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KINEMATICAL INVARIANTS AND APPLICATIONS FOR SURFACES IN THREE DIMENSIONAL EUCLIDEAN SPACE

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Abstract. In three dimensional Euclidean space we consider kinematical invariants of the surface which is generated by the motion of a planar curve, especially, the surface which is foliated by circles. At first we characterize the properties of single parameter plane with the theories of unit spherical curve in three dimensional Euclidean space. Then using these results we give the invariants and differential invariants, kinematical properties and some special examples of the surface foliated by circles. The methods established here can be used to the other kinds of the surface in three dimensional Euclidean space.

1. Introduction

The ruled surfaces are the simplest foliated submanifolds. In [6, 12], the authors defined three invariants for the non-developable ruled surfaces, called structure functions in three dimensional Euclidean space and gave also the deep relationship between the structure functions and the kinematical characterization of the structure functions of the non-developable ruled surface. Some properties and applications of these notions and theories are also given in [2–7, 12]. The structure functions of the non-developable ruled surface are related to the motion of the ruling of ruled surface and are differential invariant. By integration they are strongly related to the integral invariants given by Müller [8] in 1951 and Pottmann and Röschel [9] in 1988.

In this paper, we consider kinematical invariants of the surface which is generated by the motion of a planar curve. Especially we study the surfaces which are foliated by circles in three dimensional Euclidean space using the different ideas and methods [1, 10, 11]. Since the surfaces foliated by circles are

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generated by single parameter circles, at first we characterize the properties of single parameter plane with the theories of unit spherical curve in three dimensional Euclidean space and define associated intersection ruled surfaces of this single parameter plane. Then using these notions and conclusions we give the invariants and differential invariants of the surface foliated by circles and discuss the properties, kinematical characterization and some special examples of the surface foliated by circles. The methods established here can be used to study the surface which are foliated by a planar curve in three dimensional Euclidean space.

2. Preliminaries

In this section we briefly summarize the main notions and formulas of the space curves and spherical curves in three dimensional Euclidean space (simply Euclidean 3-space) for the convenient usage in the coming sections. We denote the Euclidean 3-space by $\mathbb{E}^3$ and standard unit sphere in $\mathbb{E}^3$ by $\mathbb{S}^2 = \mathbb{S}^2(1) \subset \mathbb{E}^3$.

Let $r(s) : \mathbf{I} \to \mathbb{E}^3$ be a regular space curve in Euclidean 3-space $\mathbb{E}^3$ parameterized by its arc length s. Denote the Frenet frame field along $r(s)$ by $\{\alpha(s), \beta(s), \gamma(s)\}$, that is, $\alpha(s) = \dot{r}(s)$ is the tangent vector field, $\beta(s) = |\dot{\alpha}(s)|^{-1}\dot{\alpha}(s)$ the (principal) normal vector field and $\gamma(s) = \alpha(s) \times \beta(s)$ the binormal vector field of $r(s)$, respectively. Here we use "dot" to denote the derivative with respect to the arc length parameter of the curve $r(s)$, and $\times$ the vector product of two vectors in $\mathbb{E}^3$. The famous Frenet formulas of the curve $r(s)$ are given by

$$\begin{cases} \dot{\alpha}(s) = \kappa(s)\beta(s), \\ \dot{\beta}(s) = -\kappa(s)\alpha(s) + \tau(s)\gamma(s), \\ \dot{\gamma}(s) = -\tau(s)\beta(s), \end{cases} \tag{2.1}$$

where $\kappa(s)$ and $\tau(s)$ are the curvature function and torsion function of the curve $r(s)$ in $\mathbb{E}^3$.

If $r(s) = x(s) : \mathbf{I} \to \mathbb{S}^2 \subset \mathbb{E}^3$ is a unit spherical curve, let $\alpha_0(s) := \dot{x}(s)$ and $y(s) := \alpha_0(s) \times x(s)$. Then $\alpha_0(s)$, $x(s)$ and $y(s)$ form an orthonormal basis along the curve $x(s)$ in $\mathbb{E}^3$. We call $\{\alpha_0(s), x(s), y(s)\}$ the spherical Frenet frame of (unit) spherical curve $x(s)$ in $\mathbb{E}^3$. Then there exists a function $\kappa_g(s)$ such that

$$\begin{cases} \dot{\alpha}_0(s) = -x(s) + \kappa_g(s)y(s), \\ \dot{x}(s) = \alpha_0(s), \\ \dot{y}(s) = -\kappa_g(s)\alpha_0(s). \end{cases} \tag{2.2}$$

We call $\kappa_g(s)$ spherical curvature function of the (unit) spherical curve $x(s)$ in $\mathbb{E}^3$. In the following we call the unit spherical curve simply as the spherical curve. By a direct calculation or from (2.2) we have

$$y(s) = \frac{1}{\sqrt{\langle \ddot{x}(s), \ddot{x}(s) \rangle - 1}}[x(s) + \ddot{x}(s)]. \tag{2.3}$$

3. Spherical curve and Frenet moving frame

In this section we give some conclusions and properties which are strongly related to the one parameter moving frame which is considered as the spherical Frenet frame of the unit spherical curve in Euclidean 3-space.

For the unit spherical curve $x(u) : \mathbf{I} \to \mathbb{S}^2 \subset \mathbb{E}^3$, choosing u as the arc length parameter of $x(u)$ and putting $x'(u) = \alpha_0(u)$, $y(u) = \alpha_0(u) \times x(u)$, then the *spherical Frenet formulas* of unit spherical curve $x(u)$ can be written as

$$\begin{cases} x'(u) = x(u) \times \Omega(u) = \alpha_0(u), \\ \alpha_0'(u) = \alpha_0(u) \times \Omega(u) = -x(u) + \kappa_g(u)y(u), \\ y'(u) = y(u) \times \Omega(u) = -\kappa_g(u)\alpha_0(u), \end{cases} \tag{3.1}$$

where $\Omega(u) = \kappa_g(u)x(u) + y(u)$ is Darboux vector field of the frame $\{\alpha_0(u), x(u), y(u)\}$. The function $\kappa_g(u)$ is called the *spherical curvature function*, and the moving frame $\{\alpha_0(u), x(u), y(u)\}$ is called the *spherical Frenet frame* of (unit) spherical curve $x(u)$ ([6]).

