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THE NORMING SET OF A SYMMETRIC n-LINEAR FORM ON THE PLANE WITH A ROTATED SUPREMUM NORM FOR $n = 3, 4, 5$

Sung Guen Kim

Abstract. Let $n \in \mathbb{N}, n \geq 2$. An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of $T \in \mathcal{L}({}^n E)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$, where $\mathcal{L}({}^n E)$ denotes the space of all continuous n-linear forms on E. For $T \in \mathcal{L}({}^n E)$, we define

$$\text{Norm}(T) = \Big\{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \text{ is a norming point of } T\Big\}.$$

Norm(T) is called the *norming set* of T.

Let $0 \leq \theta \leq \frac{\pi}{4}$ and $\ell^2_{\infty,\theta} = \mathbb{R}^2$ with the rotated supremum norm

$$\|(x, y)\|_{(\infty,\theta)} = \max\Big\{|x \cos\theta + y \sin\theta|,\ |x \sin\theta - y \cos\theta|\Big\}.$$

In this paper, we characterize the norming set of $T \in \mathcal{L}({}^n \ell^2_{(\infty,\theta)})$. Using this result, we completely describe the norming set of $T \in \mathcal{L}_s({}^n \ell^2_{(\infty,\theta)})$ for $n = 3, 4, 5$, where $\mathcal{L}_s({}^n \ell^2_{(\infty,\theta)})$ denotes the space of all continuous symmetric n-linear forms on $\ell^2_{(\infty,\theta)}$. We generalizes the results from [9] for $n = 3$ and $\theta = \frac{\pi}{4}$.

1. Introduction

In 1961, Bishop and Phelps [2] showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property

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is also sufficient for the denseness of norm attaining polynomials. Jiménez-Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}$, $n \geq 2$. We write S_E for the unit sphere of a Banach space E. We denote by $\mathcal{L}(^n E)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{(x_1,\ldots,x_n)\in S_E\times\cdots\times S_E} |T(x_1,\ldots,x_n)|$. $\mathcal{L}_s(^n E)$ denotes the closed subspace of all continuous symmetric n-linear forms on E. An element $(x_1,\ldots,x_n) \in E^n$ is called a *norming point* of T if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1,\ldots,x_n)| = \|T\|$.

For $T \in \mathcal{L}(^n E)$, we define

$$\text{Norm}(T) = \Big\{(x_1,\ldots,x_n) \in E^n : (x_1,\ldots,x_n) \text{ is a norming point of } T\Big\}.$$

Norm(T) is called the *norming set* of T. Notice that $(x_1,\ldots,x_n) \in \text{Norm}(T)$ if and only if $(\epsilon_1 x_1,\ldots,\epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1,\ldots,n)$. Indeed, if $(x_1,\ldots,x_n) \in \text{Norm}(T)$, then

$$|T(\epsilon_1 x_1,\ldots,\epsilon_n x_n)| = |\epsilon_1\cdots\epsilon_n T(x_1,\ldots,x_n)| = |T(x_1,\ldots,x_n)| = \|T\|,$$

which shows that $(\epsilon_1 x_1,\ldots,\epsilon_n x_n) \in \text{Norm}(T)$. If $(\epsilon_1 x_1,\ldots,\epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1,\ldots,n)$, then

$$(x_1,\ldots,x_n) = \Big(\epsilon_1(\epsilon_1 x_1),\ldots,\epsilon_n(\epsilon_n x_n)\Big) \in \text{Norm}(T).$$

The following examples show that Norm$(T) = \emptyset$ or an infinite set.

Examples. (a) Let

$$T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i y_i \in \mathcal{L}_s(^2 c_0).$$

We claim that Norm$(T) = \emptyset$. Obviously, $\|T\| = 1$. Assume that Norm$(T) \neq \emptyset$. Let $((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}) \in \text{Norm}(T)$. Then,

$$1 = \Big|T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big)\Big| \leq \sum_{i=1}^{\infty} \frac{1}{2^i}|x_i|\ |y_i| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1,$$

which shows that $|x_i| = |y_i| = 1$ for all $i \in \mathbb{N}$. Hence, $(x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}} \notin c_0$. This is a contradiction. Therefore, Norm$(T) = \emptyset$.

(b) Let

$$T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) = x_1 y_1 \in \mathcal{L}_s(^2 c_0).$$

Then,

$$\text{Norm}(T) = \Big\{\Big((\pm 1, x_2, x_3, \ldots),\ (\pm 1, y_2, y_3, \ldots)\Big) \in c_0 \times c_0 \\ : |x_j| \leq 1,\ |y_j| \leq 1 \text{ for } j \geq 2\Big\}.$$

A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that

$P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$.

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}(^nE)$, we define

$$\text{Norm}(P) = \Big\{x \in E : x \text{ is a norming point of } P\Big\}.$$

$\text{Norm}(P)$ is called the *norming set* of P. Notice that $\text{Norm}(P) = \emptyset$ or a finite set or an infinite set.

Kim [7] classified $\text{Norm}(P)$ for every $P \in \mathcal{P}(^2\ell_\infty^2)$, where $\ell_\infty^2 = \mathbb{R}^2$ with the supremum norm.

If $\text{Norm}(T) \neq \emptyset$, $T \in \mathcal{L}(^nE)$ is called a *norm attaining* n-linear form and if $\text{Norm}(P) \neq \emptyset$, $P \in \mathcal{P}(^nE)$ is called a *norm attaining* n-homogeneous polynomial (see [3]).

