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FUZZY SUPER SUBDIVISION MODEL WITH AN APPLICATION IN INFECTION GROWTH ANALYSIS

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Abstract. In our study, the integration of fuzzy graphs into classical graph theory gives rise to a novel concept known as "Fuzzy Super Subdivision." Let $SS_f(G)$ be the fuzzy super subdivision graphs, by substituting a complete bipartite graph $k_{(2,m)}$ $(m = 1, 2, \ldots)$ for each edge of a fuzzy graph. The attributes and properties of this newly proposed concept are briefly outlined, in addition to illustrative examples. Furthermore, significant findings are discussed on connectivity, size, degree and order of fuzzy super subdivision structures. To illustrate the practical implications of our approach, we present an application focused on analyzing the growth of infections in blood or urine samples using the Fuzzy Super Subdivision model.

1. Introduction

In between the scale of sharp or precise decisions, to address the uncertainty or vagueness within sets, Lotfi Asker Zadeh introduced "fuzzy sets and fuzzy relations" in 1965 [27], laying the groundwork for mathematical solutions. Additionally, Rosenfeld [21] extended various concepts from graph theory into the realm of fuzzy analogues and subsequent extensions by Kauffman's proposition of fuzzy graphs in 1973 [18], and Azriel Rosenfeld's integration of fuzzy relations into fuzzy sets in 1975 and Bhattacharya made noteworthy observations regarding fuzzy graphs [7]. In fuzzy graph, the membership values of edges and vertices are mathematically represented as values ranging from $[0, 1]$ $[1, 2]$. To deal with uncertainty or vagueness, fuzzy graph theory and its extensions have been applied in various domains such as image recognition, social networking, decision making and medical diagnosis [3, 5, 6, 14, 22].

The motivation stems from the constant advancement of graph theory methodologies, particularly in the context of creating new graphs. As an extension of subdivision [12], super subdivision of graphs was proposed by

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Sethuraman and Selvaraju in 2003 [23]. Replacing all edges of a graph with the complete bipartite graph $k_{(2,m)}$ for $m > 1$ yields super subdivision of a graph [4]. This operation can be used in diverse applications such as natural language processing, medical imaging, and robotics and autonomous systems [8, 10, 13, 15, 16]. On the basis of this foundation, the study on fuzzy super subdivision graph is newly presented for potential new results and proofs within the context of fuzzy graph theory.

The problem statement focuses on the gap found in the existing literature, pertaining to fuzzy super subdivision graphs. In this domain, there has been limited research and concepts, implying a need for further exploration and understanding at the intersection of fuzzy graph, subdivision, and super subdivision principles.

The objective of the paper outlines the introduction of fuzzy super subdivision graphs and its properties with practical applications. On addressing the gap which serves as the primary motivation, we introduce and define fuzzy super subdivision graphs with properties and results. This paper presents a novel insight of fuzzy graphs in the context of super subdivision.

The novelty of this research paper is to introduce fuzzy super subdivision graphs which extends graph theory into fuzzy framework with its structural properties, such as adjacency, connectivity, and comparison between degree, order and size. This research also demonstrates application in medical diagnosis related to infection growth analysis. This proposed model produces visual and analytical representation of complex nature of the bacterial growth through different phases. Basic definitions are stated in preliminaries and fuzzy super subdivision graph is presented in the main section with detailed explanation and definitions.

2. Preliminaries

In this section, we review some basic definitions for fuzzy graph theory and subdivision.

Definition 2.1 ([20])**.** A fuzzy graph $G = (V, E, \sigma, \mu)$ corresponding to the crisp graph G is a non-empty set V together with a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that for all $u, v \in V$, $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$, where $\sigma(u), \sigma(v)$ and $\mu(u, v)$ represent the membership values of the vertex u and v and (u, v) is the corresponding adjacent edge, respectively.

Definition 2.2 ([11])**.** The degree of a vertex v of a fuzzy graph G is defined by $d(v) = \sum_{(u\neq v)} \mu(u, v)$. The minimum and the maximum degree of G is defined by $\delta(G) = \bigwedge\{d(v)|v \in V\}$ and $\Delta(G) = \bigvee\{d(v)|v \in V\}$, respectively.

Definition 2.3 ([11])**.** The size $S(G)$ of a graph G is defined by $d(v) = \sum_{(u\neq v)} \mu(u, v)$ and the order $O(G)$ of a graph G is defined by $O(G) = \sum_{v\in V} \sigma(v)$.

Definition 2.4 ([20])**.** The strength of a path $P\{v_1, v_2, \ldots, v_n\}$ of G is defined as $\min\{\mu(v_i, v_{i+1}) : i = 1, 2, \ldots, n - 1\}$. The strength of connectedness

$CONN_G(v_1, v_2)$ or $\mu_\infty(v_1, v_2)$ between the vertices v_1 and v_2 is the maximum strength among all the paths between v_1 and v_2. An $v_1 - v_2$ path P is called a strongest $v_1 - v_2$ path if its strength equals $CONN_G(v_1, v_2)$.

Definition 2.5 ([25])**.** Let G be a fuzzy graph and $S_f(G)$ denote the subdivision of G that is obtained from G by subdividing each edge of G once.