Let $a(u) : \mathbf{I} \to \mathbb{E}^3$ be any regular curve in $\mathbb{E}^3$. We consider the motion of the frame $\{\alpha_0(u), x(u), y(u)\}$ along the curve $a(u)$. We take

$$\begin{aligned} a'(u) &= \frac{\mathrm{d}a(u)}{\mathrm{d}u} \\ &= [a'(u) \cdot \alpha_0(u)]\alpha_0(u) + [a'(u) \cdot x(u)]x(u) \\ &\quad + [a'(u) \cdot y(u)]y(u) \\ &= (a'\alpha_0)\alpha_0 + (a'x)x + (a'y)y \\ &= \xi\alpha_0 + \eta x + \psi y, \end{aligned} \tag{3.2}$$

where

$$\begin{cases} \xi = \xi(u) = a(u)' \cdot \alpha_0(u) = a'\alpha_0, \\ \eta = \eta(u) = a'(u) \cdot x(u) = a'x, \\ \psi = \psi(u) = a'(u) \cdot y(u) = a'y. \end{cases} \tag{3.3}$$

We use $\pi_1(u)$ to denote the plane spanned by $y(u)$ and $\alpha_0(u)$, $\pi_2(u)$ the plane spanned by $\alpha_0(u)$ and $x(u)$, $\pi_3(u)$ the plane spanned by $x(u)$ and $y(u)$.

From the point $a(u_0)$ to point $a(u_0 + \Delta u)$, we denote the straight line of the intersection of two planes $\pi_i(u_0)$ and $\pi_i(u_0 + \Delta u)$ by $l_i(u_0 + \Delta u)$, $i = 1, 2, 3$.

Definition 3.1. The ruled surface $Y_i(u, v)$ generated by

$$l_i(u) = \lim_{\Delta u \to 0} l_i(u + \Delta u) \tag{3.4}$$

is called the associated intersection ruled surface of $\pi_i(u)$ $(i = 1, 2, 3)$. The straight line $l_i(u)$ is called rolling line (or axis line) of $\pi_i(u)$ or the moving frame $\{\alpha_0(u), x(u), y(u)\}$ at u $(i = 1, 2, 3)$.

The unit vector of the direction of the intersection line $l_1(u_0 + \Delta u)$ is

$$\frac{x(u_0 + \Delta u) \times x(u_0)}{|x(u_0 + \Delta u) \times x(u_0)|} = \frac{[x(u_0 + \Delta u) - x(u_0)] \times x(u_0)}{|[x(u_0 + \Delta u) - x(u_0)] \times x(u_0)|}.$$

Therefore we have

$$
\begin{aligned}
(3.5)\qquad & \lim_{\Delta u\to 0}\frac{x(u_0+\Delta u)\times x(u_0)}{|x(u_0+\Delta u)\times x(u_0)|}\\
&= \lim_{\Delta u\to 0}\frac{[x(u_0+\Delta u)-x(u_0)]\times x(u_0)}{|[x(u_0+\Delta u)-x(u_0)]\times x(u_0)|}\\
&= \frac{\alpha_0(u_0)\times x(u_0)}{|\alpha_0(u_0)\times x(u_0)|}\\
&= y(u_0).
\end{aligned}
$$

Proposition 3.1. *Associated intersection ruled surface of* $\pi_1(u)$ *can be written as*

$$
\begin{aligned}
Y_1(u,v) &= a(u)+[a'(u)\cdot x(u)]\alpha_0(u)+vy(u)\\
&= a(u)+\eta(u)\alpha_0(u)+vy(u)
\end{aligned}
\tag{3.6}
$$

and is always developable.

Proof. Let Z be the point on the intersection line $l_1(u+\Delta u)$ of two planes $\pi_1(u)$ and $\pi_1(u+\Delta u)$. Then

$$\{Z-a(u)\}\cdot x(u)=0, \tag{3.7}$$

$$\{Z-a(u+\Delta u)\}\cdot x(u+\Delta u)=0. \tag{3.8}$$

Using

$$
\begin{cases}
a(u+\Delta u)=a(u)+a'(u)\Delta u+o(\Delta u^2),\\
x(u+\Delta u)=x(u)+\alpha_0(u)\Delta u+o(\Delta u^2),
\end{cases}
\tag{3.9}
$$

and (3.7), from (3.8) we have

$$\{Z-a(u)\}\cdot\alpha_0(u)-a'(u)\cdot x(u)+o(\Delta u)=0. \tag{3.10}$$

Therefore we get (3.6).

For the developable condition of the ruled surface $Y_1(u,v)$, we have

$$
\begin{aligned}
&\{a(u)+[a'(u)\cdot x(u)]\alpha_0(u)\}'\cdot y(u)\times y'(u)\\
&= [a+(a'\cdot x)\alpha_0]'\cdot y\times(-\kappa_g\alpha_0)\\
&= -\kappa_g[a+(a'\cdot x)\alpha_0]'\cdot x\\
&= -\kappa_g[(a'\cdot x)-(a'\cdot x)]\\
&= 0.
\end{aligned}
$$

Therefore $Y_1(u,v)$ is always a developable ruled surface. □

The striction line of $Y_1(u,v)$ is

$$
\begin{aligned}
A_1(u) = a(u)&+[a'(u)\cdot x(u)]\alpha_0(u)\\
&+\frac{a'(u)\cdot\alpha_0(u)+[a'(u)\cdot x(u)]'}{\kappa_g(u)}y(u)
\end{aligned}
\tag{3.11}
$$

$$= a(u) + \eta(u)\alpha_0(u) + \frac{\xi(u) + \eta'(u)}{\kappa_g(u)} y(u).$$

By a direct calculation we have

$$A_1'(u) = \psi y + \eta \kappa_g y + \left[\frac{\xi(u) + \eta'(u)}{\kappa_g(u)}\right]' y. \tag{3.12}$$

Proposition 3.2. *Associated intersection ruled surface of $\pi_2(u)$ can be written as*

$$\begin{aligned} Y_2(u,v) &= a(u) - \kappa_g^{-1}(u)[a'(u) \cdot y(u)]\alpha_0(u) + vx(u) \\ &= a(u) - \kappa_g^{-1}(u)\psi(u)\alpha_0(u) + vx(u) \end{aligned} \tag{3.13}$$

and is always developable, where $\kappa_g(u) \neq 0$.

