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MAPS PRESERVING GENERALIZED PROJECTION OPERATORS

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Abstract. Let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on a Hilbert space H with $\dim(H) > 2$. Let $\mathcal{GP}(H)$ be the subset of $\mathcal{B}(H)$ of all generalized projection operators. In this paper, we give a complete characterization of surjective maps $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfying $A - \lambda B \in \mathcal{GP}(H) \Leftrightarrow \Phi(A) - \lambda\Phi(B) \in \mathcal{GP}(H)$ for any $A, B \in \mathcal{B}(H)$ and $\lambda \in \mathbb{C}$.

1. Introduction and statement of the main result

One of the most fascinating areas of research in matrix theory and operator theory is the study of preserver problems, which is closely related to quantum mechanics. For many researchers, an essential approach to such a problem is to reduce it to mappings that preserve properties, sets, or relations, with their objective being to characterize these maps. The study of preserver problems dates back to Frobenius, who described the form of bijective linear maps that preserve the determinant of matrices, see [10]. Since then, several results have been established for linear preserver problems, see [5, 6, 13, 17, 18] and the references therein. More recently, there has been growing interest among mathematicians to explore preserver problems without assuming linearity as a priori, see [7, 24].

J. Groß and G. Trenkler introduced a generalization of orthogonal projection called a generalized projection. It is defined as a complex matrix A such that $A^2 = A^*$, see [12]. This concept was later extended for infinite-dimensional Hilbert spaces by H-K. Du and Y. Li in [9], who characterized this type of projection by its spectral decomposition. In [21], S. Radosavljevic and D. Djordjevic investigated the conditions under which the product, difference, and sum of these operators belong to the same class of operators. For more about generalized projection and its connection to other subjects, see, for instance, [1–4, 11, 14, 22].

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Throughout this paper, H will denote a complex Hilbert space with $\dim(H) > 2$ and $\mathcal{B}(H)$ stands for the algebra of all bounded linear operators acting on H. For an operator $A \in \mathcal{B}(H)$, by A^*, $\sigma(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$, we denote the adjoint, spectrum, range and the kernel of A, respectively. An operator $A \in \mathcal{B}(H)$ is called a generalized projection on H if $A^2 = A^*$. The set of all generalized projection operators on H is denoted by $\mathcal{GP}(H)$. Recall that, an orthogonal projection operator (or simply projection) P is a bounded linear operator satisfying $P^2 = P^* = P$. Note that $\mathcal{P}(H)$, the set of all orthogonal projection operators on H, is a subset of $\mathcal{GP}(H)$. We adopt the notation $x \otimes y$ for an operator of rank at most one, defined by $(x \otimes y)z = \langle z, y\rangle x$. It is well known that a rank-one operator T is an orthogonal projection if and only if $T = x \otimes x$ for some unit vector $x \in H$. The set of all rank-one projections on H is denoted by $\mathcal{P}_1(H)$.

Recently, linear maps preserving generalized projections were studied in [6, 13]. More precisely, the authors determined the form of any surjective linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$, where H is a separable complex Hilbert space, that preserves the generalized projection operators.

In [7], Dolinar provides the complete form of surjective maps on $\mathcal{B}(H)$, that satisfy the condition $A - \lambda B$ is idempotent if and only if $\Phi(A) - \lambda\Phi(B)$ is idempotent for every $A, B \in \mathcal{B}(H)$ and $\lambda \in \mathbb{C}$. Based on the above studies, L. Yang and L. Zhang, in [24], have characterized the form of surjective maps on $\mathcal{B}(H)$ such that $A - \lambda B$ is an involution if and only if $\Phi(A) - \lambda\Phi(B)$ is an involution for any $A, B \in \mathcal{B}(H)$ and $\lambda \in \mathbb{C}$. For more information, on preserving orthogonal projections and some expositions on non-linear preserver problems, the reader is referred to [15–17] and the references therein.

Motivated by the previous results, this paper aims to show that for a surjective map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ (not necessarily linear), the condition

$$\begin{aligned} &A - \lambda B \in \mathcal{GP}(H) \\ &\Leftrightarrow \Phi(A) - \lambda\Phi(B) \in \mathcal{GP}(H) \text{ for all } A, B \in \mathcal{B}(H) \text{ and } \lambda \in \mathbb{C}, \end{aligned} \tag{1}$$

is enough to characterize the map Φ.