Definition 2.6 ([20])**.** A fuzzy graph G is said to be a complete fuzzy graph if $\mu(uv) = \min\{\sigma(u), \sigma(v)\}$ for $u, v \in V$.

3. Methodology: Fuzzy super subdivision graphs

Definition 3.1. Let $G = (V, E, \sigma, \mu)$ be a fuzzy graph with p vertices and q edges. The super subdivision of a fuzzy graph is defined as $SS_f(G) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ by replacing each edge $v_iv_j \in E$ such that $i \neq j$ and $1 \leq i, j \leq p$, by a complete bipartite graph $k_{(2,m)}$ for $m > 1$ in such a way that the ends of v_iv_j are merged with the 2- vertices part of $k_{(2,m)}$. Here $V_{SS} = V \cup V^*$, where V^* contains super subdivided vertices $w_{(p-1)t}$ with $1 \leq (p-1) \leq q$ and $1 \leq t \leq m$ and E_{SS} is the collection of super subdivided edges e_{rs} with $1 \leq r \leq q$ and $1 \leq s \leq 2m$ satisfying the following conditions

(1) $\sigma_{SS}(v_i) < \sigma_{SS}(w_{(p-1)t}) > \sigma_{SS}(v_j)$
(2) $\mu_{SS}(v_i w_{(p-1)t}) = \sigma_{SS}(v_i) \wedge \sigma_{SS}(w_{(p-1)t})$ where $v_i \in V$ and $w_{(p-1)t} \in V_{ss}$.

Note 3.1. The super subdivided vertices of the cycle are w_{pt} with $1 \leq p \leq q$ and $1 \leq t \leq m$.

Note 3.2. We have provided new membership values for the fuzzy super subdivision graph so that it satisfies the above two conditions as the edge of the fuzzy graph is replaced with the two new edges of the complete bipartite graph.

Example 3.2. Consider the fuzzy graph of a path P_4 (in Figure 1). Fuzzy super subdivision of path graph $SS_f(P_4) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$, where

$$V_{SS} = \{v_1, w_{11}, w_{12}, v_2, w_{21}, w_{22}, v_3, w_{31}, w_{32}, v_4\}$$

and

$$E_{SS} = \{e_{11}, e_{12}, e_{13}, e_{14}, e_{21}, e_{22}, e_{23}, e_{24}, e_{31}, e_{32}, e_{33}, e_{34}\}$$

is obtained by replacing each edge with the complete bipartite graph $k_{(2,m)}$, where $m = 2$ in Figure 1.

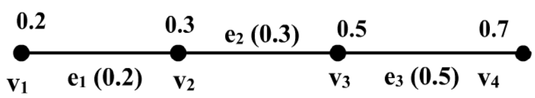


FIGURE 1. Given Fuzzy graph of a path P_4

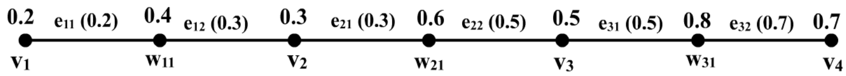


FIGURE 2. Fuzzy subdivided path graph P_4

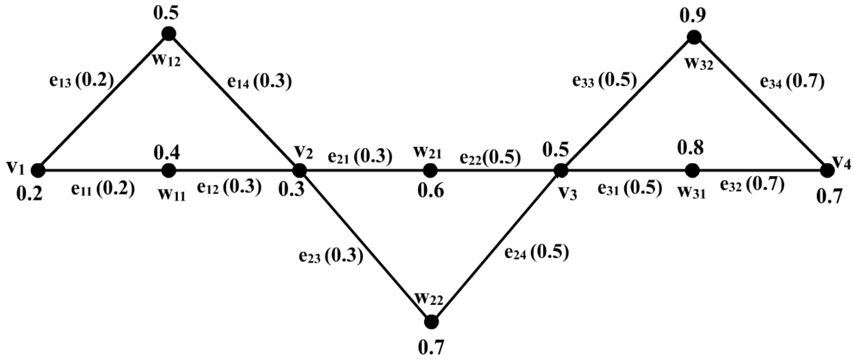


FIGURE 3. Fuzzy super subdivided path graph where $m = 2$ in $k_{2,2}$ P_4

On satisfying the conditions, the edge membership value and the vertex membership values of the fuzzy super subdivided path graph $SS_f(P_4)$ are given as follows:

Vertex membership values:

$$\sigma_{SS}(v_1) = 0.2, \sigma_{SS}(v_2) = 0.3$$

$$\sigma_{SS}(v_1) < \sigma_{SS}(w_{11}) > \sigma_{SS}(v_2) = 0.2 < 0.4 > 0.3.$$

Therefore,

$$\sigma_{SS}(w_{11}) = 0.4$$

$$\sigma_{SS}(v_1) = 0.2, \sigma_{SS}(v_2) = 0.3$$

$$\sigma_{SS}(v_1) < \sigma_{SS}(w_{12}) > \sigma_{SS}(v_2) = 0.2 < 0.5 > 0.3.$$

Therefore,

$$\sigma_{SS}(w_{12}) = 0.5.$$

Similarly, $\sigma_{SS}(w_{21})$ and $\sigma_{SS}(w_{22})$ should be greater than $\sigma_{SS}(v_2)$ and $\sigma_{SS}(v_3)$. Therefore, $w_{21} = 0.6$ and $w_{22} = 0.7$, $w_{31} = 0.8$ and $w_{32} = 0.9$.