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

It seems to be natural and interesting to study about $\text{Norm}(T)$ for $T \in \mathcal{L}(^nE)$. For $m \in \mathbb{N}$, let $\ell_1^m := \mathbb{R}^m$ with the ℓ_1-norm and $\ell_\infty^2 = \mathbb{R}^2$ with the supremum norm. Notice that if $E = \ell_1^m$ or ℓ_∞^2 and $T \in \mathcal{L}(^nE)$, $\text{Norm}(T) \neq \emptyset$ since S_E is compact. Kim [6, 8–10] classified $\text{Norm}(T)$ for every $T \in \mathcal{L}_s(^2\ell_\infty^2)$, $\mathcal{L}(^2\ell_\infty^2)$, $\mathcal{L}(^2\ell_1^2)$, $\mathcal{L}_s(^2\ell_1^3)$ or $\mathcal{L}_s(^3\ell_1^2)$. Kim [11] classified $\text{Norm}(T)$ for every $T \in \mathcal{L}(^2\mathbb{R}^2_{h(w)})$, where $\mathbb{R}^2_{h(w)}$ denotes the plane with the hexagonal norm with weight $0 < w < 1$, $\|(x, y)\|_{h(w)} = \max\{|y|, |x| + (1 - w)|y|\}$.

Let $0 \leq \theta \leq \frac{\pi}{4}$ and $\ell^2_{\infty,\theta} = \mathbb{R}^2$ with the rotated supremum norm

$$\|(x, y)\|_{(\infty,\theta)} = \max\Big\{|x\cos\theta + y\sin\theta|,\ |x\sin\theta - y\cos\theta|\Big\}.$$

Notice that $\|(x, y)\|_{(\infty,0)} = \|(x, y)\|_\infty$ and $\|(x, y)\|_{(\infty,\pi/4)} = \frac{1}{\sqrt{2}}\|(x, y)\|_1$.

In this paper, we characterize the norming sets of $\mathcal{L}(^n\ell^2_{(\infty,\theta)})$. Using this result, we completely describe the norming sets of $\mathcal{L}_s(^n\ell^2_{(\infty,\theta)})$ for $n = 3, 4, 5$. We generalize the results from [9] for $n = 3$ and $\theta = \frac{\pi}{4}$.

2. Main results

Proposition 2.1 ([10]). *Let $n, m \geq 2$. Let $T \in \mathcal{L}(^m\ell_1^n)$ with*

$$T\Big(\big(x_1^{(1)}, \ldots, x_n^{(1)}\big), \ldots, \big(x_1^{(m)}, \ldots, x_n^{(m)}\big)\Big) = \sum_{1\leq i_k\leq n,\ 1\leq k\leq m} a_{i_1\cdots i_m}\ x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}$$

for some $a_{i_1\cdots i_m} \in \mathbb{R}$. Then

$$\|T\| = \max\{|a_{i_1\cdots i_m}| : 1 \leq i_k \leq n,\ 1 \leq k \leq m\}.$$

By simplicity, we denote $T = \big(a_{i_1\cdots i_m}\big)_{1\leq i_k\leq n, 1\leq k\leq m}$. We call $a_{i_1\cdots i_m}$'s the *coefficients* of T. Notice that if $\|T\| = 1$, then $|a_{i_1\cdots i_m}| \leq 1$ for all $1 \leq i_k \leq n, 1 \leq k \leq m$.

Theorem 2.2 ([10])**.** *Let $n, m \geq 2$. Let $T \in \mathcal{L}(^m\ell_1^n)$ be the same as in Theorem A. Suppose that $((t_1^{(1)}, \ldots, t_n^{(1)}), \ldots, (t_1^{(m)}, \ldots, t_n^{(m)})) \in \mathrm{Norm}(T)$. If $|a_{i_1' \cdots i_m'}| < \|T\|$ for $1 \leq i_k' \leq n,\ 1 \leq k \leq m$, then $t_{i_1'}^{(1)} \cdots t_{i_m'}^{(m)} = 0$.*

The following characterizes the norming sets of $\mathcal{L}(^n\ell^2_{(\infty,\theta)})$.

Theorem 2.3. *Let $n \in \mathbb{N}, 0 \leq \theta \leq \frac{\pi}{4}$ and $T \in \mathcal{L}(^n\ell^2_{(\infty,\theta)})$ with $\|T\| = 1$. Then,*

$$\mathrm{Norm}(T) = \bigcup_{k=1}^{n} \big(A_k^+ \cup A_k^- \cup B_{k,1} \cup B_{k,2}\big),$$

where $W_1 = (\cos\theta - \sin\theta,\ \cos\theta + \sin\theta),\ W_2 = (\cos\theta + \sin\theta,\ -\cos\theta + \sin\theta)$,

$$\begin{aligned}
A_k^+ = \Big\{\Big(&\pm X_1, \ldots, \pm X_{k-1}, \pm(tW_1 + (1-t)W_2), \pm X_{k+1}, \ldots, \pm X_n\Big) \\
&\in \big(S_{\ell^2_{(\infty,\theta)}}\big)^n : T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, \pm X_n\big) \\
&\times T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, X_n\big) = 1,\ 0 \leq t \leq 1\Big\}, \\
A_k^- = \Big\{\Big(&\pm X_1, \ldots, \pm X_{k-1}, \pm(tW_1 - (1-t)W_2), \pm X_{k+1}, \ldots, \pm X_n\Big) \\
&\in \big(S_{\ell^2_{(\infty,\theta)}}\big)^n : T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, \pm X_n\big) \\
&\times T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, X_n\big) = -1,\ 0 \leq t \leq 1\Big\}, \\
B_{k,1} = \Big\{\Big(&\pm X_1, \ldots, \pm X_{k-1}, \pm W_1, \pm X_{k+1}, \ldots, \pm X_n\Big) \in \big(S_{\ell^2_{(\infty,\theta)}}\big)^n \\
&: 1 = \big|T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, \pm X_n\big)\big| \\
&> \big|T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, X_n\big)\big|\Big\}, \\
B_{k,2} = \Big\{\Big(&\pm X_1, \ldots, \pm X_{k-1}, \pm W_2, \pm X_{k+1}, \ldots, \pm X_n\Big) \in \big(S_{\ell^2_{(\infty,\theta)}}\big)^n \\
&: 1 = \Big|T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, \pm X_n\big)\Big| \\
&> \Big|T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, X_n\big)\Big|\Big\}.
\end{aligned}$$

Proof. Let $\mathcal{F}_k = A_k^+ \cup A_k^- \cup B_{k,1} \cup B_{k,2}$ for $k = 1, \ldots, n$.
($\subseteq$) Let $(X_1, \ldots, X_n) \in \mathrm{Norm}(T)$. Let $1 \leq k \leq n$ be fixed. Then $X_k = \lambda_1^{(k)} W_1 + \lambda_2^{(k)} W_2$ for some $\lambda_1^{(k)}, \lambda_2^{(k)} \in \mathbb{R}$ with $|\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1$.