This makes our result an extension of the main result of [6, 13] to the non-linear case and without assuming that the map Φ to be continuous. The main theorem is presented in the following.

Theorem 1.1. *Let H be a complex Hilbert space with $\dim(H) > 2$, and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective map. Then, the following assertions are equivalent.*

(1) Φ *satisfies* (1).
(2) *There exists $\alpha \in \mathbb{C}$ with $\alpha^3 = 1$, such that either*

$$\Phi(T) = \alpha UTU^* \text{ for all } T \in \mathcal{B}(H),$$

where $U : H \to H$ is a unitary operator, or

$$\Phi(T) = \alpha UT^*U^* \text{ for all } T \in \mathcal{B}(H),$$

where $U : H \to H$ is an anti-unitary operator.

In order to prove Theorem 1.1 we need some preliminary results, which are presented in the following section.

2. Preliminaries

Let us begin by the following properties of generalized projection operators which are easily checked.

Lemma 2.1. *Let $A \in \mathcal{B}(H)$.*

(1) *If A is an orthogonal projection operator, then A is a generalized projection operator.*
(2) *If A is a generalized projection operator, then A^3 is an orthogonal projection operator.*
(3) *A is a rank-one generalized projection operator if and only if $A = \lambda x \otimes x$ for some unit vector $x \in H$ and $\lambda \in \mathbb{C}$ such that $\lambda^3 = 1$.*
(4) *For $\lambda \in \mathbb{C}$ such that $\lambda^3 = 1$, we have $A \in \mathcal{GP}(H) \Leftrightarrow \lambda A \in \mathcal{GP}(H)$.*

The following theorem is the main result of [9], where the authors gave a spectral characterization of generalized projection operators that will be useful to prove our main theorem.

Theorem 2.2. *Let $A \in \mathcal{B}(H)$. Then A is a generalized projection operator if and only if A is a normal operator and $\sigma(A) \subseteq \{0, 1, \mathrm{e}^{\pm i\frac{2}{3}\pi}\}$. In this case, A has the following spectral representation*

$$A = 0E(0) \oplus E(1) \oplus \mathrm{e}^{\mathrm{i}\frac{2}{3}\pi} E\left(\mathrm{e}^{\mathrm{i}\frac{2}{3}\pi}\right) \oplus \mathrm{e}^{-\mathrm{i}\frac{2}{3}\pi} E\left(\mathrm{e}^{-\mathrm{i}\frac{2}{3}\pi}\right), \tag{2}$$

where $E(\alpha)$ denotes the spectral projection associated with a spectral point $\alpha \in \sigma(A)$ and $E(\alpha) = 0$ if $\alpha \notin \sigma(A)$.

Proof. See [9, Theorem 2]. □

As a consequence of the previous theorem (see [9, Theorem 6]), we obtain that

$$A \in \mathcal{GP}(H) \Leftrightarrow A \text{ is normal and } A^4 = A.$$

The result below, taken from [21], establishes the necessary and sufficient conditions for the difference and sum of generalized projections to also be generalized projections.

Theorem 2.3. *Let $A, B \in \mathcal{GP}(H)$ be two generalized projection operators. Then*

(1) *$A + B \in \mathcal{GP}(H)$ if and only if $AB = BA = 0$.*
(2) *$A - B \in \mathcal{GP}(H)$ if and only if $AB = BA = B^*$.*

Proof. See [21, Theorem 3.3 and 3.4]. □

The following lemma plays an important role in the proof of the main theorem.