Edge membership values:

$$\mu_{SS}(v_1 w_{11}) = \sigma_{SS}(v_1) \wedge \sigma_{SS}(w_{11})$$

$$\mu_{SS}(v_1 w_{11}) = 0.2 \wedge 0.4 = 0.2.$$

Therefore, $e_{11} = 0.2$. Similarly,

$$\mu_{SS}(v_2w_{11}) = e_{12} = 0.3, \mu_{SS}(v_1w_{12}) = e_{13} = 0.2$$
$$\mu_{SS}(v_2w_{12}) = e_{14} = 0.3, \mu_{SS}(v_2w_{21}) = e_{21} = 0.3$$
$$\mu_{SS}(v_3w_{21}) = e_{22} = 0.5, \mu_{SS}(v_2w_{22}) = e_{23} = 0.3$$
$$\mu_{SS}(v_3w_{22}) = e_{24} = 0.5, \mu_{SS}(v_3w_{31}) = e_{31} = 0.5$$
$$\mu_{SS}(v_4w_{31}) = e_{32} = 0.7, \mu_{SS}(v_3w_{32}) = e_{33} = 0.5$$
$$\mu_{SS}(v_4w_{32}) = e_{34} = 0.7.$$

Remark 3.3. For $m = 1$ in $k_{(2,m)}$ (Figure 2), the subdivision of a fuzzy graph $G = (V, E, \sigma, \mu)$ is $S_f(G) = (V_S, E_S, \sigma_S, \mu_S)$ where $V_S = V \cup E$ and E_S containing subdivided edges with $\sigma_S(v) = \sigma(v)$, $v \in V$ and $\sigma_S(q) = \mu(q), q \in E$ $\mu_S(uv) = \sigma(u) \wedge \mu(v)$ for $uv \in E_S$.

4. Connectivity

Proposition 4.1. *Every pair of adjacent vertices in a fuzzy graph is non-adjacent in its super subdivision.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a fuzzy graph with p vertices and q edges and its fuzzy super subdivision $SS_f(G)$, where $v_1, v_2, \ldots, v_p \in V$ are adjacent vertices in G. We aim to prove that every pair of adjacent vertices in G is non-adjacent in $SS_f(G)$. Assume, any two adjacent vertices v_i and $v_j \in V$ in G. When G undergoes super subdivision, each edge is replaced with a complete bipartite graph $k_{(2,m)}$ for $m > 1$. In a fuzzy graph, adjacency is denoted by a non-zero membership value associated with the edge between vertices. When v_i and v_j are adjacent in G, the associated membership value implies a degree of connection. In this fuzzy super subdivision process, a new vertex, $w_{(p-1)t}$ is introduced to subdivide the edge between v_i and v_j, creating super subdivided edges $v_iw_{(p-1)t}$ and $v_jw_{(p-1)t}$. Thus, $w_{(p-1)t}$ facilitates a connection between v_i and v_j, it does not establish direct adjacency between them in $SS_f(G)$. Therefore, the adjacent vertex pairs of G are no longer adjacent in its fuzzy super subdivision. This holds for any pair of adjacent vertices in G. □

Proposition 4.2. *The vertices that are connected in a fuzzy graph are not connected in its super subdivision.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a fuzzy graph and $SS_f(G)$ be its fuzzy super subdivided graph. To demonstrate that vertices connected in G are not directly connected in $SS_f(G)$, consider any two vertices v_i and $v_j \in V$ in G connected by an edge, i.e., $\{v_i, v_j\} \in E$. On super subdividing the fuzzy graph G by replacing this edge with $k_{(2,m)}$ for $m > 1$, a new vertex $w_{(p-1t}$ is introduced. Consequently, $w_{(p-1)t}$ is connected to v_i and v_j forming new edges $v_iw_{(p-1)t}$ and $v_jw_{(p-1)t}$ in $SS_f(G)$. This implies that v_i and v_j are indirectly connected through the super subdivided vertex $w_{(p-1)t}$. Since there is an intermediate vertex $w_{(p-1)t}$ between v_i and v_j, they are no longer directly adjacent. Therefore, the vertices connected by an edge in G are not directly connected in $SS_f(G)$. □

Theorem 4.3. *For every fuzzy graph, the number of edges in the corresponding fuzzy super subdivided graph is equal to 2qm edges.*

