\mathbf{1}_{\{\theta_R>T\}}\right]+\frac{2\delta}{p}\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^p\right]\\&\quad+\frac{1-\frac{2}{p}}{\delta^{2/(p-2)}}\mathbb{C}\left(\tau_R\le T\text{ or }\rho_R\le T\right).\end{aligned}\tag{3}$$

By Lemma 3.1, we have

$$\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^p\right]\le 2^{p-1}\mathbb{E}\left[\sup_{0\le t\le T}\left(|x(t)|^p+|y(t)|^p\right)\right]\le 2^pK(p)$$

and

$$\begin{aligned}\mathbb{C}\left(\tau_R\le T\text{ or }\rho_R\le T\right)&\le\mathbb{C}\left(\tau_R\le T\right)+\mathbb{C}\left(\rho_R\le T\right)\\&\le\mathbb{E}\left[1_{\{\tau_R\le T\}}\frac{\left|y\left(\tau_R\right)\right|^P}{R^p}\right]+\mathbb{E}\left[1_{\{\rho_R\le T\}}\frac{\left|x\left(\rho_R\right)\right|^P}{R^p}\right]\\&\le\frac{1}{R^p}\mathbb{E}\left[\sup_{0\le t\le T}|y(t)|^p\right]+\frac{1}{R^p}\mathbb{E}\left[\sup_{0\le t\le T}|x(t)|^p\right]\\&\le 2\frac{K(p)}{R^p}.\end{aligned}$$

Hence, in (3),

$$\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right] \le \mathbb{E}\left[\sup_{0\le t\le T}\left|x\left(t\wedge\theta_R\right)-y\left(t\wedge\theta_R\right)\right|^2\right] + \frac{2^{p+1}\delta K(p)}{p} + \frac{(p-2)2K(p)}{p\delta^{2/(p-2\}}R^p}. \tag{4}$$

Now we bound the first term on the right-hand side of (4). In view of

$$y\left(t\wedge\theta_R\right) := y_0 + \int_0^{t\wedge\theta_R} f(y(s))ds + \int_0^{t\wedge\theta_R} g(y(s))d\langle B\rangle_s + \int_0^{t\wedge\theta_R} h(y(s))dB_s,$$

and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&|x(t\wedge\theta_R)-y(t\wedge\theta_R)|^2\\
=&\left|\int_0^{t\wedge\theta_R} f(x(s))-f(\bar{y}(s))ds + \int_0^{t\wedge\theta_R} g(x(s))-g(\bar{y}(s))d\langle B\rangle_s\right.\\
&\left.+\int_0^{t\wedge\theta_R} h(x(s))-h(\bar{y}(s))dB_s\right|^2\\
\le& 3\left[T\int_0^{t\wedge\theta_R}\Big|f(x(s))-f(\bar{y}(s))\Big|^2 ds + \Big|\int_0^{t\wedge\theta_R} g(x(s))-g(\bar{y}(s))d\langle B\rangle_s\Big|^2\right.\\
&\left.+\Big|\int_0^{t\wedge\theta_R} h(x(s))-h(\bar{y}(s))dB_s\Big|^2\right]\\
\le& 3\left[T\int_0^t I_{[0,\theta_R]}\Big|f(x(s))-f(\bar{y}(s))\Big|^2 ds + \Big|\int_0^t I_{[0,\theta_R]}g(x(s))-g(\bar{y}(s))d\langle B\rangle_s\Big|^2\right.\\
&\left.+\Big|\int_0^t I_{[0,\theta_R]}h(x(s))-h(\bar{y}(s))dB_s\Big|^2\right].
\end{aligned}$$

Then, for any $\tau\le T$, with the help of (**A1**) and Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le\tau}\left|x\left(t\wedge\theta_R\right)-y\left(t\wedge\theta_R\right)\right|^2\right]\\
&\le 3\left(L_1^2(R)T+L_2^2(R)\bar{\sigma}^2C(2)+L_3^2(R)\bar{\sigma}^4T\right)\mathbb{E}\int_0^{\tau\wedge\theta_R}|x(s)-\bar{y}(s)|^2ds\\
&\le 6L^2(R)\left(T+\bar{\sigma}^2C(2)+\bar{\sigma}^4T\right)\\
&\quad\times\mathbb{E}\int_0^{\tau\wedge\theta_R}\left[|x(s)-y(s)|^2+|y(s)-\bar{y}(s)|^2\right]ds\\
&\le 6L^2(R)\left(T+\bar{\sigma}^2C(2)+\bar{\sigma}^4T\right)\\
&\quad\times\left[\mathbb{E}\int_0^{\tau}\left|x\left(s\wedge\theta_R\right)-y\left(s\wedge\theta_R\right)\right|^2ds+\mathbb{E}\int_0^{\tau\wedge\theta_R}|y(s)-\bar{y}(s)|^2ds\right],
\end{aligned} \tag{5}$$