Lemma 2.4. *If P is an orthogonal projection operator and Q is a generalized projection operator such that $2P - Q \in \mathcal{GP}(H)$, then $Q = P$.*

Proof. Let $P \in \mathcal{P}(H)$ and $Q \in \mathcal{GP}(H)$ such that $2P - Q \in \mathcal{GP}(H)$. Under the direct sum decomposition $H = \mathcal{R}(P) \oplus \mathcal{N}(P)$, we can suppose that

$$P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Since $2P - Q \in \mathcal{GP}(H)$, then $(2P - Q)^2 = (2P - Q)^*$, which proves that

$$P + Q^* = PQ + QP.$$

Using matrix writing, we get $I + A^* = 2A$, $C = B^*$ and $D = 0$. Thus

$$Q = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}.$$

From $Q \in \mathcal{GP}(H)$, we obtain $B = 0$. Hence

$$Q = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

Taking * on $I + A^* = 2A$ and then subtracting it from the original equation leads to $A = A^*$. By plugging $A = A^*$ to $I + A^* = 2A$, we have $A = I$. Therefore

$$Q = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

This means $Q = P$. □

For two self-adjoint operators $A, B \in \mathcal{B}(H)$, we write $A \leq B$ if $B - A$ is a positive operator. In the case of orthogonal projections, we have the following well known result.

Lemma 2.5. *Let $P, Q \in \mathcal{P}(H)$ be two orthogonal projection operators. Then the following assertions are equivalent.*

(1) $P \leq Q$.
(2) $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$.
(3) $PQ = QP = P$.
(4) $Q - P \in \mathcal{P}(H)$.

Proof. See [19, Theorem 2.3.2]. □

The star partial order, which was defined by Drazin [8] as

$$A \overset{*}{\leq} B \Leftrightarrow A^*A = A^*B \text{ and } AA^* = BA^*,$$

where $A, B \in \mathcal{B}(H)$, can be characterized on the set of generalized projection operators as follows.

Lemma 2.6. *Let $A, B \in \mathcal{GP}(H)$ be two generalized projection operators. Then the following assertions are equivalent.*

(1) $A \overset{*}{\leq} B$.
(2) $B - A \in \mathcal{GP}(H)$.

Proof. Let $A, B \in \mathcal{GP}(H)$. Then we have

$$\begin{aligned}
A \overset{*}{\leq} B &\Leftrightarrow A^*A = A^*B \quad \text{and} \quad AA^* = BA^* \\
&\Leftrightarrow A^3 = A^2B \quad \text{and} \quad A^3 = BA^2 \\
&\Leftrightarrow A^3 = A^2B = BA^2 \\
&\Leftrightarrow A^5 = A^4B = BA^4 \quad \big(\text{since } A^4 = A \text{ when } A \in \mathcal{GP}(H)\big) \\
&\Leftrightarrow A^2 = AB = BA \\
&\Leftrightarrow A^* = AB = BA \\
&\Leftrightarrow B - A \quad \text{is a generalized projection operator.}
\end{aligned}$$

□

An immediate consequence of the two previous lemmas is

$$P \overset{*}{\leq} Q \quad \Leftrightarrow \quad P \leq Q, \quad \text{for every} \quad P, Q \in \mathcal{P}(H).$$

In the next lemma, we characterize the rank-one generalized projection operators by the star order.

Lemma 2.7. *Let $A \in \mathcal{GP}(H)$ be a nonzero generalized projection operator. Then the following assertions are equivalent.*

(1) *A is a rank-one operator.*
(2) *$\forall B \in \mathcal{GP}(H)$, $B \overset{*}{\leq} A \implies B = 0$ or $B = A$.*

Proof. (1) ⇒ (2) Suppose that A is a rank-one operator, by Lemma 2.1, $A = \lambda a \otimes a$ for some unit vector $a \in H$ and a scalar $\lambda \in \mathbb{C}$ such that $\lambda^3 = 1$. Let B be a nonzero generalized projection operator such that $B \overset{*}{\leq} A$. It follows, from Lemma 2.6, that $A - B$ is a generalized projection operator. By Theorem 2.3, we get that $B^* = BA = AB$. This implies that

$$B = \overline{\lambda} a \otimes Ba = \overline{\lambda} B^* a \otimes a.$$

Hence B is a rank-one generalized projection operator. Then $B = \alpha a \otimes a$ for some scalar $\alpha \in \mathbb{C}$ such that $\alpha^3 = 1$. Moreover $B^3 = B^*B = ABB = \lambda\alpha^2 a \otimes a$ is a nonzero projection, so $\lambda\alpha^2 = 1$, which (with $\alpha^3 = 1$) means $\alpha = \lambda$. Therefore $B = A$.