Proof. Let us consider $G = (V, E, \sigma, \mu)$ be the fuzzy graph and $SS_f(G) = (V_{SS}, E_{SS})$ be the fuzzy super subdivided graph. Let $v_1, v_2, \ldots, v_p \in V$ and $e_1, e_2, \ldots, e_q \in E$ be the vertices and edges of the fuzzy graph respectively. When super subdividing each edge in G, it involves replacing each edge with a complete bipartite graph $k_{(2,m)}$ for $m > 1$. This replacement introduces $(p-1)t$ vertices where $1 \leq (p-1) \leq q$ and $1 \leq t \leq m$ i.e., m copies of the super subdivided vertices. Additionally, e_{rs} edges are introduced, where $1 \leq r \leq q$ and $1 \leq s \leq 2m$ i.e., representing $2m$ new edges of a path. Then we have

$$\{w_{11}, w_{12}, \ldots, w_{1t}, w_{21}, w_{22}, \ldots, w_{2t}, \ldots, w_{(p-1)1}, w_{(p-1)2}, \ldots, w_{(p-1)t}\} \in V_{SS}$$

and

$$\{e_{11}, e_{12}, \ldots, e_{1s}, e_{21}, e_{22}, \ldots, e_{2s}, \ldots, e_{r1}, e_{r2}, \ldots, e_{rs}\} \in E_{SS}$$

as vertices and edges of the fuzzy super subdivided graphs respectively. Therefore, the q number of edges in G after m super subdivision, the total number of new edges introduced is $q \times 2m = 2qm$. This concludes the proof. □

5. Degree, order and size

Definition 5.1. Let $SS_f(G) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ be the fuzzy super subdivided graph. The degree of a vertex v is defined by the sum of the membership value of super subdivided edges incident with v and it is denoted by $d_{SS}(v) = \sum_{(u \neq v)} \sigma_{SS}(v_i w_{(p-1)t})$. The minimum and the maximum degree is defined by $\delta_{SS}(G) = \bigwedge\{d_{SS}(w_{(p-1)t}) | w_{(p-1)t}\} \in V_{SS}$ and $\Delta_{SS}(G) = \bigvee\{d_{SS}(w_{(p-1)t} | w_{(p-1)t}\} \in V_{SS}$, respectively.

Definition 5.2. The size $S_{SS}(G)$ of fuzzy super subdivided graph $SS_f(G)$ is defined by $S_{SS}(G) = \sum_{(v_i \neq w_{p-1}t)} \mu_{SS}(v_i, w_{(p-1)t})$ and the order $O_{SS}(G)$ of fuzzy super subdivided fuzzy graph $SS_f(G)$ is defined by $O_{SS}(G) = \sum_{(v_i \neq w_{(p-1)t})} \sigma_{SS}(v_i)$.

Example 5.3. Consider the fuzzy super subdivided twig graph

$$SS_f(Tg_2) = (V_{SS}, E_{SS}, \sigma_{SS}, \sigma_{SS})$$

in Figure 4. The degree, order and size of the fuzzy super subdivided twig graph $SS_f(Tg_2)$ are given as follows:

$$d(v_1) = 0.4, d(w_{11}) = 0.5, d(w_{12}) = 0.5$$

$$d(v_2) = 2.4, d(w_{21}) = 0.7, d(w_{22}) = 0.7$$

$$d(v_3) = 3.2, d(w_{31}) = 0.9, d(w_{32}) = 0.9$$

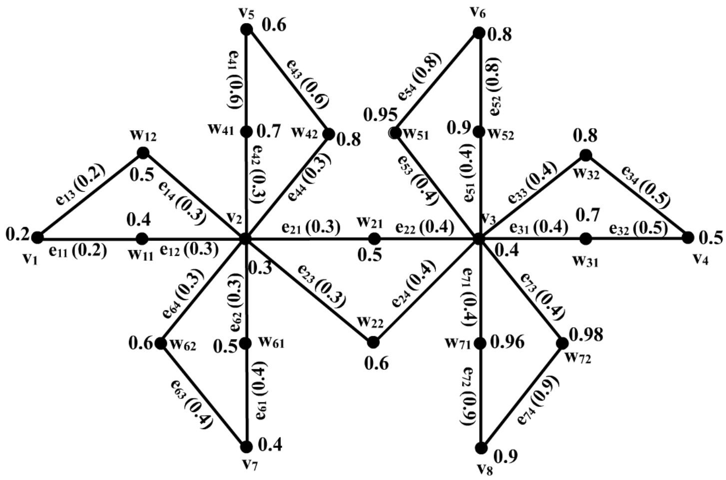


FIGURE 4. Fuzzy super subdivided twig graph $SS_f(Tg_2)$ for $k_{2,2}$

$$d(v_4) = 1.0, d(w_{41}) = 0.9, d(w_{42}) = 0.9$$
$$d(v_5) = 1.2, d(w_{51}) = 1.2, d(w_{52}) = 1.2$$
$$d(v_6) = 1.6, d(w_{61}) = 0.7, d(w_{62}) = 0.7$$
$$d(v_7) = 0.8, d(w_{71}) = 1.3, d(w_{72}) = 1.3, d(v_8) = 1.8$$
$$\sum_{(v_1 \neq w_{11})} \mu_{SS}(v_1 w_{11}) = 12.4$$
$$\sum_{(v_1 \neq w_{11})} \sigma_{SS}(v_1) = 13.99.$$

Proposition 5.4. *The maximum degree of any vertex in fuzzy super subdivided graph with p vertices is less than or equal to* $p - 1$.