Case 1.

$$\begin{aligned}
&T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, \pm X_n\big) \\
&\times T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, X_n\big) = 1.
\end{aligned}$$

Since $\|T\| = 1$, we have

$$1 = T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, \pm X_n\big)$$

$$= T\big(X_1, \dots, X_{k-1}, W_2, X_{k+1}, \dots, X_n\big)$$

or

$$\begin{aligned} -1 &= T\big(X_1, \dots, X_{k-1}, W_1, X_{k+1}, \dots, \pm X_n\big) \\ &= T\big(X_1, \dots, X_{k-1}, W_2, X_{k+1}, \dots, X_n\big). \end{aligned}$$

Claim 1. $X_k \in \big\{ \pm (tW_1 + (1-t)W_2) : 0 \le t \le 1 \big\}$. By n-linearity of T, it follows that

$$\begin{aligned} 1 &= T\big(X_1, \dots, X_n\big) = T\big(X_1, \dots, X_{k-1}, (\lambda_1^{(k)} W_1 + \lambda_2^{(k)} W_2), X_{k+1}, \dots, X_n\big) \\ &= \big|\lambda_1^{(k)} T\big(X_1, \dots, X_{k-1}, W_1, X_{k+1}, \dots, X_n\big) \\ &\quad + \lambda_2^{(k)} T\big(X_1, \dots, X_{k-1}, W_2, X_{k+1}, \dots, X_n\big)\big| \\ &= |\lambda_1^{(k)} + \lambda_2^{(k)}| \le |\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1. \end{aligned}$$

Thus, $|\lambda_1^{(k)} + \lambda_2^{(k)}| = |\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1$. Hence, $\mathrm{sign}(\lambda_1^{(k)}) = \mathrm{sign}(\lambda_1^{(k)})$. Thus,

$$\begin{aligned} X_k &\in \big\{|\lambda_1^{(k)}| W_1 + |\lambda_2^{(k)}| W_2, -(|\lambda_1^{(k)}| W_1 + |\lambda_2^{(k)}| W_2)\big\} \\ &\subseteq \big\{ \pm (te_1 + (1-t)e_2) : 0 \le t \le 1 \big\}. \end{aligned}$$

Therefore, $X \in A_k^+ \subseteq \mathcal{F}_k \subseteq \bigcup_{j=1}^n \mathcal{F}_j$.

Case 2.

$$\begin{aligned} &T\big(X_1, \dots, X_{k-1}, W_1, X_{k+1}, \dots, \pm X_n\big) \\ &\times T\big(X_1, \dots, X_{k-1}, W_2, X_{k+1}, \dots, X_n\big) = -1. \end{aligned}$$

Since $\|T\| = 1$, we have

$$\begin{aligned} 1 &= T\big(X_1, \dots, X_{k-1}, W_1, X_{k+1}, \dots, \pm X_n\big) \\ &= -T\big(X_1, \dots, X_{k-1}, W_2, X_{k+1}, \dots, X_n\big) \end{aligned}$$

or

$$\begin{aligned} -1 &= T\big(X_1, \dots, X_{k-1}, W_1, X_{k+1}, \dots, \pm X_n\big) \\ &= -T\big(X_1, \dots, X_{k-1}, W_2, X_{k+1}, \dots, X_n\big). \end{aligned}$$

Claim 2. $X_k \in \big\{ \pm (tW_1 - (1-t)W_2) : 0 \le t \le 1 \big\}$. It follows that

$$\begin{aligned} 1 &= T\big(X_1, \dots, X_n\big) = T\big(X_1, \dots, X_{k-1}, (\lambda_1^{(k)} W_1 + \lambda_2^{(k)} W_2), X_{k+1}, \dots, X_n\big) \\ &= \big|\lambda_1^{(k)} T\big(X_1, \dots, X_{k-1}, W_1, X_{k+1}, \dots, X_n\big) \\ &\quad + \lambda_2^{(k)} T\big(X_1, \dots, X_{k-1}, W_2, X_{k+1}, \dots, X_n\big)\big| \\ &= |\lambda_1^{(k)} - \lambda_2^{(k)}| \le |\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1. \end{aligned}$$

Thus, $|\lambda_1^{(k)} - \lambda_2^{(k)}| = |\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1$. Hence, $\mathrm{sign}(\lambda_1^{(k)}) = -\mathrm{sign}(\lambda_2^{(k)})$. Thus,

$$X_k \in \big\{|\lambda_1^{(k)}| W_1 - |\lambda_2^{(k)}| W_2, -(|\lambda_1^{(k)}| W_1 - |\lambda_2^{(k)}| W_2)\big\}$$

$$\subseteq \big\{ \pm(tW_1 - (1-t)W_2) : 0 \le t \le 1 \big\}.$$

Therefore, $X \in A_k^- \subseteq \mathcal{F}_k \subseteq \bigcup_{j=1}^n \mathcal{F}_j$.

Case 3.

$$\begin{aligned} 1 &= \big|T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, \pm X_n\big)\big| \\ &> \big|T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, X_n\big)\big|. \end{aligned}$$

Claim 3. $\lambda_2^{(k)} = 0$. Assume that $\lambda_2^{(k)} \neq 0$. It follows that

$$\begin{aligned} 1 &= \big|T\big(X_1, \ldots, X_n\big)\big| = \big|T\big(X_1, \ldots, X_{k-1}, (\lambda_1^{(k)} W_1 + \lambda_2^{(k)} W_2), X_{k+1}, \ldots, X_n\big)\big| \\ &\le |\lambda_1^{(k)}| \, \big|T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, X_n\big)\big| \\ &\quad + |\lambda_2^{(k)}| \, \big|T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, X_n\big)\big| \\ &< |\lambda_1^{(k)}| \big|T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, X_n\big)\big| + |\lambda_2^{(k)}| \\ &\le |\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1, \end{aligned}$$

which is a contradiction. Thus, $\lambda_2^{(k)} = 0$ and so $X_k = W_1$. Therefore, $X \in B_{k,1} \subseteq \mathcal{F}_k \subseteq \bigcup_{j=1}^n \mathcal{F}_j$.