(2) ⇒ (1) Assume that the rank of A is at least 2. From Theorem 2.2, there exists $\alpha \in \sigma(A) \setminus \{0\}$ such that $E(\alpha)$ is a nonzero spectral projection. Let x be a unit vector in the range $\mathcal{R}(E(\alpha))$. From the equation (2), we obtain $Ax = \alpha x$. For $B = \alpha x \otimes x$, a simple calculation shows that $B^* = BA = AB$. This proves that $B \overset{*}{\leq} A$. Therefore $A = B = \alpha x \otimes x$. This contradiction shows that A is a rank-one operator and finishes the proof. □

In [23], Uhlhorn presents an important result that characterizes bijective maps $\Phi : \mathcal{P}_1(H) \to \mathcal{P}_1(H)$ which preserve orthogonality. This result can be summarized as follows.

Lemma 2.8 (Uhlhorn's theorem)**.** *Let H be a complex Hilbert space with $\dim(H) \geq 3$. Let $\Phi : \mathcal{P}_1(H) \to \mathcal{P}_1(H)$ be a bijective map satisfying the condition*

$$PQ = 0 \Leftrightarrow \Phi(P)\Phi(Q) = 0 \quad for\ all\ P, Q \in \mathcal{P}_1(H).$$

Then there exists an unitary or anti-unitary operator $U : H \to H$ such that $\Phi(P) = UPU^*$ for every $P \in \mathcal{P}_1(H)$.

3. Proof of the main theorem

Note that the sufficiency of Theorem 1.1 is clear. To prove the necessity, we will first go through a few propositions. Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective map satisfying the condition (1).

Proposition 3.1. *The following assertions hold.*

(1) Φ *is injective.*
(2) Φ *is homogeneous, i.e., $\Phi(\lambda A) = \lambda\Phi(A)$ for every $\lambda \in \mathbb{C}$ and $A \in \mathcal{B}(H)$.*
(3) $\Phi(P)^* = \Phi(P)\Phi(I) = \Phi(I)\Phi(P)$ *for all orthogonal projection operator $P \in \mathcal{P}(H)$.*

Proof. (1) Let $A, B \in \mathcal{B}(H)$ such that $\Phi(A) = \Phi(B)$. Then $\Phi(A) - \Phi(B) = \Phi(B) - \Phi(A) = 0 \in \mathcal{GP}(H)$. By the condition (1), $A - B \in \mathcal{GP}(H)$ and $B - A \in \mathcal{GP}(H)$. It follows that $(A - B)^* = (A - B)^2 = (B - A)^*$. Hence $A = B$ and Φ is injective. Thus, Φ is bijective since it is assumed to be surjective, and moreover, Φ^{-1} satisfies the condition (1).

(2) Let $A \in \mathcal{B}(H)$ and $\lambda \in \mathbb{C}$.

Case 1. $\lambda \neq 0$ and $\lambda^3 \neq -1$. Set $X = \Phi(\lambda A) - \lambda\Phi(A)$. Since $(\lambda A) - \lambda A \in \mathcal{GP}(H)$, then $X \in \mathcal{GP}(H)$. Similarly, we have $A - \left(\frac{1}{\lambda}\right)(\lambda A) \in \mathcal{GP}(H)$. Thus

$$\frac{-1}{\lambda}X = \Phi(A) - \left(\frac{1}{\lambda}\right)\Phi(\lambda A) \in \mathcal{GP}(H).$$

It follows that

$$-\frac{1}{\lambda}X = \left(-\frac{1}{\lambda}X\right)^4 = \frac{1}{\lambda^4}X$$

and

$$\left(\frac{1}{\lambda} + \frac{1}{\lambda^4}\right)X = 0.$$

Therefore $X = 0$ and $\Phi(\lambda A) = \lambda\Phi(A)$.

Case 2. $\lambda \in \{-1, -e^{i\frac{2}{3}\pi}, -e^{-i\frac{2}{3}\pi}\}$. By case 1, we have $\Phi(\lambda A) = 2\Phi(\frac{\lambda}{2}A) = \lambda\Phi(A)$.