Proof. Let $SS_f(G) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ be the fuzzy super subdivided graph. Proof by contradiction: Suppose there exists a fuzzy super subdivided graph with p vertices where the maximum degree of any vertex is greater than $p-1$. In a fuzzy super subdivided graph, vertices are introduced during the subdivision process. Each subdivision introduces a new vertex and connects it with a path to the original vertices. Assume there exists a vertex v_i in the graph with a degree greater than $p - 1$. Let d be the degree of v_i, where $d > (p - 1)$. Since v_i has a degree greater than $p - 1$, it is connected to more than $p - 1$ vertices. Since $d > (p - 1)$, there must be at least one vertex v_j that v_i is connected to and such that v_i has more than one edge connecting to v_j. The assumption that there exists a vertex with a degree greater than $p - 1$ leads to a contradiction. □

Proposition 5.5. *Fuzzy super subdivided vertices within the path exhibit equal degrees, when an edge is super subdivided.*

Proof. Let $SS_f(G)=(V_{SS},E_{SS},\sigma_{SS},\mu_{SS})$ be the fuzzy super subdivided graph. Consider an edge $(v_i,v_j)\in E$ in fuzzy graph G and by super subdividing the edge, forms two paths: $v_i - w_{(p-1)1} - v_j$ and $v_i - w_{(p-1)2} - v_j$. Since both the super subdivided vertices $w_{(p-1)1}$ and $w_{(p-1)2}$ are introduced by super subdividing the same edge, they connect to the same set of existing vertices v_i and v_j in path. As a result of connecting to the same vertices, the degrees of $w_{(p-1)1}$ and $w_{(p-1)2}$ are equal within the same path. Therefore, by the process of super subdivision, the degrees of these super subdivided vertices are identical since they connect to the same existing vertex. This completes the proof, demonstrating that fuzzy super subdivided vertices introduced along the same path in a fuzzy super subdivided graph have equal degrees. □

Theorem 5.6. *In $SS_f(G)$, the sum of the degrees of all the vertices is twice the sum of the membership values of all the edges.*

Proof. Let $SS_f(G)=(V_{SS},E_{SS},\sigma_{SS},\mu_{SS})$ be the fuzzy super subdivided graph. Here

$$\begin{aligned}V_{SS} &= V\cup V^*\\ &= \{v_1,w_{11},w_{12},\dots,w_{1t},v_2,w_{21},w_{22},\dots,w_{2t},\dots,v_{(p-1)},\\ &\quad w_{(p-1)1},w_{(p-1)2},\dots,w_{(p-1)t},v_p\}.\end{aligned}$$

The degree of all vertices $v\in V_{ss}$ is defined as

$$\sum_{v\in V_{ss}} d_{ss}(v) = \Big[\sum_{i=1}^{p} d_{ss}(v_i),\ \sum_{(p-1)=1}^{q}\sum_{t=1}^{m} d_{ss}(w_{(p-1)t})\Big]$$

$$\begin{aligned}\sum_{v\in V_{ss}} d_{ss}(v) &= d_{ss}(v_1)+d_{ss}(w_{11})+\cdots+d_{ss}(w_{1t})+d_{ss}(v_2)+d_{ss}(w_{21})\\ &\quad+\cdots+d_{ss}(w_{2t})+\cdots+d_{ss}(v_{(p-1)})+d_{ss}(w_{(p-1)1})\\ &\quad+\cdots+d_{ss}(w_{(p-1)t})+d_{ss}(v_p).\end{aligned}$$

By the definition of the degree of a vertex,

$$d_{SS}(v)=\sum_{v_i\neq w_{(p-1)t}}\mu_{SS}(v_iw_{(p-1)t}).$$

We have

$$\begin{aligned}\sum_{v_i\neq w_{(p-1)t}} d_{ss}(v) &= \mu_{SS}(v_1w_{11})+\mu_{SS}(v_1w_{12})+\cdots+\mu_{SS}(v_1w_{1t})+\mu_{SS}(v_1w_{11})\\ &\quad+\mu_{SS}(v_2w_{11})+\cdots+\mu_{SS}(v_1w_{1t})+\mu_{SS}(v_2w_{1t})+\cdots\\ &\quad+\mu_{SS}(v_2w_{21})+\mu_{SS}(v_2w_{22})+\cdots+\mu_{SS}(v_2w_{2t})\\ &\quad+\mu_{SS}(v_2w_{21})+\mu_{SS}(v_3w_{21})+\cdots+\mu_{SS}(v_2w_{2t})\end{aligned}$$

$$+ \mu_{SS}(v_3 w_{2t}) + \cdots + \mu_{SS}(v_{(p-1)} w_{(p-1)1})$$
$$+ \mu_{SS}(v_{(p-1)} w_{(p-1)2}) + \cdots + \mu_{SS}(v_{(p-1)} w_{(p-1)t})$$
$$+ \mu_{SS}(v_{(p-1)} w_{(p-1)1}) + \mu_{SS}(v_p w_{(p-1)1}) + \cdots$$
$$+ \mu_{SS}(v_{(p-1)} w_{(p-1)t}) + \mu_{SS}(v_p w_{(p-1)t}) + \mu_{SS}(v_p w_{(p-1)1})$$
$$+ \cdots + \mu_{SS}(v_p w_{(p-1)t}).$$

$$\begin{aligned}\sum_{v_i \neq w_{(p-1)t}} d_{ss}(v) = 2[&\mu_{SS}(v_1 w_{11}) + \cdots + \mu_{SS}(v_1 w_1) + \mu_{SS}(v_2 w_{21}) \\ &+ \mu_{SS}(v_2 w_{22}) + \cdots + \mu_{SS}(v_2 w_{2t}) + \mu_{SS}(v_{(p-1)} w_{(p-1)1}) \\ &+ \mu_{SS}(v_{(p-1)} w_{(p-1)2}) + \cdots + \mu_{SS}(v_{(p-1)} w_{(p-1)t}) \\ &+ \mu_{SS}(v_p w_{(p-1)1}) + \cdots + \mu_{SS}(v_p w_{(p-1)t})].\end{aligned}$$