Case 4.

$$\begin{aligned} 1 &= \big|T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, \pm X_n\big)\big| \\ &> \big|T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, X_n\big)\big|. \end{aligned}$$

Claim 4. $\lambda_1^{(k)} = 0$. Assume that $\lambda_1^{(k)} \neq 0$. It follows that

$$\begin{aligned} 1 &= \big|T\big(X_1, \ldots, X_n\big)\big| \\ &= \big|T\big(X_1, \ldots, X_{k-1}, (\lambda_1^{(k)} W_1 + \lambda_2^{(k)} W_2), X_{k+1}, \ldots, X_n\big)\big| \\ &\le |\lambda_1^{(k)}| \, \big|T\big(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, X_n\big)\big| \\ &\quad + |\lambda_2^{(k)}| \, \big|T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, X_n\big)\big| \\ &< |\lambda_1^{(k)}| + |\lambda_2^{(k)}| \big|T\big(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, X_n\big)\big| \\ &\le |\lambda_1^{(k)}| + |\lambda_2^{(k)}| \\ &\le 1, \end{aligned}$$

which is a contradiction. Thus, $\lambda_1^{(k)} = 0$ and so $X_k = W_2$. Therefore, $X \in B_{k,2} \subseteq \mathcal{F}_k \subseteq \bigcup_{j=1}^n \mathcal{F}_j$.

($\supseteq$) We will show that $\mathcal{F}_k \subseteq \text{Norm}(T)$ for every $1 \le k \le n$.

Let $1 \le k \le n$ be fixed and $Y = (Y_1, \ldots, Y_n) \in \mathcal{F}_k$. Suppose that $Y \in A_k^+$. Then $Y_k = \pm(t_k W_1 + (1-t_k) W_2)$ for some $0 \le t_k \le 1$ and

$$T\big(Y_1, \ldots, Y_{k-1}, W_1, Y_{k+1}, \ldots, \pm Y_n\big) \cdot T\big(Y_1, \ldots, Y_{k-1}, W_2, Y_{k+1}, \ldots, Y_n\big) = 1.$$

It follows that

$$\begin{aligned}1 &= T\big(Y_1, \dots, Y_n\big) = T\big(Y_1, \dots, Y_{k-1}, t_k W_1 + (1-t_k)W_2, Y_{k+1}, \dots, Y_n\big)\\ &= \big|t_k T\big(Y_1, \dots, Y_{k-1}, W_1, Y_{k+1}, \dots, Y_n\big)\\ &\quad + (1-t_k)T\big(Y_1, \dots, Y_{k-1}, W_2, Y_{k+1}, \dots, Y_n\big)\big|\\ &= |t_k + (1-t_k)| = 1.\end{aligned}$$

Thus, $Y \in \mathrm{Norm}(T)$.

Suppose that $Y \in A_k^-$. Then $Y_k = \pm(t_k W_1 - (1-t_k)W_2)$ for some $0 \le t_k \le 1$ and $T\big(Y_1, \dots, Y_{k-1}, W_1, Y_{k+1}, \dots, \pm Y_n\big) \cdot T\big(Y_1, \dots, Y_{k-1}, W_2, Y_{k+1}, \dots, Y_n\big) = -1$. It follows that

$$\begin{aligned}1 &= T\big(Y_1, \dots, Y_n\big) = T\big(Y_1, \dots, Y_{k-1}, t_k W_1 - (1-t_k)W_2, Y_{k+1}, \dots, Y_n\big)\\ &= \big|t_k T\big(Y_1, \dots, Y_{k-1}, W_1, Y_{k+1}, \dots, Y_n\big)\\ &\quad - (1-t_k)T\big(Y_1, \dots, Y_{k-1}, W_2, Y_{k+1}, \dots, Y_n\big)\big|\\ &= |t_k + (1-t_k)| = 1.\end{aligned}$$

Thus, $Y \in \mathrm{Norm}(T)$.

Suppose that $Y \in B_{k,1}$. Then $Y_k = \pm W_1$ and

$$\big|T\big(Y_1, \dots, Y_n\big)\big| = \big|T\big(Y_1, \dots, Y_{k-1}, W_1, Y_{k+1}, \dots, \pm Y_n\big)\big| = 1.$$

Thus, $Y \in \mathrm{Norm}(T)$.

Suppose that $Y \in B_{k,2}$. Then $Y_k = \pm W_2$ and

$$\big|T\big(Y_1, \dots, Y_n\big)\big| = \big|T\big(Y_1, \dots, Y_{k-1}, W_2, Y_{k+1}, \dots, \pm Y_n\big)\big| = 1.$$

Thus, $Y \in \mathrm{Norm}(T)$. We complete the proof. □

Let $\mathcal{W} \subseteq (S_{\ell^2_{(\infty,\theta)}})^n$. We denote

$$\begin{aligned}&\mathrm{Sym}(\mathcal{W})\\ &= \Big\{\big((x_{\sigma(1)}, y_{\sigma(1)}), \dots, (x_{\sigma(n)}, y_{\sigma(n)})\big) : X = \big((x_1, y_1), \dots, (x_n, y_n)\big) \in \mathcal{W},\\ &\quad \sigma \text{ is a permutation on } \{1, \dots, n\}\Big\}.\end{aligned}$$

We are in a position to classify $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}_s({}^5\ell^2_{(\infty,\theta)})$.