Case 3. $\lambda = 0$. We have $\Phi(0) = \Phi(-0) = -\Phi(0)$. Then $\Phi(0) = 0$.

(3) Let P be an orthogonal projection operator. Then $I - P$ is also an orthogonal projection. Hence $I - P \in \mathcal{GP}(H)$. Thus $\Phi(I) - \Phi(P) \in \mathcal{GP}(H)$. Since $\Phi(I)$ and $\Phi(P)$ are generalized projections, it follows, by Theorem 2.3, that $\Phi(P)^* = \Phi(P)\Phi(I) = \Phi(I)\Phi(P)$. □

Proposition 3.2. *Let $A, B \in \mathcal{GP}(H)$ be generalized projections. Then the following assertions hold.*

(1) $AB = BA = 0 \Leftrightarrow \Phi(A)\Phi(B) = \Phi(B)\Phi(A) = 0$.
(2) $A \overset{*}{\leq} B \Leftrightarrow \Phi(A) \overset{*}{\leq} \Phi(B)$.
(3) *If* $AB = BA = 0$, *then* $\Phi(A + B) = \Phi(A) + \Phi(B)$.

Proof. (1) Follows from Theorem 2.3 and (1).

(2) Follows from Lemma 2.6 and (1).

(3) Let $A, B \in \mathcal{GP}(H)$ such that $AB = BA = 0$. By Theorem 2.3, we have $A + B \in \mathcal{GP}(H)$. Thus $\Phi(A) + \Phi(B) \in \mathcal{GP}(H)$. Note that $A \overset{*}{\leq} A + B$ and $B \overset{*}{\leq} A + B$, which implies, by the second assertion, that

$$\Phi(A) \overset{*}{\leq} \Phi(A + B) \textit{ and } \Phi(B) \overset{*}{\leq} \Phi(A + B).$$

Using Theorem 2.3 and Lemma 2.6, we get

$$\begin{aligned} \Phi(A + B)(\Phi(A) + \Phi(B)) &= \Phi(A + B)\Phi(A) + \Phi(A + B)\Phi(B) \\ &= \Phi(A)\Phi(A + B) + \Phi(B)\Phi(A + B) \\ &= (\Phi(A) + \Phi(B))\Phi(A + B) \\ &= \Phi(A)^* + \Phi(B)^* \\ &= (\Phi(A) + \Phi(B))^*. \end{aligned}$$

This shows that

$$\Phi(A) + \Phi(B) \overset{*}{\leq} \Phi(A + B).$$

On the other hand, since Φ^{-1} satisfies the same assumptions as Φ, we have

$$\begin{aligned} \Phi(A + B) &= \Phi\left[\Phi^{-1}(\Phi(A)) + \Phi^{-1}(\Phi(B))\right] \\ &\overset{*}{\leq} \Phi\left[\Phi^{-1}(\Phi(A) + \Phi(B))\right] \\ &\overset{*}{\leq} \Phi(A) + \Phi(B). \end{aligned}$$

Finally,

$$\Phi(A + B) = \Phi(A) + \Phi(B).$$ □

Proposition 3.3. *The following statements hold.*

(1) Φ *preserves the set of rank-one generalized projections in both directions.*
(2) $\Phi(I) = \alpha I$ *for some scalar* $\alpha \in \mathbb{C}$ *for which* $\alpha^3 = 1$.

Proof. (1) Let A be a rank-one generalized projection. Using Lemma 2.7, we get

$$\forall B \in \mathcal{GP}(H),\ B \overset{*}{\leq} A \implies B = 0 \quad \text{or} \quad B = A.$$

Since Φ preserves the start-order relation on the set of generalized projections,

$$\forall S \in \mathcal{GP}(H),\ S \overset{*}{\leq} \Phi(A) \implies S = 0 \quad \text{or} \quad S = \Phi(A).$$

Once again, Lemma 2.7 implies that $\Phi(A)$ is a rank-one generalized projection. The converse holds since Φ and Φ^{-1} have the same properties.