Therefore,

$$\sum_{(v_i \neq w_{(p-1)t})} d_{ss}(v) = 2 \sum_{(v_i \neq w_{(p-1)t})} \mu_{SS}(v_i w_{(p-1)t}).$$

□

Corollary 5.7. *The sum of the degrees of all the vertices is equal to twice the size of the fuzzy graph.*

Proof. Considering the fuzzy super subdivided graph $SS_f(G)$. The sum of the degree of the membership value of vertices v_i is $\sum_{(v_i \neq w_{(p-1)t})} d_{ss}(v_i)$. By the definition of the size of the super subdivided fuzzy graph,

$$S_{SS}(G) = \sum_{(v_i \neq w_{(p-1)t})} \mu_{SS}(v_i w_{(p-1)t}).$$

From the above theorem,

$$\sum_{(v_i \neq w_{(p-1)t})} d_{ss}(v) = 2 \sum_{(v_i \neq w_{(p-1)t})} \mu_{SS}(v_i w_{(p-1)t}).$$

Therefore, we have

$$\sum_{(v_i \neq w_{(p-1)t})} d_{ss}(v_i) = 2S_{SS}(G).$$

□

Proposition 5.8. *In every $SS_f(G)$, the order is greater than the size of the graph.*

Proof. Let $SS_f(G) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ be the fuzzy super subdivided graph. By the definition of the order and size of the fuzzy super subdivided graph, the order is

$$O_{SS}(G) = \sum_{(v_i \neq w_{(p-1)t})} \sigma_{SS}(v_i)$$

and the size is

$$S_{SS}(G) = \sum_{(v_i \neq w_{(p-1)t})} \mu_{SS}(v_i, w_{(p-1)t}).$$

Considering that the membership value of vertices is less than or equal to the membership value of edges, we have:

$$\sum_{(v_i \neq w_{(p-1)t})} \sigma_{SS}(v_i) \geq \sum_{(v_i \neq w_{(p-1)t})} \mu_{SS}(v_i, w_{(p-1)t}).$$

Hence,

$$O_{SS}(G) \geq S_{SS}(G).$$ □

6. Complete fuzzy super subdivided graph

Definition 6.1. The fuzzy super subdivided graph $SS_f(G)$ is said to be a complete fuzzy super subdivided graph $C_f(G)$ if

$$\mu_{SS}(v_i w_{(pt)}) = \min\{\sigma_{SS}(v_i), \sigma_{SS}(w_{(pt)})\}$$

where $v_i \in V$ and $w_{pt} \in V_{ss}$ for $1 \leq i \leq$, $1 \leq p \leq q$ and $1 \leq t \leq m$.

Example 6.2. Consider the complete fuzzy super subdivided triangle (k_3) graph $SS_f(k_3) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ in Figure 5.

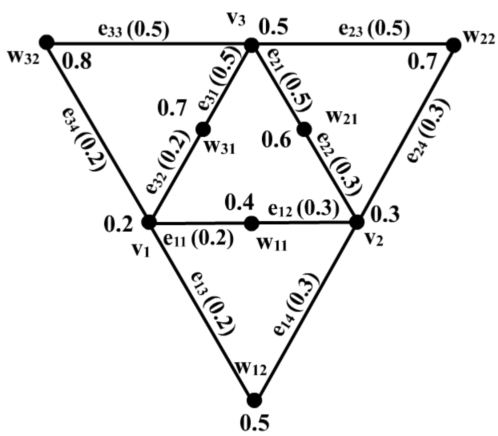


FIGURE 5. Complete Fuzzy super subdivided triangle $SS_f(k_3)$ for $k_{2,2}$

The edge membership values for the graph are assigned as follows (by Definition 3.1 of $SS_f(G)$)

$\min\{v_1(0.2), w_{11}(0.4)\} = e_{11}(0.2); \min\{v_2(0.3), w_{11}(0.4)\} = e_{12}(0.3)$

$\min\{v_1(0.2), w_{12}(0.5)\} = e_{13}(0.2); \min\{v_2(0.3), w_{12}(0.5)\} = e_{14}(0.3)$

$\min\{v_2(0.3), w_{21}(0.6)\} = e_{21}(0.3); \min\{v_3(0.5), w_{21}(0.6)\} = e_{22}(0.5)$

$\min\{v_2(0.3), w_{23}(0.7)\} = e_{23}(0.3); \min\{v_3(0.5), w_{22}(0.7)\} = e_{24}(0.5)$

$\min\{v_1(0.2), w_{31}(0.7)\} = e_{31}(0.2); \min\{v_3(0.5), w_{31}(0.7)\} = e_{32}(0.5)$

$$\min\{v_1(0.2), w_{32}(0.8)\} = e_{33}(0.2); \min\{v_3(0.5), w_{32}(0.8)\} = e_{34}(0.5)$$

These edge values satisfy the condition of complete fuzzy super subdivided graph.