Theorem 2.4. *Let $0 \le \theta \le \frac{\pi}{4}$ and*

$$W_1 = (-\sin\theta + \cos\theta,\ \sin\theta + \cos\theta) \text{ and } W_2 = (\sin\theta + \cos\theta,\ \sin\theta - \cos\theta).$$

Let

$$T\big((x_1^{(1)}, x_2^{(1)}), \dots, (x_1^{(5)}, x_2^{(5)})\big) = \sum_{i_k = 1,2, k=1,\dots,5} a_{i_1 i_2 i_3 i_4 i_5} x_{i_1}^{(1)} x_{i_2}^{(2)} x_{i_3}^{(3)} x_{i_4}^{(4)} x_{i_5}^{(5)} \in \mathcal{L}_s({}^5\ell^2_{(\infty,\theta)})$$

with $\|T\| = 1$. Then the following assertions hold: Let $A_{i_1 i_2 i_3 i_4 i_5} = T(W_{i_1}, W_{i_2}, W_{i_3}, W_{i_4}, W_{i_5})$ for $i_k = 1, 2$ and $A_{11111} \ge 0$.

Case 1. $A_{11111} = |A_{i_1i_2i_3i_4i_5}| = 1$ *for all* $(i_1, i_2, i_3, i_4, i_5) \neq (1,1,1,1,1,1)$.

1.1. $A_{11111} = A_{11112} = A_{22222} = A_{11122} = A_{11222} = A_{12222} = 1$

$$\begin{aligned}\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\big(&\pm (tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2),\\ &\pm (uW_1 + (1-u)W_2), \pm(vW_1 + (1-v)W_2),\\ &\pm (wW_1 + (1-w)W_2)\big) : 0 \leq t, s, u, v, w \leq 1\Big\}\Big).\end{aligned}$$

1.2. $A_{11111} = -A_{11112} = A_{22222} = A_{11122} = A_{11222} = A_{12222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm (tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2), \pm W_1, \pm W_1, \pm W_1\big),\\ &\quad \big(\pm (tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2), \pm W_1, \pm W_2, \pm W_2\big),\\ &\quad \big(\pm (tW_1 - (1-t)W_2), \pm W_2, \pm W_2, \pm W_2, \pm W_2\big) : 0 \leq t, s \leq 1\Big\}\Big).\end{aligned}$$

1.3. $A_{11111} = A_{11112} = -A_{22222} = A_{11122} = A_{11222} = A_{12222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm (tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2), \pm(uW_1 + (1-u)W_2),\\ &\quad \pm (vW_1 + (1-v)W_2), \pm W_1\big), \big(\pm (tW_1 - (1-t)W_2), \pm W_2, \pm W_2, \pm W_2,\\ &\quad \pm W_2\big) : 0 \leq t, s, u, v \leq 1\Big\}\Big).\end{aligned}$$

1.4. $A_{11111} = A_{11112} = A_{22222} = -A_{11122} = A_{11222} = A_{12222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm (tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2), \pm W_1, \pm W_1, \pm W_2\big),\\ &\quad \big(\pm (tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2), \pm W_2, \pm W_2, \pm W_2\big),\\ &\quad \big(\pm (tW_1 + (1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_1\big) : 0 \leq t, s \leq 1\Big\}\Big).\end{aligned}$$

1.5. $A_{11111} = A_{11112} = A_{22222} = A_{11122} = -A_{11222} = A_{12222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm (tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2), \pm W_1, \pm W_1, \pm W_1\big),\\ &\quad \big(\pm (tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2), \pm W_1, \pm W_2, \pm W_2\big),\\ &\quad \big(\pm (tW_1 + (1-t)W_2), \pm W_2, \pm W_2, \pm W_2, \pm W_2\big) : 0 \leq t, s \leq 1\Big\}\Big).\end{aligned}$$

1.6. $A_{11111} = A_{11112} = A_{22222} = A_{11122} = A_{11222} = -A_{12222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm (tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2), \pm(uW_1 + (1-u)W_2),\end{aligned}$$

$$\pm W_1, \pm W_1\big), \big(\pm (tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2), \pm W_2, \pm W_2, \pm W_2\big) : 0 \le t, s, u \le 1\Big\}\Big).$$

1.7. $A_{11111} = -A_{11112} = -A_{22222} = A_{11122} = A_{11222} = A_{12222} = 1$

$$\begin{aligned}&\text{Norm}(T)\\ &= \text{Sym}\Big(\Big\{\big(\pm (tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2), \pm(uW_1 + (1-u)W_2),\\ &\quad \pm W_2, \pm W_2\big), \big(\pm (tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2),\\ &\quad \pm W_1, \pm W_1, \pm W_1\big) : 0 \le t, s, u \le 1\Big\}\Big).\end{aligned}$$

1.8. $A_{11111} = -A_{11112} = A_{22222} = -A_{11122} = A_{11222} = A_{12222} = 1$

$$\begin{aligned}&\text{Norm}(T)\\ &= \text{Sym}\Big(\Big\{\big(\pm (tW_1 + (1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_2\big),\\ &\quad \big(\pm (tW_1 + (1-t)W_2), \pm W_1, \pm W_2, \pm W_2, \pm W_2\big),\\ &\quad \big(\pm (tW_1 - (1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_1\big),\\ &\quad \big(\pm (tW_1 - (1-t)W_2), \pm W_1, \pm W_1, \pm W_2, \pm W_2\big),\\ &\quad \big(\pm (tW_1 - (1-t)W_2), \pm W_2, \pm W_2, \pm W_2, \pm W_2\big) : 0 \le t \le 1\Big\}\Big).\end{aligned}$$

1.9. $A_{11111} = -A_{11112} = A_{22222} = A_{11122} = -A_{11222} = A_{12222} = 1$

$$\begin{aligned}&\text{Norm}(T)\\ &= \text{Sym}\Big(\Big\{\big(\pm (tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2), \pm(uW_1 - (1-u)W_2),\\ &\quad \pm (vW_1 - (1-v)W_2), \pm(wW_1 - (1-w)W_2)\big) : 0 \le t, s, u, v, w \le 1\Big\}\Big).\end{aligned}$$