(2) Set $A = \Phi(I)$, and let $x \in H$ be a unit vector. Note that $x \otimes x$ is a rank-one projection operator. It follows, by the first assertion, that there exist a unit vector $y \in H$ and $\lambda \in \mathbb{C}$ such that $\lambda^3 = 1$ and $\Phi(y \otimes y) = \lambda x \otimes x$. By the third assertion in Proposition 3.1, we get that

$$A\Phi(y \otimes y) = \Phi(y \otimes y)A = \Phi(y \otimes y)^*.$$

Hence, $A\Phi(y \otimes y) = \lambda Ax \otimes x$ is a nonzero generalized projection operator. This implies that x and Ax are linearly dependent. Consequently, there exists a nonzero scalar $\alpha \in \mathbb{C}$ such that $A = \alpha I$. Since A is a nonzero generalized projection, then $\alpha^3 = 1$, as desired. $\square$

Proposition 3.4. *Let $\Phi(I) = I$. Then the following statements hold.*

(1) Φ *preserves the set of orthogonal projection operators in both directions.*
(2) Φ *preserves the orthogonality between the projections in both directions.*

$$PQ = QP = 0 \Leftrightarrow \Phi(P)\Phi(Q) = \Phi(Q)\Phi(P) = 0.$$

(3) Φ *preserves the order relation on the set of orthogonal projection operators in both directions.*

$$Q \leq P \Leftrightarrow \Phi(Q) \leq \Phi(P).$$

(4) *For all orthogonal projection operators P, Q such that $PQ = QP = 0$ we have,*

$$\Phi(P + Q) = \Phi(P) + \Phi(Q).$$

(5) Φ *preserves rank-one projections in both directions.*
(6) *There exists a unitary or anti-unitary operator $U : H \to H$ such that*

$$\Phi(P) = UPU^* \textit{ for all } P \in \mathcal{P}(H).$$

Proof. (1) Let P be an orthogonal projection operator. Then, $I - P$ is also an orthogonal projection. In particular, $P, I - P \in \mathcal{GP}(H)$. Therefore, $\Phi(P), I - \Phi(P) \in \mathcal{GP}(H)$, which implies, by Theorem 2.3, that

$$\Phi(P) = \Phi(P)^*.$$

Finally, $\Phi(P)$ is an orthogonal projection operator. The converse is also true because both Φ and Φ^{-1} possess the same properties.

(2) This assertion follows from the first assertion in Proposition 3.2.
(3) This assertion follows from the second assertion in Proposition 3.2.
(4) This assertion follows from the third assertion in Proposition 3.2.

(5) This assertion follows from the fact that Φ preserves the set of rank-one generalized projections in both directions and $\Phi(\mathcal{P}(H)) = \mathcal{P}(H)$.

(6) Note that Φ is a bijection on $\mathcal{P}_1(H)$ that preserves orthogonality in both directions. It follows from the Uhlhorn's Theorem (Lemma 2.8) that there is a unitary or anti-unitary operator U on H such that

$$\Phi(P) = UPU^* \text{ for all } P \in \mathcal{P}_1(H).$$

Since every finite rank projection is sum of pairwise orthogonal rank-one projections, then $\Phi(P) = UPU^*$ for every finite rank projection P.

Now, let P be an infinite rank projection. Then

$$UPU^* = \sup\{Q : Q \leq UPU^*,\, Q \text{ is a finite rank projection}\}.$$

Since Φ preserves the order of projections in both directions, we conclude that

$$\begin{aligned}\Phi(P) &= \sup\{Q : Q \leq \Phi(P),\, Q \text{ is a finite rank projection}\}\\ &= \sup\{Q : \Phi^{-1}(Q) \leq P,\, Q \text{ is a finite rank projection}\}\\ &= \sup\{Q : Q \leq UPU^*,\, Q \text{ is a finite rank projection}\}\\ &= UPU^*.\end{aligned}$$