where $L(R) := \max\{L_1(R), L_2(R), L_3(R)\}$. Given $s \in [0, T \wedge \theta_R)$, let k_s be the integer for which $s \in [t_{k_s}, t_{k_s+1})$, and noting that $\bar{y}(s) = y_{k_s}$, we have

$$\begin{aligned} y(s) - \bar{y}(s) &= \int_{t_{k_s}}^{s} f(\bar{y}(s))ds + \int_{t_{k_s}}^{s} g(\bar{y}(s))d\langle B\rangle_s + \int_{t_{k_s}}^{s} h(\bar{y}(s))dB_s \\ &= f(y_{k_s})(s - t_{k_s}) + g(y_{k_s})\left(\langle B\rangle_s - \langle B\rangle_{t_{k_s}}\right) + h(y_{k_s})\left(B_s - B_{t_{k_s}}\right). \end{aligned}$$

Thus,

$$\begin{aligned} |y(s) - \bar{y}(s)|^2 \leq 3\Big[&|f(y_{k_s})|^2 \Delta^2 + |g(y_{k_s})|^2 \left|\langle B\rangle_s - \langle B\rangle_{t_{k_s}}\right|^2 \\ &+ |h(y_{k_s})|^2 \left|B_s - B_{t_{k_s}}\right|^2\Big]. \end{aligned} \tag{6}$$

By (**A1**), for $|y| \leq R$ we have

$$|f(y)|^2 \vee |g(y)|^2 \vee |h(y)|^2 \leq 2\left(L^2(R)|y|^2 + m_0^2\right),$$

where $m_0 := |f(0)| \vee |g(0)| \vee |h(0)|$. Hence, in (6),

$$|y(s) - \bar{y}(s)|^2 \leq 6(L^2(R)|y|^2 + m_0^2)(\Delta^2 + |\langle B\rangle_s - \langle B\rangle_{t_{k_s}}|^2 + \left|B_s - B_{t_{k_s}}\right|^2).$$

Integrating and then taking G-expectation on both sides, for any $\tau \leq T$, by Corollary 5.5 in [20], we get

$$\begin{aligned} &\mathbb{E}\int_0^{\tau\wedge\theta_R} |y(s) - \bar{y}(s)|^2 ds \\ &\leq 6\int_0^{\tau} \mathbb{E}\left(L^2(R)|y_{k_s}|^2 + m_0^2\right)\left(\Delta^2 + |\langle B\rangle_s - \langle B\rangle_{t_{k_s}}|^2 + |B_s - B_{t_{k_s}}|^2\right) ds \\ &\leq 6\int_0^{T} \left(L^2(R)\mathbb{E}|y_{k_s}|^2 + m_0^2\right)\left(\Delta^2 + \bar{\sigma}^4\Delta^2 + \bar{\sigma}^2\Delta\right) ds \\ &\leq 6T\left(L^2(R)K(p)^{\frac{2}{p}} + m_0^2\right)\left(\Delta + \bar{\sigma}^4\Delta + \bar{\sigma}^2\right)\Delta \\ &\leq 6T\left(L^2(R)K(p)^{\frac{2}{p}} + m_0^2\right)\left(1 + \bar{\sigma}^4 + \bar{\sigma}^2\right)\Delta. \end{aligned}$$

Substituting it into (5), we have

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\leq t\leq\tau} |x(t\wedge\theta_R) - y(t\wedge\theta_R)|^2\right] \\ &\leq U_1\left(L^4(R)K(p)^{\frac{2}{p}} + L^2(R)m_0^2\right)\Delta \\ &\quad + 6L^2(R)\left(T + \bar{\sigma}^2 C(2) + \bar{\sigma}^4 T\right)\int_0^{\tau} \mathbb{E}[\sup_{0\leq r\leq s} |x(r\wedge\theta_R) - y(r\wedge\theta_R)|^2]ds, \end{aligned}$$

where $U_1 := 36T\left(T + \bar{\sigma}^2 C(2) + \bar{\sigma}^4 T\right)\left(1 + \bar{\sigma}^4 + \bar{\sigma}^2\right)$. Applying the Gronwall inequality, we have