Proof. The direction of the ruling $l_2(u)$ of $Y_2(u,v)$ is

$$\begin{aligned} &\lim_{\Delta u \to 0} \frac{y(u_0 + \Delta u) \times y(u_0)}{|y(u_0 + \Delta u) \times y(u_0)|} \\ =\ &\lim_{\Delta u \to 0} \frac{[y(u_0 + \Delta u) - y(u_0)] \times y(u_0)}{|[y(u_0 + \Delta u) - y(u_0)] \times y(u_0)|} \\ =\ &\frac{-\kappa_g(u_0)\alpha_0(u_0) \times y(u_0)}{|-\kappa_g(u_0)\alpha_0(u_0) \times y(u_0)|} \\ =\ &\pm x(u_0). \end{aligned} \tag{3.14}$$

Let Z be the point on the intersection line $l_2(u + \Delta u)$ of two planes $\pi_2(u)$ and $\pi_2(u + \Delta u)$. Then

$$\{Z - a(u)\} \cdot y(u) = 0, \tag{3.15}$$

$$\{Z - a(u + \Delta u)\} \cdot y(u + \Delta u) = 0. \tag{3.16}$$

Using

$$\begin{cases} a(u + \Delta u) = a(u) + a'(u)\Delta u + o(\Delta u^2), \\ y(u + \Delta u) = y(u) - \kappa_g(u)\alpha_0(u)\Delta u + o(\Delta u^2), \end{cases} \tag{3.17}$$

and (3.15), from (3.16) we have

$$-\kappa_g(u)\{Z - a(u)\} \cdot \alpha_0(u) - a'(u) \cdot y(u) + o(\Delta u) = 0. \tag{3.18}$$

Therefore we get (3.13).

For the developable condition of the ruled surface $Y_2(u,v)$, we have

$$\begin{aligned} &\{a(u) - \kappa_g^{-1}(u)[a'(u) \cdot y(u)]\alpha_0(u)\}' \cdot x(u) \times x'(u) \\ &= [a - \kappa_g^{-1}(a' \cdot y)\alpha_0]' \cdot x \times \alpha_0 \\ &= -[a - \kappa_g^{-1}(a' \cdot y)\alpha_0]' \cdot y \\ &= -[(a' \cdot y) - (a' \cdot y)] \\ &= 0. \end{aligned}$$

Therefore $Y_2(u,v)$ is always a developable ruled surface. □

The striction line of $Y_2(u,v)$ is

$$A_2(u) = a(u) - \kappa_g^{-1}(u)\psi(u)\alpha_0(u) - \{a'(u)\cdot\alpha_0(u) - [\kappa_g^{-1}(u)\psi(u)]'\}x(u) = a(u) - \kappa_g^{-1}(u)\psi(u)\alpha_0(u) - \{\xi(u) - [\kappa_g^{-1}(u)\psi(u)]'\}x(u). \tag{3.19}$$

By a direct calculation we have

$$A_2'(u) = \eta(u)x(u) + \kappa_g^{-1}(u)\psi(u)x(u) - \{\xi(u) - [\kappa_g^{-1}(u)\psi(u)]'\}'x(u). \tag{3.20}$$

Proposition 3.3. *Associated intersection ruled surface of $\pi_3(u)$ can be written as*

$$Y_3(u,v) = a(u) + \left[\frac{a'(u)\cdot\alpha_0(u)}{\sqrt{1+\kappa_g^2(u)}}\right]\left[\frac{-x(u)+\kappa_g(u)y(u)}{\sqrt{1+\kappa_g^2(u)}}\right] + v\Omega_0(u) = a(u) + \left[\frac{\xi(u)}{1+\kappa_g^2(u)}\right][-x(u)+\kappa_g(u)y(u)] + v\Omega_0(u) \tag{3.21}$$

and is always developable, where $\Omega = \sqrt{1+\kappa_g^2}\Omega_0 = \kappa_g(u)x(u) + y(u)$.

Proof. The direction of the ruling $l_3(u)$ of $Y_3(u,v)$ is

$$\lim_{\Delta u\to 0}\frac{\alpha_0(u_0+\Delta u)\times\alpha_0(u_0)}{|\alpha_0(u_0+\Delta u)\times\alpha_0(u_0)|} = \lim_{\Delta u\to 0}\frac{[\alpha_0(u_0+\Delta u)-\alpha_0(u_0)]\times\alpha_0(u_0)}{|[\alpha_0(u_0+\Delta u)-\alpha_0(u_0)]\times\alpha_0(u_0)|} = \frac{[-x(u_0)+\kappa_g(u_0)y(u_0)]\times\alpha_0(u_0)}{|[-x(u_0)+\kappa_g(u_0)y(u_0)]\times\alpha_0(u_0)|} = \frac{\kappa_g(u_0)x(u_0)+y(u_0)}{\sqrt{1+\kappa_g^2(u_0)}} = \Omega_0(u_0). \tag{3.22}$$

Let Z be the point on the intersection line $l_3(u+\Delta u)$ of two planes $\pi_3(u)$ and $\pi_3(u+\Delta u)$. Then

$$\{Z - a(u)\}\cdot\alpha_0(u) = 0, \tag{3.23}$$

$$\{Z - a(u+\Delta u)\}\cdot\alpha_0(u+\Delta u) = 0. \tag{3.24}$$

Using

$$\begin{cases} a(u+\Delta u) = a(u) + a'(u)\Delta u + o(\Delta u^2), \\ \alpha_0(u+\Delta u) = \alpha_0(u) + [-x(u)+\kappa_g(u)y(u)]\Delta u + o(\Delta u^2), \end{cases} \tag{3.25}$$

and (3.23), from (3.24) we have

$$\{Z - a(u)\}\cdot[-x(u)+\kappa_g(u)y(u)] - a'(u)\cdot\alpha_0(u) + o(\Delta u) = 0. \tag{3.26}$$

Therefore we get (3.21).

For the developable condition of the ruled surface $Y_3(u,v)$, we have

$$\begin{aligned}&\left\{a(u)+\frac{a'(u)\cdot\alpha_0(u)}{1+\kappa_g^2(u)}[-x(u)+\kappa_g(u)y(u)]\right\}'\cdot\Omega_0(u)\times\Omega_0'(u)\\ =&-\kappa_g'(1+\kappa_g^2)^{-1}\left\{a(u)+\frac{a'(u)\cdot\alpha_0(u)}{1+\kappa_g^2(u)}[-x(u)+\kappa_g(u)y(u)]\right\}'\cdot\alpha_0\\ =&-\kappa_g'(1+\kappa_g^2)^{-1}[(a'\cdot\alpha_0)-(a'\cdot\alpha_0)]\\ =&\ 0.\end{aligned}$$