Hence, $\Phi(P) = UPU^*$ holds for every projection P, as desired. □

Proof of Theorem 1.1. It remains to prove the necessity. From Proposition 3.3 and Proposition 3.4, we have found that there exists $\alpha \in \mathbb{C}$ with $\alpha^3 = 1$ and a unitary or anti-unitary operator U on H such that

$$\Phi(P) = \alpha UPU^* \text{ for all } P \in \mathcal{P}(H).$$

Case 1. If U is unitary, we will show that $\Phi(T) = \alpha UTU^*$ for all $T \in \mathcal{B}(H)$. To do that, define $\Psi : \mathcal{B}(H) \to \mathcal{B}(H)$ as $\Psi(T) = \alpha^{-1}U^*\Phi(T)U$. Note that Ψ is a bijective homogeneous map that satisfies the condition (1) and

$$\Psi(P) = P \text{ for all } P \in \mathcal{P}(H).$$

We claim that we have

$$\Psi(A) = A \text{ for all } A \in \mathcal{GP}(H).$$

Indeed, let $A \in \mathcal{GP}(H)$. Then by Theorem 2.2, we have

$$A = \bigoplus_{\lambda \in \sigma(A)} \lambda E(\lambda).$$

Using Proposition 3.2, we get that

$$\Psi(A) = \bigoplus_{\lambda \in \sigma(A)} \Psi(\lambda E(\lambda)).$$

Since Ψ is homogeneous, then

$$\Psi(A) = \bigoplus_{\lambda \in \sigma(A)} \lambda \Psi(E(\lambda)).$$

Note that $E(\alpha)$ is an orthogonal projection, so

$$\Psi(A) = \bigoplus_{\lambda\in\sigma(A)} \lambda E(\lambda) = A.$$

Now, let us prove, by induction, that for all positive integers n

$$\Psi\left(A + \sum_{i=1}^{n} \lambda_i P_i\right) = A + \sum_{i=1}^{n} \lambda_i P_i \tag{3}$$

for every $A \in \mathcal{GP}(H)$, $P_i \in \mathcal{P}(H)$ and $\lambda_i \in \mathbb{C}$.

For $n = 1$, let $A \in \mathcal{GP}(H)$, $P_1 \in \mathcal{P}(H)$, and λ_1 be a nonzero complex scalar. Since $\lambda_1^{-1}(A + \lambda_1 P_1) - \lambda_1^{-1}A = P_1 \in \mathcal{GP}(H)$, then

$$Q = \lambda_1^{-1}\Psi\left(A + \lambda_1 P_1\right) - \lambda_1^{-1}\Psi(A) = \lambda_1^{-1}\Psi\left(A + \lambda_1 P_1\right) - \lambda_1^{-1}A \in \mathcal{GP}(H). \tag{4}$$

Similarly, we have $(A + \lambda_1 P_1) - \lambda_1 P_1 = A \in \mathcal{GP}(H)$. Then

$$D = \Psi\left(A + \lambda_1 P_1\right) - \lambda_1\Psi(P_1) = \Psi\left(A + \lambda_1 P_1\right) - \lambda_1 P_1 \in \mathcal{GP}(H).$$

This implies that $\lambda_1^{-1}\Psi\left(A + \lambda_1 P_1\right) - \lambda_1^{-1}D \in \mathcal{GP}(H)$. Therefore

$$\lambda_1^{-1}(A + \lambda_1 P_1) - \lambda_1^{-1}D = \lambda_1^{-1}\left(2\lambda_1 P_1 + A - \Psi(A + \lambda_1 P_1)\right) \in \mathcal{GP}(H). \tag{5}$$

It follows from (4) and (5) that

$$2P_1 - Q \in \mathcal{GP}(H).$$

Using Lemma 2.4, we obtain that $Q = P_1$. Thus $\Psi(A + \lambda_1 P_1) = A + \lambda_1 P_1$.

Now, assume that the formula (3) holds for some arbitrary positive integer n. Let $\lambda_1, \ldots, \lambda_{n+1}$ be nonzero scalars in $\mathbb{C}$, $A \in \mathcal{GP}(H)$ and $P_1, \ldots, P_{n+1} \in \mathcal{P}(H)$. We have $\lambda_1^{-1}(A + \sum_{i=1}^{n+1} \lambda_i P_i) - \lambda_1^{-1}(A + \sum_{i=2}^{n+1} \lambda_i P_i) \in \mathcal{GP}(H)$, then

$$Q' = \lambda_1^{-1}\Psi\left(A + \sum_{i=1}^{n+1} \lambda_i P_i\right) - \lambda_1^{-1}\Psi\left(A + \sum_{i=2}^{n+1} \lambda_i P_i\right) \in \mathcal{GP}(H). \tag{6}$$