1.10. $A_{11111} = -A_{11112} = A_{22222} = A_{11122} = A_{11222} = -A_{12222} = 1$

$$\begin{aligned}\text{Norm}(T) = \text{Sym}\Big(\Big\{&\big(\pm (tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2),\\ &\pm W_1, \pm W_1, \pm W_1\big), \big(\pm (tW_1 + (1-t)W_2), \pm W_2, \pm W_2, \pm W_2,\\ &\pm W_2\big), \big(\pm (tW_1 - (1-t)W_2), \pm W_1, \pm W_2, \pm W_2, \pm W_2\big)\\ &: 0 \le t, s \le 1\Big\}\Big).\end{aligned}$$

1.11. $A_{11111} = A_{11112} = -A_{22222} = -A_{11122} = A_{11222} = A_{12222} = 1$

$$\begin{aligned}&\text{Norm}(T)\\ &= \text{Sym}\Big(\Big\{\big(\pm (tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2), \pm W_1, \pm W_1, \pm W_2\big),\\ &\quad \big(\pm (tW_1 - (1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_1\big),\\ &\quad \big(\pm (tW_1 - (1-t)W_2), \pm W_2, \pm W_2, \pm W_2, \pm W_2\big),\end{aligned}$$

$$\big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_2, \pm W_2, \pm W_2\big) : 0 \le t, s \le 1\Big\}\Big).$$

1.12. $A_{11111} = A_{11112} = -A_{22222} = A_{11122} = -A_{11222} = A_{12222} = 1$

$$\begin{aligned}\mathrm{Norm}(T) = \ &\mathrm{Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2), \pm(sW_1-(1-s)W_2),\\ &\pm(uW_1-(1-u)W_2), \pm W_2, \pm W_2\big), \big(\pm(tW_1+(1-t)W_2),\\ &\pm(sW_1+(1-s)W_2), \pm W_1, \pm W_1, \pm W_1\big) : 0 \le t, s, u \le 1\Big\}\Big).\end{aligned}$$

1.13. $A_{11111} = A_{11112} = -A_{22222} = A_{11122} = A_{11222} = -A_{12222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ =\ &\mathrm{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2), \pm(sW_1+(1-s)W_2), \pm(uW_1+(1-u)W_2),\\ &\pm W_1, \pm W_1\big), \big(\pm(tW_1-(1-t)W_2), \pm W_1, \pm W_2, \pm W_2, \pm W_2\big),\\ &\big(\pm(tW_1+(1-t)W_2), \pm W_2, \pm W_2, \pm W_2, \pm W_2\big) : 0 \le t, s \le 1\Big\}\Big).\end{aligned}$$

1.14. $A_{11111} = A_{11112} = A_{22222} = -A_{11122} = -A_{11222} = A_{12222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ =\ &\mathrm{Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_2\big),\\ &\big(\pm(tW_1-(1-t)W_2), \pm W_1, \pm W_2, \pm W_2, \pm W_2\big),\\ &\big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm_1\big),\\ &\big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_1, \pm W_2, \pm W_2\big),\\ &\big(\pm(tW_1+(1-t)W_2), \pm W_2, \pm W_2, \pm W_2, \pm W_2\big) : 0 \le t \le 1\Big\}\Big).\end{aligned}$$

1.15. $A_{11111} = A_{11112} = A_{22222} = -A_{11122} = A_{11222} = -A_{12222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ =\ &\mathrm{Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2), \pm(sW_1-(1-s)W_2), \pm(uW_1-(1-u)W_2),\\ &\pm W_1, \pm W_2\big), \big(\pm(tW_1-(1-t)W_2), \pm W_1, \pm W_2, \pm W_2, \pm W_2\big),\\ &\big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_1\big) : 0 \le t, s, u \le 1\Big\}\Big).\end{aligned}$$

1.16. $A_{11111} = A_{11112} = A_{22222} = A_{11122} = -A_{11222} = -A_{12222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ =\ &\mathrm{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2), \pm(sW_1+(1-s)W_2),\\ &\pm W_1, \pm W_1, \pm W_1\big), \big(\pm(tW_1-(1-t)W_2), \pm W_2, \pm W_2, \pm W_2, \pm W_2\big),\\ &\big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_2, \pm W_2, \pm W_2\big) : 0 \le t, s \le 1\Big\}\Big).\end{aligned}$$

1.17. $A_{11111} = -A_{11112} = -A_{22222} = -A_{11122} = A_{11222} = A_{12222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2), \pm W_2, \pm W_2, \pm W_2\big),\\ &\big(\pm(tW_1 + (1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_2\big),\\ &\big(\pm(tW_1 - (1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_1\big),\\ &\big(\pm(tW_1 - (1-t)W_2), \pm W_1, \pm W_1, \pm W_2, \pm W_2\big) : 0 \le t, s \le 1\Big\}\Big).\end{aligned}$$

1.18. $A_{11111} = -A_{11112} = -A_{22222} = A_{11122} = -A_{11222} = A_{12222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2), \pm(uW_1 - (1-u)W_2),\\ &\pm(vW_1 - (1-v)W_2), \pm W_1\big), \big(\pm(tW_1 + (1-t)W_2), \pm W_2, \pm W_2, \pm W_2,\\ &\pm W_2\big) : 0 \le t, s, u, v \le 1\Big\}\Big).\end{aligned}$$

1.19. $A_{11111} = -A_{11112} = -A_{22222} = A_{11122} = A_{11222} = -A_{12222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2), \pm W_1, \pm W_1, \pm W_1\big),\\ &\big(\pm(tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2), \pm W_1, \pm W_2, \pm W_2\big),\\ &\big(\pm(tW_1 - (1-t)W_2), \pm W_1, \pm W_2, \pm W_2, \pm W_2\big) : 0 \le t, s \le 1\Big\}\Big).\end{aligned}$$

1.20. $A_{11111} = -A_{11112} = A_{22222} = -A_{11122} = -A_{11222} = A_{12222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2), \pm W_1, \pm W_1, \pm W_2\big),\\ &\big(\pm(tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2), \pm W_2, \pm W_2, \pm W_2\big),\\ &\big(\pm(tW_1 + (1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_1\big) : 0 \le t, s \le 1\Big\}\Big).\end{aligned}$$