$$\mathbb{E}\left[\sup_{0\leq t\leq\tau} |x(t\wedge\theta_R) - y(t\wedge\theta_R)|^2\right]$$

$$\le U_1\left(L^4(R)K(p)^{\frac{2}{p}}+L^2(R)m_0^2\right)\Delta e^{6L^2(R)T\left(T+\bar{\sigma}^2C(2)+\bar{\sigma}^4T\right)}$$

and then

$$\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right]\le U_1\left(L^4(R)K(p)^{\frac{2}{p}}+L^2(R)m_0^2\right)\Delta e^{6L^2(R)T\left(T+\bar{\sigma}^2C(2)+\bar{\sigma}^4T\right)}$$
$$+\frac{2^{p+1}\delta K(p)}{p}+\frac{(p-2)2K(p)}{p\delta^{2/(p-2\}}R^p}.$$

Given any $\epsilon>0$, we can choose $\delta>0$ and R so that $\left(2^{p+1}\delta K(p)\right)/p<\epsilon/3$, and

$$\frac{(1-\frac{2}{p})2K(p)}{\delta^{2/(p-2)}R^p}<\frac{\epsilon}{3},$$

and then choose $\Delta>0$ sufficiently small for

$$U_1\left(L^4(R)K(p)^{\frac{2}{p}}+L^2(R)m_0^2\right)\Delta e^{6L^2(R)T\left(T+\bar{\sigma}^2C(2)+\bar{\sigma}^4T\right)}<\frac{\epsilon}{3}.$$

As a result, we obtain $\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right]<\epsilon$. The proof is complete. □

4. Convergence rate

In this section, we aim to establish the rate of convergence.

Lemma 4.1. *Assume that there are positive constants L_1, L_2 and L_3 such that*

$$|f(x)-f(y)|\le L_1|x-y|,\ |g(x)-g(y)|\le L_2|x-y|,\ |h(x)-h(y)|\le L_3|x-y|$$

for all $x,y\in\mathbb{R}^n$. Then, there exists a constant $D>0$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T}|y(t)-x(t)|^4\right]\le DT\Delta^2.$$

Proof. By the G-Itô's formula, we have

$$\begin{aligned}|y(t)-x(t)|^4=&\int_0^t 4|y(s)-x(s)|^2\langle y(s)-x(s),f(\bar{y}(s))-f(x(s))\rangle ds\\&+\int_0^t\big[4|y(s)-x(s)|^2\langle y(s)-x(s),g(\bar{y}(s))-g(x(s))\rangle\\&+2|y(s)-x(s)|^2|h(\bar{y}(s))-h(x(s))|^2\\&+4\langle y(s)-x(s),h(\bar{x}(s))-h(x(s))\rangle^2\big]\,d\langle B\rangle_s\\&+\int_0^t 4|y(s)-x(s)|^2\langle y(s)-x(s),h(\bar{y}(s))-h(x(s))\rangle dB_s.\end{aligned}\tag{7}$$

Next, we consider (7) on the interval $[0, t_1)$. Using Lemma 2.3, we have for any $\tau \in [0, t_1)$

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le\tau}\left|\int_0^t 4|y(s)-x(s)|^2\langle y(s)-x(s), h(\bar{y}(s))-h(x(s))\rangle dB_s\right|\right]\\
&\le 4C(1)\bar{\sigma}\mathbb{E}\left[\left|\int_0^\tau |y(s)-x(s)|^4\langle y(s)-x(s), h(\bar{y}(s))-h(x(s))\rangle^2 ds\right|\right]^{\frac{1}{2}}\\
&\le \frac{1}{2}\mathbb{E}\left[\sup_{0\le t\le\tau}|y(t)-x(t)|^4\right]\\
&\quad + 8C^2(1)\bar{\sigma}^2\mathbb{E}\left[\int_0^\tau \langle y(s)-x(s), h(\bar{y}(s))-h(x(s))\rangle^2 ds\right]\\
&\le \frac{1}{2}\mathbb{E}\left[\sup_{0\le t\le\tau}|y(t)-x(t)|^4\right]\\
&\quad + 8C^2(1)\bar{\sigma}^2L_3^2\mathbb{E}\left[\int_0^\tau |y(s)-x(s)|^2|\bar{y}(s)-x(s)|^2 ds\right]\\
&\le \frac{1}{2}\mathbb{E}\left[\sup_{0\le t\le\tau}|y(t)-x(t)|^4\right]\\
&\quad + 4C^2(1)\bar{\sigma}^2L_3^2\mathbb{E}\left[\int_0^\tau (|y(s)-x(s)|^4+|\bar{y}(s)-x(s)|^4) ds\right]\\
&\le \frac{1}{2}\mathbb{E}\left[\sup_{0\le t\le\tau}|y(t)-x(t)|^4\right] + 36C^2(1)\bar{\sigma}^2L_3^2\mathbb{E}\left[\int_0^\tau |y(s)-x(s)|^4 ds\right]\\
&\quad + 32C^2(1)\bar{\sigma}^2L_3^2\mathbb{E}\left[\int_0^\tau |\bar{y}(s)-x(s)|^4 ds\right].
\end{aligned}
\tag{8}
$$