Therefore $Y_3(u,v)$ is always a developable ruled surface. □

Since

$$\Omega_0'=\kappa_g'(1+\kappa_g^2)^{-\frac{3}{2}}(x-\kappa_g y).$$

The striction line of $Y_3(u,v)$ is

$$\begin{aligned}A_3(u)=&\ a(u)+\left[\frac{\xi(u)}{1+\kappa_g^2(u)}\right][-x(u)+\kappa_g(u)y(u)]\\ &-[\kappa_g'(u)]^{-1}\left[1+\kappa_g^2(u)\right]^{\frac{1}{2}}\left\{\eta(u)-\left[1+\kappa_g(u)^2\right]\left[\frac{\xi(u)}{1+\kappa_g^2(u)}\right]'\right.\\ &\left.-\kappa_g(u)\psi(u)-\kappa_g(u)\kappa_g'(u)\left[\frac{\xi(u)}{1+\kappa_g^2(u)}\right]\right\}\Omega_0(u).\end{aligned}\tag{3.27}$$

By a direct calculation we have

$$\begin{aligned}A_3'(u)=&\left[1+\kappa_g^2(u)\right]^{-\frac{1}{2}}\left\{\psi(u)+\eta(u)\kappa_g(u)+\kappa_g'(u)\left[\frac{\xi(u)}{1+\kappa_g^2(u)}\right]\right\}\\ &\times\Omega_0(u)-F'(u)\Omega_0(u),\end{aligned}\tag{3.28}$$

where

$$\begin{aligned}F(u)=&\ [\kappa_g'(u)]^{-1}\left[1+\kappa_g^2(u)\right]^{\frac{1}{2}}\times\\ &\left\{\eta(u)-\left[1+\kappa_g(u)^2\right]\left[\frac{\xi(u)}{1+\kappa_g^2(u)}\right]'\right.\\ &\left.-\kappa_g(u)\psi(u)-\kappa_g(u)\kappa_g'(u)\left[\frac{\xi(u)}{1+\kappa_g^2(u)}\right]\right\}.\end{aligned}\tag{3.29}$$

Proposition 3.4. *Let* $\mathrm{dist}_{u_0}(l_i,l_j)=\mathrm{dist}(l_i(u_0),l_j(u_0))$. *Denote the signed distance of the rolling lines from* $l_i(u_0)$ *to* $l_j(u_0)$, $i\neq j$, $i,j=1,2,3$. *Then*

$$\mathrm{dist}_{u_0}(l_1,l_2)=\eta(u_0)+\frac{\psi(u_0)}{\kappa_g(u_0)},\tag{3.30}$$

$$\mathrm{dist}_{u_0}(l_2,l_3)=\frac{\psi(u_0)}{\kappa_g(u_0)\sqrt{1+\kappa_g^2(u_0)}},\tag{3.31}$$

$$\operatorname{dist}_{u_0}(l_3, l_1) = \frac{\kappa_g(u_0)\eta(u_0)}{\sqrt{1+\kappa_g^2(u_0)}}. \tag{3.32}$$

Proposition 3.5. *The rolling lines $l_1(u)$ and $l_2(u)$ are always perpendicular. The intersection angle of rolling lines $l_1(u)$ and $l_3(u)$ is*

$$\operatorname{angle}_u(l_1, l_3) = \operatorname{angle}(l_1(u), l_3(u)) = \arccos\left(\frac{1}{\sqrt{1+\kappa_g^2(u)}}\right) \tag{3.33}$$

and the intersection angle of rolling lines $l_2(u)$ and $l_3(u)$ is

$$\operatorname{angle}_u(l_2, l_3) = \operatorname{angle}(l_2(u), l_3(u)) = \arccos\left(\frac{\kappa_g(u)}{\sqrt{1+\kappa_g^2(u)}}\right). \tag{3.34}$$

4. Surfaces foliated by circles.

In this section we consider the surfaces foliated by circles in Euclidean 3-space $\mathbb{E}^3$. We denote the regular unit spherical curve with arc length parameter u on $\mathbb{S}^2 \subset \mathbb{E}^3$ by $x(u)$ and the regular parameter surface with parameters u and v in $\mathbb{E}^3$ by $X(u,v)$. We use the notions and conclusions as in above section for the spherical Frenet moving frame $\{\alpha_0(u), x(u), y(u)\}$ of the spherical curve $x(u)$.

Definition 4.1. If the surface $X(u,v)$ is foliated by circles in Euclidean 3-space $\mathbb{E}^3$, denote the center curve of the circles by $a(u)$, the unit normal vector field of the planes that the circles lie in by $x(u)$ and the spherical Frenet moving frame of $x(u)$ by $\{\alpha_0(u), x(u), y(u)\}$. Then $X(u,v)$ can be written as

$$X(u,v) = a(u) + \rho(u)[(\cos v)\alpha_0(u) + (\sin v)y(u)]. \tag{4.1}$$

This expression is called standard equation of the surfaces foliated by circles in Euclidean 3-space. The surface foliated by circles is simply called circled surface in Euclidean 3-space $\mathbb{E}^3$.

Remark 4.1. For the unit vector field $x(u)$, it is usually considered as a spherical curve. In a very special case, if $x(u)$ is a constant vector, then we know that the circled surface $X(u,v)$ is generated by parallel circles. Especially if $X(u,v)$ is rotational surface, the curve $a(u)$ is a straight line. In the case of that $x(u)$ is a constant vector, $X(u,v)$ is called parallel circled surface and can be written as (4.1) with α_0 and y are constant vectors. Therefore in the following we always assume that $x(u)$ is not a constant vector. That means we always assume that the circled surface $X(u,v)$ is not generated by parallel circles in this paper.

From the definition and standard equation (4.1) of the circled surface $X(u,v)$, we know that the function $\rho(u)$ is the radius of the circles which generate the surface. Obviously $\rho(u)$ is an invariant of $X(u,v)$. In the following we study another invariants of the circled surface $X(u,v)$.

Denote the unit normal vector field of $X(u,v)$ by

$$N = N(u,v) = \frac{X_u \times X_v}{|X_u \times X_v|},$$

and also put

$$\begin{aligned} a'(u) &= [a'(u)\cdot\alpha_0(u)]\alpha_0(u) + [a'(u)\cdot x(u)]x(u) + [a'(u)\cdot y(u)]y(u) \\ &= \xi(u)\alpha_0(u) + \eta(u)x(u) + \psi(u)y(u). \end{aligned} \tag{4.2}$$