By the inductive hypothesis, we get that

$$\Psi\left(A + \sum_{i=2}^{n+1} \lambda_i P_i\right) = A + \sum_{i=2}^{n+1} \lambda_i P_i.$$

It follows from (6), that

$$\Psi\left(A + \sum_{i=1}^{n+1} \lambda_i P_i\right) = \lambda_1 Q' + (A + \sum_{i=2}^{n+1} \lambda_i P_i). \tag{7}$$

Similarly, we have $(A + \sum_{i=1}^{n+1} \lambda_i P_i) - (\sum_{i=1}^{n+1} \lambda_i P_i) \in \mathcal{GP}(H)$. Then

$$D' = \Psi\left(A + \sum_{i=1}^{n+1} \lambda_i P_i\right) - \Psi\left(\sum_{i=1}^{n+1} \lambda_i P_i\right) \in \mathcal{GP}(H).$$

Once again, by the inductive hypothesis, we have $\Psi(\sum_{i=1}^{n+1}\lambda_i P_i) = \sum_{i=1}^{n+1}\lambda_i P_i$, which implies that

$$\Psi\left(A + \sum_{i=1}^{n+1}\lambda_i P_i\right) = D' + \sum_{i=1}^{n+1}\lambda_i P_i. \tag{8}$$

Since

$$\lambda_1^{-1}\Psi\left(A + \sum_{i=1}^{n+1}\lambda_i P_i\right) - \lambda_1^{-1}\left(D' + \sum_{i=2}^{n+1}\lambda_i P_i\right) = P_1 \in \mathcal{GP}(H).$$

Then

$$\lambda_1^{-1}\left(A - D' + \lambda_1 P_1\right) \in \mathcal{GP}(H) \tag{9}$$

Combining (7), (8) and (9) we get that

$$2P_1 - Q' \in \mathcal{GP}(H).$$

By Lemma 2.4, we obtain $Q' = P_1$. Finally

$$\Psi\left(A + \sum_{i=1}^{n+1}\lambda_i P_i\right) = A + \sum_{i=1}^{n+1}\lambda_i P_i.$$

To finish the proof for this case, note that, by [20, Corollary 2.3], every operator $T \in \mathcal{B}(H)$ is a finite linear combination of orthogonal projection operators. Then $\Psi(T) = T$, which shows that

$$\Phi(T) = \alpha U T U^*, \quad (T \in \mathcal{B}(H)).$$

Case 2. If U is anti-unitary, it is easy to see that the map

$$T \to \left(\bar{\alpha}^{-1} U^* \Phi(T) U\right)^*$$

satisfies the proprieties of Ψ in the case 1. Hence $\left(\bar{\alpha}^{-1} U^* \Phi(T) U\right)^* = T$ for every $T \in \mathcal{B}(H)$. Thus $\Phi(T) = \alpha U T^* U^*$ for every $T \in \mathcal{B}(H)$. The proof is now complete. □

Let μ be a nonzero complex number, and let Γ_μ be the subset of $\mathcal{B}(H)$ defined by

$$\Gamma_\mu = \left\{T \in \mathcal{B}(H) : \ \mu^2 T^2 = \bar{\mu} T^*\right\}.$$

It is evident that $\Gamma_1 = \mathcal{GP}(H)$ and $T \in \Gamma_\mu$ if and only if $\mu T \in \mathcal{GP}(H)$.

The following corollary is a consequence of the main theorem and provides a characterization of surjective maps satisfying

$$A - \lambda B \in \Gamma_\mu \Leftrightarrow \Phi(A) - \lambda \Phi(B) \in \Gamma_\mu \text{ for all } A, B \in \mathcal{B}(H) \text{ and } \lambda \in \mathbb{C}. \tag{10}$$

Corollary 3.5. *Let H be a complex Hilbert space with* $\dim(H) > 2$*, and let* $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ *be a surjective map. Then Φ satisfies* (10) *if and only if it takes one of the two forms as described in Theorem 1.1.*

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