1.21. $A_{11111} = -A_{11112} = A_{22222} = -A_{11122} = A_{11222} = -A_{12222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2), \pm W_1, \pm W_2, \pm W_2\big),\\ &\big(\pm(tW_1 + (1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_2\big),\\ &\big(\pm(tW_1 + (1-t)W_2), \pm W_2, \pm W_2, \pm W_2, \pm W_2\big),\\ &\big(\pm(tW_1 - (1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_1\big) : 0 \le t, s \le 1\Big\}\Big).\end{aligned}$$

1.22. $A_{11111} = -A_{11112} = A_{22222} = A_{11122} = -A_{11222} = -A_{12222} = 1$

$\mathrm{Norm}(T)$

$$= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2),\pm(sW_1-(1-s)W_2),\pm(uW_1-(1-u)W_2),$$
$$\pm W_1,\pm W_1\big),\big(\pm(tW_1+(1-t)W_2),\pm(sW_1+(1-s)W_2),$$
$$\pm W_2,\pm W_2,\pm W_2\big) : 0\le t,s,u\le 1\Big\}\Big).$$

1.23. $A_{11111} = A_{11112} = -A_{22222} = -A_{11122} = -A_{11222} = A_{12222} = 1$

$$\mathrm{Norm}(T)$$
$$= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2),\pm(sW_1-(1-s)W_2),\pm W_2,\pm W_2,\pm W_2\big),$$
$$\big(\pm(tW_1-(1-t)W_2),\pm W_1,\pm W_1,\pm W_1,\pm W_2\big),$$
$$\big(\pm(tW_1+(1-t)W_2),\pm W_1,\pm W_1,\pm W_1,\pm W_1\big),$$
$$\big(\pm(tW_1+(1-t)W_2),\pm W_1,\pm W_1,\pm W_2,\pm W_2\big) : 0\le t,s\le 1\Big\}\Big).$$

1.24. $A_{11111} = A_{11112} = -A_{22222} = -A_{11122} = A_{11222} = -A_{12222} = 1$

$$\mathrm{Norm}(T)$$
$$= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2),\pm(sW_1-(1-s)W_2),\pm(uW_1-(1-u)W_2),$$
$$\pm W_2,\pm W_2\big),\big(\pm(tW_1-(1-t)W_2),\pm W_1,\pm W_1,\pm W_1,\pm W_1\big)$$
$$: 0\le t,s,u\le 1\Big\}\Big).$$

1.25. $A_{11111} = A_{11112} = -A_{22222} = A_{11122} = -A_{11222} = -A_{12222} = 1$

$$\mathrm{Norm}(T)$$
$$= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2),\pm(sW_1+(1-s)W_2),\pm W_1,\pm W_1,\pm W_1\big),$$
$$\big(\pm(tW_1-(1-t)W_2),\pm(sW_1-(1-s)W_2),\pm W_1,\pm W_2,\pm W_2\big),$$
$$\big(\pm(tW_1+(1-t)W_2),\pm W_1,\pm W_2,\pm W_2,\pm W_2\big) : 0\le t,s\le 1\Big\}\Big).$$

1.26. $A_{11111} = A_{11112} = -A_{22222} = -A_{11122} = A_{11222} = -A_{12222} = 1$

$$\mathrm{Norm}(T)$$
$$= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2),\pm(sW_1-(1-s)W_2),\pm W_1,\pm W_1,\pm W_2\big),$$
$$\big(\pm(tW_1-(1-t)W_2),\pm W_2,\pm W_2,\pm W_2,\pm W_2\big),$$
$$\big(\pm(tW_1+(1-t)W_2),\pm W_1,\pm W_1,\pm W_1,\pm W_1\big) : 0\le t,s\le 1\Big\}\Big).$$

1.27. $A_{11111} = A_{11112} = A_{22222} = -A_{11122} = -A_{11222} = -A_{12222} = 1$

$$\mathrm{Norm}(T)$$
$$= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2),\pm(sW_1+(1-s)W_2),\pm W_1,\pm W_2,\pm W_2\big),$$
$$\big(\pm(tW_1-(1-t)W_2),\pm W_1,\pm W_1,\pm W_1,\pm W_2\big),$$

$\big(\pm(tW_1-(1-t)W_2), \pm W_2, \pm W_2, \pm W_2, \pm W_2\big),$

$\big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_1\big) : 0 \le t, s \le 1\Big\}\Big).$

1.28. $A_{11111} = -A_{11112} = -A_{22222} = -A_{11122} = -A_{11222} = A_{12222} = 1$

$$\begin{aligned}
&\mathrm{Norm}(T)\\
&= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2), \pm(sW_1+(1-s)W_2), \pm W_1, \pm W_1, \pm W_2\big),\\
&\big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_1\big),\\
&\big(\pm(tW_1+(1-t)W_2), \pm W_2, \pm W_2, \pm W_2, \pm W_2\big),\\
&\big(\pm(tW_1-(1-t)W_2), \pm W_1, \pm W_2, \pm W_2, \pm W_2\big) : 0 \le t, s \le 1\Big\}\Big).
\end{aligned}$$

1.29. $A_{11111} = -A_{11112} = -A_{22222} = A_{11122} = -A_{11222} = -A_{12222} = 1$

$$\begin{aligned}
&\mathrm{Norm}(T)\\
&= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2), \pm(sW_1-(1-s)W_2), \pm(uW_1-(1-u)W_2),\\
&\pm W_1, \pm W_1\big), \big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_2, \pm W_2, \pm W_2\big),\\
&\big(\pm(tW_1-(1-t)W_2), \pm W_2, \pm W_2, \pm W_2, \pm W_2\big) : 0 \le t, s \le 1\Big\}\Big).
\end{aligned}$$

1.30. $A_{11111} = A_{11112} = -A_{22222} = -A_{11122} = -A_{11222} = -A_{12222} = 1$

$$\begin{aligned}
&\mathrm{Norm}(T)\\
&= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2), \pm(sW_1+(1-s)W_2), \pm(uW_1+(1-u)W_2),\\
&\pm W_2, \pm W_2\big), \big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_1\big)\\
&: 0 \le t, s, u \le 1\Big\}\Big).
\end{aligned}$$