Then we have

$$\left\{\begin{aligned} X_u =&\ a' + \rho'[(\cos v)\alpha_0 + (\sin v)y] + \rho[(\cos v)\alpha_0' + (\sin v)y'] \\ =&\ a' + [\rho'\cos v - \kappa_g\rho\sin v]\alpha_0 - \rho(\cos v)x \\ &+[\rho'\sin v + \kappa_g\rho\cos v]y \\ =&\ (a'\alpha_0 + \rho'\cos v - \kappa_g\rho\sin v)\alpha_0 + (a'x - \rho\cos v)x \\ &+(a'y + \rho'\sin v + \kappa_g\rho\cos v)y \\ =&\ (\xi + \rho'\cos v - \kappa_g\rho\sin v)\alpha_0 + (\eta - \rho\cos v)x \\ &+(\psi + \rho'\sin v + \kappa_g\rho\cos v)y, \\ X_v =&\ \rho[-(\sin v)\alpha_0 + (\cos v)y], \\ DN =&\ \rho\cos v(a'x - \rho\cos v)\alpha_0 - \rho[(a'y)\sin v + (a'\alpha_0)\cos v + \rho']x \\ &+\rho\sin v(a'x - \rho\cos v)y \\ =&\ \rho(a'x - \rho\cos v)(\alpha_0\cos v + y\sin v) \\ &-\rho[(a'y)\sin v + (a'\alpha_0)\cos v + \rho']x \\ =&\ \rho(\eta - \rho\cos v)(\alpha_0\cos v + y\sin v) - \rho(\psi\sin v + \xi\cos v + \rho')x, \end{aligned}\right. \tag{4.3}$$

where

$$\begin{aligned} D &= D(u,v) = |X_u \times X_v| \\ &= \rho\sqrt{(a'x - \rho\cos v)^2 + [(a'y)\sin v + (a'\alpha_0)\cos v + \rho']^2} \\ &= \rho\sqrt{(\eta - \rho\cos v)^2 + (\psi\sin v + \xi\cos v + \rho')^2} > 0. \end{aligned} \tag{4.4}$$

$$N(u,v) = \frac{(\eta - \rho\cos v)(\alpha_0\cos v + y\sin v) - (\psi\sin v + \xi\cos v + \rho')x}{\sqrt{(\eta - \rho\cos v)^2 + (\psi\sin v + \xi\cos v + \rho')^2}}. \tag{4.5}$$

Then the first fundamental form of $X(u,v)$ is

$$\left\{\begin{aligned} I =&\ E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2, \\ E =&\ (\xi + \rho'\cos v - \kappa_g\rho\sin v)^2 + (\eta - \rho\cos v)^2 \\ &+(\psi + \rho'\sin v + \kappa_g\rho\cos v)^2, \\ F =&\ -\rho\sin v(\xi + \rho'\cos v - \kappa_g\rho\sin v) \\ &+\rho\cos v(\psi + \rho'\sin v + \kappa_g\rho\cos v) \\ =&\ \rho^2\kappa_g - \rho\xi\sin v + \rho\psi\cos v, \\ G =&\ \rho^2. \end{aligned}\right. \tag{4.6}$$

At first we consider the rotation of the points along the circle in the plane. We give the following definition.

Definition 4.2. Let $X(u,v)$ be the surface foliated by circles in $\mathbb{E}^3$ and $x(u)$ the unit normal vector field of the planes that the circles lie in. The rotational density function of the circled surface $X(u,v)$ is defined by the rotational angle density of the circles

$$\varphi(u) = -[x'(u) \times x(u)]' \cdot x'(u) = -y'(u) \cdot \alpha_0(u) = \kappa_g(u). \tag{4.7}$$

Remark 4.2. The rotational angle density function of the unit vector field is defined in [12, Definition 2.2]. The function $\varphi(u)$ characterizes the state of points on the circles since the points on the circles can rotate with respect to the axis $x(u)$.

Now we consider the curves which intersect all circles on the circled surface $X(u,v)$.

Definition 4.3. The striction point $S_-(u_0)$ of the circled surface $X(u,v)$ at point u_0 in $\mathbb{E}^3$ is defined by

$$\begin{aligned} &S_-(u_0) \\ &= \lim_{\Delta u \to 0} \left\{ X(u_0 + \Delta u, v_{\min}) \;\middle|\; \min_{v_1, v_2 \in [0,2\pi]} \operatorname{dist}(X(u_0+\Delta u, v_1), X(u_0, v_2)) \right\}. \end{aligned} \tag{4.8}$$

The striction point $S_+(u_0)$ of $X(u,v)$ at u_0 is defined by

$$S_+(u_0) = \lim_{\Delta u \to 0} \left\{ X(u_0 + \Delta u, v) \;\middle|\; \max_{v \in [0,2\pi]} \operatorname{dist}(X(u_0+\Delta u, v), X(u_0, v)) \right\}, \tag{4.9}$$

where dist denotes the distance of two points $X(u_0+\Delta u, v_1)$ and $X(u+\Delta u, v_2)$, that is,

$$\operatorname{dist}(X(u_0+\Delta u, v_1), X(u_0, v_2)) = |X(u_0+\Delta u, v_1) - X(u+\Delta u, v_2)|.$$

Proposition 4.1. *The two striction curves* (*maximal and minimal*) *of the circled surface* $X(u,v)$ *are*

$$S_\pm(u) = a(u) \pm \rho(u)\alpha_0(u) = a(u) + \varepsilon\rho(u)\alpha_0(u) = S_\varepsilon(u). \tag{4.10}$$

Proof. By Proposition 3.1 we know that the striction curves of $X(u,v)$ can be written as (4.10). In this case, by a parameter transformation $s \to -s$ if necessary, we assume that the direction of the vector $\alpha_0(u)$ is always pointing to the anti-direction of the intersection line $l_1(u)$ (cf. Definition 3.1) of two planes. □

Remark 4.3. From (4.5), if ρ is constant, we know that

$$\lim_{v \to -\frac{\pi}{2}} N(u,v) = -\lim_{v \to \frac{\pi}{2}} N(u,v).$$

Then we can also define the striction points by

$$S_-(u_0) = \lim_{v \to 0} X(u_0, v)$$

and

$$S_+(u_0) = \lim_{v\to\pi} X(u_0, v).$$

Therefore we have also

$$S_\pm(u) = a(u) \pm \rho(u)\alpha_0(u).$$

Now we consider the motion of the planes that the circles lie in. This can be characterized by the changes of the centers of the circles.

Definition 4.4. For the circled surface $X(u,v)$, at the point u_0, (signed) translation density of circle center along the direction of the intersection $l_1(u_0)$ of two planes $\pi_1(u_0)$ and $\pi_1(u_0+\Delta u)$ is defined by

$$\begin{aligned}(4.11)\qquad &\lim_{\Delta u\to 0}\frac{a(u_0+\Delta u)-a(u_0)}{\Delta u}\cdot\frac{x(u_0+\Delta u)\times x(u_0)}{|x(u_0+\Delta u)\times x(u_0)|}\\ &= a'(u_0)\cdot y(u_0) = \psi(u_0).\end{aligned}$$

The function $\psi(u)$ is called first translation density function (or first kind of translation density function).