Note 6.1. As all complete graphs are cycle, the super subdivided vertices of the complete fuzzy super subdivided graph is taken as w_{pt} for $1 \leq p \leq q$ and $1 \leq t \leq m$.

Theorem 6.3. *Every fuzzy super subdivided graph is a complete fuzzy super subdivided graph but the converse need not be true.*

Proof. Let $SS_f(G) = (V_{SS}, E_{SS}, \sigma_{SS}, \mu_{SS})$ be the fuzzy super subdivided graph where V_{SS} and E_{SS} are super subdivided vertices and edges. To prove: $SS_f(G)$ is a complete fuzzy super subdivided graph. By the definition of $SS_f(G)$ every edge in G is replaced with the complete bipartite graph $k_{(2,m)}$ for $m > 1$. This implies that there is an edge between each pair of vertices, forming a fully connected graph without loops. According to condition (ii) of Definition 3.1, $\mu_{SS}(v_i w_{(p-1)t}) = \sigma_{SS}(v_i) \wedge \sigma_{SS}(w_{(p-1)t})$ where $w_{(p-1)t} \in V_{ss}$, the equality condition holds for all pairs of vertices in the graph, establishing that the fuzzy super subdivided graph is a complete fuzzy graph.

Conversely, consider the complete fuzzy super subdivided graph $C_f(G)$, to prove it to be the fuzzy super subdivided graph. In the given $C_f(G)$, the condition

$$\mu_{SS}(v_i w_{(p-1)t}) = \sigma_{SS}(v_i) \wedge \sigma_{SS}(w_{(p-1)t})$$

already exists.

Case (i). Suppose the given $C_f(G)$ satisfies the condition

$$\sigma_{SS}(v_i) < \sigma_{SS}(w_{(p-1)t}) > \sigma_{SS}(v_j)$$

of Definition 3.1 of $SS_f(G)$, then the assumed complete fuzzy super subdivided graph is indeed a fuzzy super subdivided graph.

Case (ii). Suppose the given $C_f(G)$ does not satisfy the condition

$$\sigma_{SS}(v_i) < \sigma_{SS}(w_{(p-1)t}) > \sigma_{SS}(v_j)$$

of Definition 3.1 of $SS_f(G)$, in this case, the assumed complete fuzzy super subdivided graph is not a fuzzy super subdivided graph. □

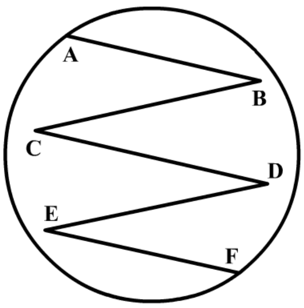
7. Application: Analysis of the infection's growth in urine

Mathematics applied to medical sciences and the use of mathematical models by medical researchers are gaining benefits to both sectors [19]. Fuzzy super subdivision, finds novel applications in medical research, particularly in urine culture testing.

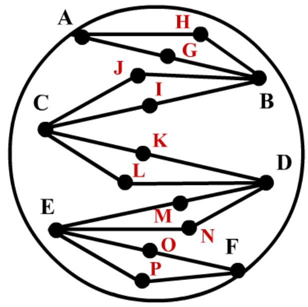
One of the most common diagnostic procedures conducted globally is Urinary microbiological culture [9, 17, 28]. The concept of fuzzy super subdivision model can be implemented in the culture test. In this application, the fuzzy super subdivision graph serves as a dynamic and detailed representation of bacterial colonies and their interactions in a urine culture. This model captures the temporal aspects of bacterial growth, allowing for the observation of changes in colony development.

Bacterial growth can be promoted by the nutrient media, including blood agar (SBA), cysteine lactose electrolyte deficient medium (CLED), Eosin methylene blue (EMB), and MacConkey (MAC) [24, 26]. Selective media like EMB and MAC are essential for fostering specific growth conditions. Once the growth-promoting substances and nutrient media are added into the system, bacteria or yeast within the culture exhibit distinct growth patterns.

In this model, urine samples are pictured as fuzzy graphs. The culture medium and the urine sample are represented as the vertices and edges of the fuzzy graph. By utilizing the nutrients in the culture medium, the growth of bacteria in the urine can be visualized and envisioned as fuzzy super subdivision. The membership values assigned to super subdivided vertices is the concentration of nutrients in the medium, while the membership values of super subdivided edges reflect the rate at which bacteria proliferate in the medium, forming colonies. The parameter “m” signifies the number of copies in the super subdivision, representing the count of bacterial colonies. Notably, the fuzzy super subdivision model employs unique pattern to prevent overlapping growth, providing non-overlapping and clearly defined bacterial colonies.