Similarly, using the global Lipschitz condition, the Hölder inequality and Lemma 2.2, we get

$$
\begin{aligned}
&4\mathbb{E}\left[\sup_{0\le t\le\tau}\int_0^t \left||y(s)-x(s)|^2\langle y(s)-x(s), f(\bar{y}(s))-f(x(s))\rangle\right| ds\right]\\
&\le 4\mathbb{E}\int_0^\tau |y(s)-x(s)|^2\,|y(s)-x(s)|\,|f(\bar{y}(s))-f(y(s))|\,ds\\
&\le 4L_1\mathbb{E}\int_0^\tau |y(s)-x(s)|^2\,|y(s)-x(s)|\,|\bar{y}(s)-x(s)|\,ds\\
&\le 3L_1\mathbb{E}\int_0^\tau |y(s)-x(s)|^4 ds + L_1\mathbb{E}\int_0^\tau |\bar{y}(s)-x(s)|^4\,ds\\
&\le 11L_1\mathbb{E}\int_0^\tau |y(s)-x(s)|^4 ds + 8L_1\mathbb{E}\int_0^\tau |\bar{y}(s)-x(s)|^4\,ds
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
&\mathbb{E}[\sup_{0\le t\le\tau}\int_0^t (4|y(s)-x(s)|^2\langle y(s)-x(s), g(\bar{y}(s))-g(x(s))\rangle \\
&\quad + 2|y(s)-x(s)|^2|h(\bar{y}(s))-h(x(s))|^2 \\
&\quad + 4\langle y(s)-x(s), h(\bar{y}(s))-h(x(s))\rangle^2)d\langle B\rangle_s] \\
&\le \bar{\sigma}^2\mathbb{E}[\int_0^\tau (4|y(s)-x(s)|^2|y(s)-x(s)||g(\bar{y}(s))-g(x(s))| \\
&\quad + 2|y(s)-x(s)|^2|h(\bar{y}(s))-h(x(s))|^2 \\
&\quad + 4\langle y(s)-x(s), h(\bar{y}(s))-h(x(s))\rangle^2)ds] \\
&\le \bar{\sigma}^2[3L_2\mathbb{E}\int_0^\tau |y(s)-x(s)|^4 ds + L_2\mathbb{E}\int_0^\tau |\bar{y}(s)-x(s)|^4 ds \\
&\quad + 6L_3^2\mathbb{E}\int_0^\tau |y(s)-x(s)|^2|\bar{y}(s)-x(s)|^2 ds] \\
&\le \bar{\sigma}^2[11L_2\mathbb{E}\int_0^\tau |y(s)-x(s)|^4 ds + 8L_2\mathbb{E}\int_0^\tau |\bar{y}(s)-y(s)|^4 ds \\
&\quad + 27L_3^2\mathbb{E}\int_0^\tau |y(s)-x(s)|^4 ds + 24L_3^2\mathbb{E}\int_0^\tau |\bar{y}(s)-y(s)|^4 ds].
\end{aligned} \tag{10}
$$

Substituting (8)-(10) into (7) yields

$$
\begin{aligned}
&\mathbb{E}[\sup_{0\le t\le\tau}|y(t)-x(t)|^4] \\
&\le U_2\mathbb{E}\int_0^\tau |y(s)-x(s)|^4 ds + U_3\mathbb{E}\int_0^\tau |\bar{y}(s)-y(s)|^4 ds,
\end{aligned} \tag{11}
$$

where $U_2 := 22L_1 + \bar{\sigma}^2(72C^2(1)+54)L_2^2 + 22\bar{\sigma}^2L_3$, $U_3 := 16L_1 + 16\bar{\sigma}^2L_2 + \bar{\sigma}^2(64C^2(1)+48)L_3^2$.