Definition 4.5. For the circled surface $X(u,v)$, at the point u_0, (signed) translation density of circle center along the direction of the intersection $l_2(u_0)$ of two planes $\pi_2(u_0)$ and $\pi_2(u_0+\Delta u)$ is defined by

$$\begin{aligned}(4.12)\qquad &\lim_{\Delta u\to 0}\frac{a(u_0+\Delta u)-a(u_0)}{\Delta u}\cdot\frac{y(u_0+\Delta u)\times y(u_0)}{|y(u_0+\Delta u)\times y(u_0)|}\\ &= \pm\, a'(u_0)\cdot x(u_0) = \pm\eta(u_0).\end{aligned}$$

The function $\eta(u)$ is called second translation density function (or second kind of translation density function).

Definition 4.6. For the circled surface $X(u,v)$, at the point u_0, (signed) translation density of circle center along the common perpendicular direction of l_1 and l_2 i.e. $\alpha_0(u)$ direction is defined by

$$\begin{aligned}(4.13)\qquad &\lim_{\Delta u\to 0}\frac{a(u_0+\Delta u)-a(u_0)}{\Delta u}\cdot\frac{x(u_0+\Delta u)\times[x(u_0+\Delta u)\times x(u_0)]}{|x(u_0+\Delta u)\times[x(u_0+\Delta u)\times x(u_0)]|}\\ &= a'(u_0)\cdot\alpha_0(u_0) = \xi(u_0).\end{aligned}$$

The function $\xi(u)$ is called third translation density function (or third kind of translation density function).

Definition 4.7. Associated ruled surfaces of the circled surface $X(u,v)$ are defined by (3.6), (3.13) and (3.21). Associated ruled surface $Y_1(u,v)$ defined by (3.6) is called axis ruled surface of $X(u,v)$. Associated ruled surface $Y_2(u,v)$ defined by (3.13) is called position ruled surface of $X(u,v)$. Associated ruled surface $Y_3(u,v)$ defined by (3.21) is called Darboux ruled surface of $X(u,v)$.

Proposition 4.2. *If the center curve $a(u)$ of $X(u,v)$ is the striction line of the axis ruled surface $Y_1(u,v)$, then we have $a'\cdot\alpha_0 = \xi = a'\cdot x = \eta \equiv 0$.*

Proof. By (3.11) we can easily get the conclusion. □

Proposition 4.3. *If the center curve $a(u)$ of $X(u,v)$ is the striction line of the position ruled surface $Y_2(u,v)$, then we have $a' \cdot \alpha_0 = \xi = a' \cdot y = \psi \equiv 0$.*

Proof. By (3.19) we can get the conclusion. □

Proposition 4.4. *If the center curve $a(u)$ of $X(u,v)$ is the striction line of the Darboux ruled surface $Y_3(u,v)$, then we have $a' \cdot \alpha_0 = \xi = a' \cdot x = \eta = a' \cdot y = \psi \equiv 0$. In this case $a(u)$ is a constant vector, i.e. the center curve $a(u)$ of $X(u,v)$ is a point.*

Proof. By (3.27) we can get the conclusion. □

5. Some special surfaces foliated by circles

In this section, as the examples, we consider three kinds of special circled surfaces in Euclidean 3-space $\mathbb{E}^3$. Usually they have the explicitly geometric characteristics.

5.1. Osculating circle surfaces

At first we consider the special circled surface which is generated by the circles they lie in the osculating planes of a curve in $\mathbb{E}^3$.

Definition 5.1. If the center curve $a(u)$ of the circled surface $X(u,v)$ is the striction line of its axis ruled surface $Y_1(u,v)$, then the surface $X(u,v)$ is called osculating circle surface in Euclidean 3-space $\mathbb{E}^3$.

For the osculating circle surfaces, from Definition 5.1 and Proposition 3.1 we know that

$$a'(u) = \psi(u)y(u). \tag{5.1}$$

Therefore we have

Proposition 5.1. *The circled surface $X(u,v)$ is an osculating circle surface if and only if its center curve $a(u)$ is tangent to the planes along the intersection direction that the circles lie in.*

Remark 5.1. From Equation (5.1) and Proposition 5.1 we know that the plane that the circle lies in is the osculating plane of the center curve at the corresponding point for the osculating circle surface.

We denote the arc length parameter of the curve $a(u)$ by $\bar{s}$, Frenet frame of $a(u)$ by $\{\bar{\alpha}, \bar{\beta}, \bar{\gamma}\}$, the curvature and torsion of $a(u)$ by $\bar{\kappa}$ and $\bar{\tau}$. Then from (5.1) we have

$$\begin{cases} \bar{\alpha}(u) = \varepsilon y(u), \\ \dfrac{\mathrm{d}\bar{s}}{\mathrm{d}u} = \varepsilon \psi(u), \quad \varepsilon = \pm 1. \end{cases} \tag{5.2}$$

Without loss of generality we may assume that $\varepsilon = +1$. From $\bar{\alpha}(u) = y(u)$ and also putting $\bar{\beta}(u) = \alpha_0(u)$ we get

$$\begin{cases} \bar{\beta}(u) = \alpha_0(u), \\ \bar{\kappa}(u)\psi(u) = -\kappa_g(u). \end{cases} \tag{5.3}$$

From $\bar{\beta}(u) = \alpha_0(u)$, by a direct calculation we have

$$\begin{cases} \bar{\gamma}(u) = x(u), \\ \bar{\tau}(u)\psi(u) = -1. \end{cases} \tag{5.4}$$

Then we have

$$\begin{cases} \bar{\alpha}(u) = y(u), \\ \bar{\beta}(u) = \alpha_0(u), \\ \bar{\gamma}(u) = x(u), \end{cases} \tag{5.5}$$

and

$$\begin{cases} \dfrac{\bar{\tau}(u)}{\bar{\kappa}(u)} = \dfrac{1}{\kappa_g(u)}, \\ \dfrac{\mathrm{d}\bar{s}}{\mathrm{d}u} = \psi(u) = -\dfrac{1}{\bar{\tau}(u)} = -\dfrac{\kappa_g(u)}{\bar{\kappa}(u)}. \end{cases} \tag{5.6}$$

From (4.10) and (5.5) we know that the striction curves of the surface $X(u, v)$ are

$$S_\varepsilon = a(u) + \varepsilon\rho(u)\alpha_0(u) = a(u) + \varepsilon\rho(u)\bar{\beta}(u). \tag{5.7}$$

If $\varepsilon\rho(u)\bar{\kappa}(u) = 1$, the striction curve S_ε is the curvature center curve of $a(u)$.