(A) Zig-Zag pattern in the culture medium



(B) Illustration of analysis of fuzzy super subdivision model in bacterial colony growth $SS_f(G)$

FIGURE 6. Illustration of visual and analytical representation of bacterial growth

In this experimental setup, we employ agar plate as a medium for bacterial growth. The culture medium in the agar plate is considered as vertices A, B, C, D, E and F. A loop with the urine sample is used to create a zigzag streaking pattern, which forms a fuzzy graph. The streaking pattern represents the isolated bacterial distribution. As bacteria in the sample utilize the nutrients in the medium, they initiate a super subdivision process. This process results in the formation of distinct colonies. The nutrient concentration in the medium evaluates the membership values of the super subdivided vertices and the super subdivided edges indicate the rate at which bacteria propagate in the medium, forming colonies. The increasing colony count of bacteria is represented by G, H, I, J, K, L, M, N, O and P, illustrating different stages of growth. Complex relationships within a urine culture can be represented and analyzed by fuzzy super subdivision model. This model allows us to produce visual and analytical representation of bacterial growth and interactions by observing their characteristics.

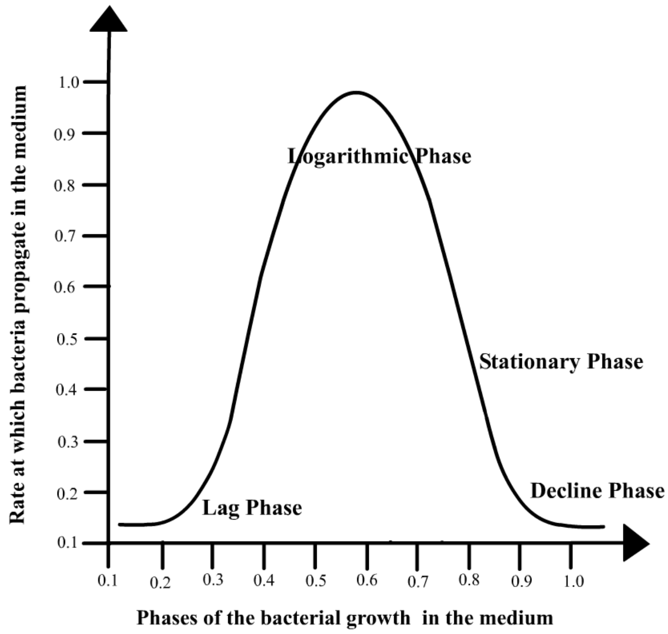


FIGURE 7. Graphical representation of phases of the bacterial growth of a fuzzy super subdivision model

By the above representation of colonial count of bacteria, the graph represents the rate at which the bacteria grow in each phase. The growth phases of bacterial colonies, including lag phase, logarithmic growth phase, and decline phase, can be represented (Table 1) as dynamic changes in the membership values over time. This adds a temporal dimension to the fuzzy super subdivision graph.

Table 1. Range of the membership values of bacterial colonies

Growth phases of bacterial colonies	**Number of Colonies**	**Growth Stages**
Lag phase	0.1 – 0.3	Little or no visible growth
Logarithmic phase	0.6 – 1.0	Rapid bacterial multiplication
Stationary phase	0.5 – 0.3	Relatively stable bacterial population
Decline phase	0.3 – 0.1	Decrease in the bacterial population.

The edge membership values are calculated in Table 2 below based on the rate at which the bacteria grow in the medium.

Table 2. Membership values of vertices and edges

Membership values of vertices	**Phases of the growth**	**Rate of the bacterial colony growth (Edge membership values)**
0.1	Lag Phase	0.1
0.2		0.1
0.3		0.2
0.4	Logarithmic Phase	0.6
0.5		0.9
0.6		1.0
0.7		0.8
0.8	Stationary Phase	0.5
0.9		0.2
1.0	Decline Phase	0.1

The information extracted from the model provides valuable insights for healthcare providers. Further integration with advanced diagnostic technologies can enhance the capabilities of fuzzy super subdivision in providing diagnostic insights. Therefore, with the analysis given above, using the fuzzy super subdivision model, the fastest probability and the infection level in the urine is stated.

8. Conclusion

In this paper, we have examined the concept of fuzzy super subdivision graph and its application in representing uncertain complex systems. Graph theory is extended with fuzzy graphs to address the vagueness of real-world problems.

In our discussion, we have established key properties and findings of fuzzy super subdivision graphs. We have demonstrated that every pair of adjacent vertices in a fuzzy graph becomes non-adjacent in its fuzzy super subdivision. Further, we have extended our study on super subdivision complete fuzzy graphs. The degree, order and size of the fuzzy super subdivision graph are analyzed and compared to gain insights on the structural properties of fuzzy super subdivision graphs. Furthermore, the application of the fuzzy super subdivision model is illustrated as infection growth analysis in the medical diagnosis domain. With fuzzy super subdivision models, we are able to interpret urine samples accurately, considering the complex and uncertain nature of urine analysis which improves decision-making processes.

In summary, fuzzy super subdivision graphs are newly introduced to advance the study of fuzzy graph theory, providing a deep and better understanding of complex structures and fuzzy graphs. This research also paves the way to investigate further and establish concepts on families of super subdivision graphs in fuzzy.

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