On the other hand, since the global Lipschitz condition implies the linear growth condition

$$|f(x)|\vee|g(x)|\vee|h(x)|\le U_4(1+|x|)$$

with $U_4 := \max\{L_1, L_2, L_3, m_0\}$. Hence, for given stepsize $0<\Delta<1$ and $t\in[0,t_1)$, we have

$$
\begin{aligned}
&|\bar{y}(t)-y(t)|^4 \\
&= \left|\int_0^t f(\bar{y}(s))ds + \int_0^t g(\bar{y}(s))d\langle B\rangle_s + \int_0^t h(\bar{y}(s))dB_s\right|^4 \\
&\le 27\left(t^3\int_0^t |f(\bar{y}(s))|^4 ds + \left|\int_0^t g(\bar{y}(s))d\langle B\rangle_s\right|^4 + \left|\int_0^t h(\bar{y}(s))dB_s\right|^4\right).
\end{aligned}
$$

Taking G-expectation and applying the Burkholder-Davis-Gundy inequality in G-framework, we get

$$\begin{aligned}
&\mathbb{E}|\bar{y}(t)-y(t)|^4\\
\leq\ & 27\mathbb{E}\left(t^3\int_0^t U_8^4(1+|\bar{y}(s)|)^4ds+\bar{\sigma}^8t^3\int_0^t U_4^4(1+|\bar{y}(s)|)^4ds\right.\\
&\left.+C(4)\bar{\sigma}^4t\int_0^t U_4^4(1+|\bar{y}(s)|)^4ds\right)\\
\leq\ & 216\mathbb{E}\left(t^3\int_0^t U_4^4(1+|\bar{y}(s)|^4)ds+\bar{\sigma}^8t^3\int_0^t U_4^4(1+|\bar{y}(s)|^4)ds\right.\\
&\left.+C(4)\bar{\sigma}^4t\int_0^t U_4^4(1+|\bar{y}(s)|^4)ds\right)\\
\leq\ & 216U_4^4(1+K(4))(t^4+\bar{\sigma}^8t^4+C(4)\bar{\sigma}^4t^2)\\
\leq\ & 216U_4^4(1+K(4))(1+\bar{\sigma}^8+C(4)\bar{\sigma}^4)\Delta^2.
\end{aligned}$$

Substituting this into (11), by the Grownwall inequality, we obtain

$$\mathbb{E}[\sup_{0\leq t\leq\tau}|y(t)-x(t)|^4]\leq DT\Delta^2,$$

where $D:=216U_2U_4^4(1+K(4))(1+\bar{\sigma}^8+C(4)\bar{\sigma}^4)e^{U_3T}$. Repeating the above procedure on each interval $[t_k,t_{k+1})$, $k\geq 1$, we get that

$$\mathbb{E}[\sup_{0\leq t\leq T}|y(t)-x(t)|^4]\leq DT\Delta^2. \tag{12}$$

Therefore, the proof is complete. □

Theorem 4.2. *Let assumptions* **(A1)** *and* **(A2)** *hold. Suppose that there are some positive constants* a_1,a_2 *and* α_3 *such that* $L_1(R)\leq\alpha_1\log R$, $\max\{L_2(R),L_2^2(R)\}\leq\alpha_2\log R$ *and* $\max\{L_3(R),L_3^2(R)\}\leq\alpha_3\log R$. *Then*

$$E\left[\sup_{0\leq t\leq T}|y(t)-x(t)|^2\right]=O(\Delta).$$

Proof. For each $R\geq 1$, define the function

$$f_R(x)=\begin{cases}f(x), & \text{if } |x|\leq R,\\ f(Rx/|x|), & \text{if } |x|>R\end{cases}$$

and $g_R(x)$, $h_R(x)$ similarly. Let $x_R(t)$ be the solution to the following stochastic differential equation

$$dx_R(t)=f_R\left(x_R(t)\right)dt+g_R\left(x_R(t)\right)d\langle B\rangle_t+h_R\left(x_R(t)\right)dB_t$$