Theorem 5.1. *If the circled surface*

$$X(u, v) = a(u) + \rho(u)[(\cos v)\alpha_0(u) + (\sin v)y(u)]$$

is an osculating circle surface, then the ruled surface

$$X_0(u, v) = a(u) + vx(u) \tag{5.8}$$

is non pitched and $X_0(u, v)$ is the binormal ruled surface of the curve $a(u)$.

Proof. If $X(u, v)$ is an osculating circle surface, from (5.5) we know that $x(u)$ is the binormal of the curve $a(u)$. Therefore $X_0(u, v)$ is the binormal ruled surface of the curve $a(u)$ and by [6], Definition 5.1 and Theorem 5.1, we know that ruled surface $X_0(u, v)$ is non pitched. □

Theorem 5.2. *If the circled surface $X(u, v)$ is an osculating circle surface, then the center curve $a(u)$ is the involute of the striction line $A_2(u)$ of position ruled surface $Y_2(u, v)$ of $X(u, v)$.*

Proof. If $X(u, v)$ is an osculating circle surface, from (3.19), (5.5) and (5.6), we have

$$\begin{aligned} A_2(u) &= a(u) - \kappa_g^{-1}(u)\psi(u)\alpha_0(u) - \{\xi(u) - [\kappa_g^{-1}(u)\psi(u)]'\}x(u) \\ &= a(u) - \kappa_g^{-1}(u)\psi(u)\alpha_0(u) + [\kappa_g^{-1}(u)\psi(u)]'x(u) \end{aligned} \tag{5.9}$$

$$= a(u) + \bar{R}(u)\bar{\beta}(u) - [\bar{R}(u)]'\bar{\gamma}(u),$$

where $\bar{R}(u) = [\bar{\kappa}(u)]^{-1}$ is the curvature radius. By (3.20) (or a direct calculation) we know that $a(u)$ is the involute of $A_2(u)$ and $A_2(u)$ is the evolute of $a(u)$. □

5.2. Normal circle surfaces

Now we consider another kind of special circled surfaces. Such a surface is generated by the circles they lie in the normal planes of a curve in $\mathbb{E}^3$.

Definition 5.2. If the center curve $a(u)$ of the circled surface $X(u,v)$ is the striction line of its position ruled surface $Y_2(u,v)$, then the surface $X(u,v)$ is called normal circle surface in Euclidean 3-space $\mathbb{E}^3$.

For the normal circle surfaces, from Definition 5.2 and Proposition 3.2, we know that

$$a'(u) = \eta(u)x(u). \tag{5.10}$$

Therefore we have

Proposition 5.2. *The circled surface $X(u,v)$ is a normal circle surface if and only if its center curve $a(u)$ is perpendicular to the planes that the circles lie in.*

Remark 5.2. From Proposition 5.2 we know that the plane that the circle lies in is the normal plane of the center curve at the corresponding point for the normal circle surface.

Remark 5.3. The normal circle surfaces defined in Definition 5.2 are called canal surface by some authors.

We denote the arc length parameter of $a(u)$ by $\bar{s}$, Frenet frame of $a(u)$ by $\{\bar{\alpha}, \bar{\beta}, \bar{\gamma}\}$, the curvature and torsion of $a(u)$ by $\bar{\kappa}$ and $\bar{\tau}$. Then from (5.10) we have

$$\begin{cases} \bar{\alpha}(u) = \varepsilon x(u), \\ \dfrac{\mathrm{d}\bar{s}}{\mathrm{d}u} = \varepsilon\eta(u), \qquad \varepsilon = \pm 1. \end{cases} \tag{5.11}$$

We also put $\varepsilon = +1$. From $\bar{\alpha}(u) = x(u)$, putting $\bar{\beta}(u) = \alpha_0(u)$ we have

$$\begin{cases} \bar{\beta}(u) = \alpha_0(u), \\ \bar{\kappa}(u)\eta(u) = 1. \end{cases} \tag{5.12}$$

From $\bar{\beta}(u) = \alpha_0(u)$, by a direct calculation we have

$$\begin{cases} \bar{\gamma}(u) = -y(u), \\ \bar{\tau}(u) = -\bar{\kappa}(u)\kappa_g(u). \end{cases} \tag{5.13}$$

Then we have

$$\begin{cases} \bar{\alpha}(u) = x(u), \\ \bar{\beta}(u) = \alpha_0(u), \\ \bar{\gamma}(u) = -y(u), \end{cases} \tag{5.14}$$

and

$$\left\{\begin{aligned} &\frac{\bar{\tau}(u)}{\bar{\kappa}(u)} = -\kappa_g(u), \\ &\frac{\mathrm{d}\bar{s}}{\mathrm{d}u} = \eta(u) = \frac{1}{\bar{\kappa}(u)} = -\frac{\kappa_g(u)}{\bar{\tau}(u)}. \end{aligned}\right. \tag{5.15}$$

From (4.10) and (5.14) we know that the striction curves of the surface $X(u,v)$ are

$$S_\varepsilon = a(u) + \varepsilon\rho(u)\alpha_0(u) = a(u) + \varepsilon\rho(u)\bar{\beta}(u). \tag{5.16}$$

If $\varepsilon\rho(u)\bar{\kappa}(u) = 1$, the striction curve S_ε is the curvature center curve of $a(u)$.

Theorem 5.3. *If the circled surface*

$$X(u,v) = a(u) + \rho(u)[(\cos v)\alpha_0(u) + (\sin v)y(u)]$$

is a normal circle surface, then the ruled surface

$$Y_0(u,v) = a(u) + vy(u) \tag{5.17}$$

is non pitched and $Y_0(u,v)$ is the binormal ruled surface of the curve $a(u)$.

Proof. If $X(u,v)$ is a normal circle surface, from (5.14) we know that $y(u)$ is the binormal of the curve $a(u)$. Therefore $Y_0(u,v)$ is the binormal ruled surface of the curve $a(u)$ and by [6], Definition 5.1 and Theorem 5.1, we know that ruled surface $Y_0(u,v)$ is non pitched. □

Theorem 5.4. *If the circled surface $X(u,v)$ is a normal circle surface, then the center curve $a(u)$ is the involute of the striction line $A_1(u)$ of axis ruled surface $Y_1(u,v)$ of $X(u,v)$.*

Proof. If $X(u,v)$ is a normal circle surface, from (3.11) and (5.14), (5.15) we have

$$\begin{aligned} A_1(u) &= a(u) + \eta(u)\alpha_0(u) + \left(\frac{\xi(u)+\eta'(u)}{\kappa_g(u)}\right)y(u) \\ &= a(u) + \eta(u)\alpha_0(u) + [\kappa_g^{-1}(u)\eta'(u)]y(u) \\ &= a(u) + \bar{R}(u)\bar{\beta}(u) - [\kappa_g^{-1}(u)\bar{R}'(u)]\bar{\gamma}(u). \end{aligned} \tag{5.18}$$

Where $\bar{R}(u) = [\bar{\kappa}(u)]^{-1}$ is the curvature radius. By (3.12) (or directly calculation) we know that $a(u)$ is the involute of $A_1(u)$ and $A_1(u)$ is the evolute of $a(u)$. □

5.3. Curvature circle surfaces

In this subsection we consider the circled surface generated by the curvature circles of a space curve in Euclidean 3-space $\mathbb{E}^3$. It is well known that the curvature circle of any regular space curve $r(s) = r(u)$ at point $s_0 = s(u_0)$ in $\mathbb{E}^3$ can be written as

$$X(u_0,v) = r(u_0) + R(u_0)\beta(u_0) + R(u_0)\{\alpha(u_0)\sin v + \beta(u_0)\cos v\}, \tag{5.19}$$

where $R(u) = \kappa^{-1}(u)$ is the curvature radius of $r(u)$; s is the arc length parameter of $r(u)$. We denote the Frenet frame field along $r(s)$ by $\{\alpha(s), \beta(s), \gamma(s)\}$ and the curvature function and torsion function of $r(s)$ by $\kappa(s)$ and $\tau(s)$.

From (5.19), we know that $\gamma(u) \parallel x(u)$, u is the arc length parameter of $x(u)$. Then by derivation and Frenet formula we get $\beta(u) \parallel \alpha_0(u)$. Therefore we have $\alpha(u) = \beta(u) \times \gamma(u) \parallel y(u)$. Putting

$$\begin{cases} \alpha(u) = y(u), \\ \beta(u) = \alpha_0(u), \\ \gamma(u) = x(u), \end{cases} \tag{5.20}$$

then (5.19) can be written as

$$X(u_0, v) = r(u_0) + R(u_0)\alpha_0(u_0) + R(u_0)\{y(u_0)\sin v + \alpha_0(u_0)\cos v\}. \tag{5.21}$$

From Proposition 4.1, we know that the striction curves of this kind of circled surface $X(u, v)$ are

$$S_\pm(u) = r(u) + R(u)\alpha_0(u) \mp R(u)\alpha_0(u).$$

Remark 5.4. From the geometric meaning of the principal normal $\bar{\beta}$, we know that the direction of α_0 here is different as in Proposition 4.1.

By (5.20) we have

$$\begin{cases} \kappa(u)\beta(u)\dfrac{\mathrm{d}s}{\mathrm{d}u} = -\kappa_g(u)\alpha_0(u), \\ -\tau(u)\beta(u)\dfrac{\mathrm{d}s}{\mathrm{d}u} = \alpha_0(u). \end{cases} \tag{5.22}$$

Then we get

$$\begin{cases} \dfrac{\tau(u)}{\kappa(u)} = \dfrac{1}{\kappa_g(u)}, \\ \dfrac{\mathrm{d}s}{\mathrm{d}u} = -\dfrac{1}{\tau(u)} = -\dfrac{\kappa_g(u)}{\kappa(u)}. \end{cases} \tag{5.23}$$

Theorem 5.5. *The circled surface $X(u, v)$ is a curvature circles surface of a space curve $r(s)$ if and only if the invariants $\xi(u), \eta(u), \psi(u), \kappa_g(u), \rho(u)$ of $X(u, v)$ and the invariants $s, \kappa(s), \tau(s)$ of $r(s)$ satisfy*

$$\begin{cases} \xi(u) = \dot{R}(u)\,\dfrac{\mathrm{d}s}{\mathrm{d}u} = R'(u) = \dfrac{\mathrm{d}R(u)}{\mathrm{d}u}, \\ \eta(u) = \tau(u)R(u)\,\dfrac{\mathrm{d}s}{\mathrm{d}u} = -R(u), \\ \psi(u) = 0, \\ \rho(u) = R(u), \\ \dfrac{\mathrm{d}s}{\mathrm{d}u} = -\dfrac{1}{\tau(u)} = -\dfrac{\kappa_g(u)}{\kappa(u)}. \end{cases} \tag{5.24}$$

Proof. If $X(u,v)$ is the curvature circles surface of $r(s)$, then

$$a'(u) = \frac{\mathrm{d}}{\mathrm{d}s}[r(s) + R(s)\beta(s)]\frac{\mathrm{d}s}{\mathrm{d}u} = [\dot{R}(s)\beta(s) + R(s)\tau(s)\gamma(s)]\frac{\mathrm{d}s}{\mathrm{d}u}.$$

By (4.2), (5.20) and (5.23) we have (5.24).

On the other hand, from (5.24) we have

$$a(u) = \int [R'(u)\alpha_0(u) - R(u)x(u)]\mathrm{d}u.$$

Put

$$r(s) = r(u) = a(u) - R(u)\alpha_0(u),$$

then

$$\begin{aligned}
&\alpha(u)\\
&= \{R'(u)\alpha_0(u) - R(u)x(u) - R'(u)\alpha_0(u) - R(u)[-x(u) + \kappa_g(u)y(u)]\}\frac{\mathrm{d}u}{\mathrm{d}s}\\
&= - R(u)\kappa_g(u)y(u)\frac{\mathrm{d}u}{\mathrm{d}s}\\
&= y(u).
\end{aligned}$$

And

$$\kappa(u)\beta(u) = -\kappa_g(u)\alpha_0(u)\frac{\mathrm{d}u}{\mathrm{d}s}$$

yields

$$\beta(u) = \alpha_0(u).$$

Therefore

$$\gamma(u) = \alpha(u) \times \beta(u) = y(u) \times \alpha_0(u) = x(u).$$

The circled surface

$$\begin{aligned}
X(u,v) &= a(u) + \rho(u)[\alpha_0(u)\cos v + y(u)\sin v]\\
&= \int [R'(u)\alpha_0(u) - R(u)x(u)]\mathrm{d}u + R(u)[\alpha_0(u)\cos v + y(u)\sin v]
\end{aligned}$$

is the curvature circles surface

$$\int [R'(u)\alpha_0(u) - R(u)x(u)]\mathrm{d}u + R(u)[\beta(u)\cos v + \alpha(u)\sin v]$$

of the curve

$$r(s) = a(u) - R(u)\beta(u) = a(u) - R(u)\alpha_0(u).$$

□

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