with $x_R(0) = x_0$ and $y_R(t)$ be the corresponding EM solution with the stepsize $0 < \Delta < 1$. Using (12), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}|y_R(t)-x_R(t)|^4\right]\\
&\le 216U_2U_4^4(1+K(4))(1+\bar\sigma^8+C(4)\bar\sigma^4)\\
&\quad\times\left[(64C^2(1)+48)\bar\sigma^2L_3^2(R)+16L_1(R)+\bar\sigma^2 16L_2(R)\right]T\Delta^2\\
&\quad\times\exp\left\{[72C^2(1)\bar\sigma^2L_2^2(R)+22L_1(R)+\bar\sigma^2(22L_3(R)+54L_2^2(R))]T\right\}\\
&\le 216U_2U_4^4(1+K(4))(1+\bar\sigma^8+C(4)\bar\sigma^4)T\Delta^2\\
&\quad\times\exp\left\{[38L_1(R)+(54+72C^2(1))\bar\sigma^2L_2^2(R)+16\bar\sigma^2L_2(R)\right.\\
&\quad\left.+(48+64C^2(1))\bar\sigma^2L_3^2(R)+22\bar\sigma^2L_3(R)]T\right\}\\
&\le 216U_2U_4^4(1+K(4))(1+\bar\sigma^8+C(4)\bar\sigma^4)T\Delta^2R^\Gamma,
\end{aligned}$$

where $\Gamma := [38\alpha_1 + (70+72C^2(1))\bar\sigma^2\alpha_2 + (70+64C^2(1))\bar\sigma^2\alpha_3]T$.

Set

$$\widehat{y}(T) = \sup_{0\le r\le T}|y(t)| \quad\text{and}\quad \widehat{x}(T) = \sup_{0\le r\le T}|x(t)|.$$

Define the stopping time

$$\tau_R = T\wedge\inf\{t\in[0,T] : |x_R(t)|\vee|y_R(t)|\ge R\}.$$

It is easy to show that

$$x(t)=x_R(t)=x_{R+1}(t)\quad\text{and}\quad y(t)=y_R(t)=y_{R+1}(t)\quad\text{if}\quad 0\le t\le\tau_R.$$

This implies that τ_R is non-decreasing and, by Lemma 3.1, $\lim_{R\to\infty}\tau_R = T$ q.s. Let $\tau_0 = 0$, for $t\in[0,T)$, we have

$$\begin{aligned}
|y(t)-x(t)|^2 &= \sum_{R=1}^{\infty}|y(t)-x(t)|^2 I_{\{\tau_{R-1}\le t<\tau_R\}}\\
&= \sum_{R=1}^{\infty}|y_R(t)-x_R(t)|^2 I_{\{\tau_{R-1}\le t<t_R)}\\
&\le \sum_{R=1}^{\infty}|y_R(t)-x_R(t)|^2 I_{(R-1\le\widehat{y}(T)\vee\widehat{x}(T)\le R)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le r\le T}|y(t)-x(t)|^2\right]\\
&\le \sum_{R=1}^{\infty}\left(\mathbb{E}|y_R(t)-x_R(t)|^4\right)^{\frac12}\left(\mathbb{E}I_{\{R-1\le\widehat{y}(T)\vee\widehat{x}(T)\le R\}}\right)^{\frac12}\\
&\le \sum_{R=1}^{\infty}\left(\mathbb{E}|y_R(t)-x_R(t)|^4\right)^{\frac12}\sqrt{\mathbb{C}(R-1\le\widehat{y}(T)\vee\widehat{x}(T)\le R)}.
\end{aligned}$$

On the other hand, by the G-Markov inequality, for any $q \geq 2$

$$\begin{aligned}\mathbb{C}(\widehat{y}(T) \vee \widehat{x}(T) \geq R-1) &\leq \frac{\mathbb{E}|\widehat{y}(T)+1|^q + \mathbb{E}|\widehat{x}(T)+1|^q}{R^q} \\ &\leq \frac{2^{q-1}[\mathbb{E}(1+|\widehat{y}(T)|^q) + \mathbb{E}(1+|\widehat{x}(T)|^q)]}{R^q} \\ &\leq \frac{2^q[1+K(q)]}{R^q},\end{aligned}$$

where $K(q)$ is defined in Lemma 3.1. Thus

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)-x(t)|^2\right] \\ &\leq \sum_{R=1}^{\infty} \sqrt{216U_2U_4^4(1+K(4))(1+\bar{\sigma}^8+C(4)\bar{\sigma}^4)T}\Delta R^{\Gamma} \frac{[2^q(1+K(q))]^{\frac{1}{2}}}{R^{q/2}}.\end{aligned} \tag{13}$$

Letting q be sufficiently large for $\frac{q}{2} > 1+\Gamma$, we see that the right-hand side of (13) is convergent, we get the rate of convergence is 1/2. The proof is complete. □

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