1.31. $A_{11111} = -A_{11112} = A_{22222} = -A_{11122} = -A_{11222} = -A_{12222} = 1$

$$\begin{aligned}
&\mathrm{Norm}(T)\\
&= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2), \pm(sW_1+(1-s)W_2), \pm(uW_1+(1-u)W_2),\\
&\pm W_1, \pm W_2\big), \big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_2, \pm W_2, \pm W_2\big),\\
&\big(\pm(tW_1-(1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_1\big) : 0 \le t, s, u \le 1\Big\}\Big).
\end{aligned}$$

1.32. $A_{11111} = -A_{11112} = -A_{22222} = -A_{11122} = -A_{11222} = -A_{12222} = 1$

$$\begin{aligned}
&\mathrm{Norm}(T)\\
&= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2), \pm(sW_1+(1-s)W_2), \pm W_1, \pm W_1, \pm W_2\big),\\
&\big(\pm(tW_1+(1-t)W_2), \pm W_2, \pm W_2, \pm W_2, \pm W_2\big),\\
&\big(\pm(tW_1-(1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_1\big) : 0 \le t, s \le 1\Big\}\Big).
\end{aligned}$$

Case 2. *$|A_{i_1\cdots i_5}|<1$ for some $i_k\in\{1,2\}$ $(k=1,\dots,5)$.*

Let $M=\{(i_1,\dots,i_5):|A_{i_1\cdots i_5}|<1\}$ and define $S=\big(b_{i_1\cdots i_5}\big)\in\mathcal{L}_s({}^5\ell^2_{(\infty,\theta)})$ be such that $b_{i_1\cdots i_5}=A_{i_1\cdots i_5}$ if $(i_1,\dots,i_5)\notin M$ and $b_{i_1\cdots i_5}=1$ if $(i_1,\dots,i_5)\in M$. (Notice that S is included in Case 1.) *Then,*

$$\begin{aligned}&\mathrm{Norm}(T)\\&=\bigcap_{(i_1,\dots,i_5)\in M}\mathrm{Sym}\Big(\Big\{\big(t_1^{(1)}W_1+t_2^{(1)}W_2,\dots,t_1^{(5)}W_1+t_2^{(5)}W_2\big)\in\mathrm{Norm}(S)\\&\quad:t_{i_1}^{(1)}\cdots t_{i_5}^{(5)}=0\Big\}\Big).\end{aligned}$$

Proof. We define $S_T\in\mathcal{L}_s({}^5\ell_1^2)$ by

$$\begin{aligned}S_T\Big((t_1^{(1)},t_2^{(1)}),\dots,(t_1^{(5)},t_2^{(5)})\Big)&=T\Big(t_1^{(1)}W_1+t_2^{(1)}W_2,\dots,t_1^{(5)}W_1+t_2^{(5)}W_2\Big)\\&=\sum_{1\le k\le 5,\ i_k=1,2}A_{i_1\cdots i_5}t_{i_1}^{(1)}\cdots t_{i_5}^{(5)}.\end{aligned}$$

Notice that

$$\begin{aligned}\mathrm{Norm}(T)=\Big\{&\Big(t_1^{(1)}W_1+t_2^{(1)}W_2,\dots,t_1^{(5)}W_1+t_2^{(5)}W_2\Big)\\&:\Big((t_1^{(1)},t_2^{(1)}),\dots,(t_1^{(5)},t_2^{(5)})\Big)\in\mathrm{Norm}(S_T)\Big\}.\end{aligned}$$

Note that

$$\begin{aligned}(\star)\qquad &S_T\big((x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4),(x_5,y_5)\big)\\&=x_1\Big\{x_2\big[x_3(x_4[A_{11111}x_5+A_{11112}y_5]+y_4[A_{11112}x_5+A_{11122}y_5])\\&\quad+y_3(x_4[A_{11112}x_5+A_{11122}y_5]+y_4[A_{11122}x_5+A_{11222}y_5])\big]\\&\quad+y_2\big[x_3(x_4[A_{11112}x_5+A_{11122}y_5]+y_4[A_{11122}x_5+A_{11222}y_5])\\&\quad+y_3(x_4[A_{11112}x_5+A_{11222}y_5]+y_4[A_{11222}x_5+A_{12222}y_5])\big]\Big\}\\&\quad+y_1\Big\{x_2\big[x_3(x_4[A_{11112}x_5+A_{11122}y_5]+y_4[A_{11122}x_5+A_{11222}y_5])\\&\quad+y_3(x_4[A_{11122}x_5+A_{11222}y_5]+y_4[A_{11222}x_5+A_{12222}y_5])\big]\\&\quad+y_2\big[x_3(x_4[A_{11122}x_5+A_{11222}y_5]+y_4[A_{11222}x_5+A_{12222}y_5])\\&\quad+y_3(x_4[A_{11222}x_5+A_{12222}y_5]+y_4[A_{12222}x_5+A_{22222}y_5]]\Big\}.\end{aligned}$$

By $(\star)$, it follows that

$$\begin{aligned}\mathrm{Norm}(S_T)\supseteq\ \mathrm{Sym}\Big(\Big\{&\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_1,\pm e_1,\pm e_1\big),\\&\big(\pm(te_1+A_{11122}(1-t)e_2),\pm e_1,\pm e_1,\pm e_1,\pm e_2\big),\\&\big(\pm(te_1+A_{11122}e(1-t)e_2),\pm e_1,\pm e_1,\pm e_2,\pm e_2\big),\\&\big(\pm te_1+A_{22222}A_{12222}(1-t)e_2),\pm e_1,\pm e_2,\pm e_2,\pm e_2\big),\end{aligned}$$

$$\big(\pm (te_1 + A_{22222}A_{12222}(1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) \\ : 0 \le t \le 1\Big\}\Big),$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Thus,

$$(\star\star) \qquad \mathrm{Norm}(T) \\ \supseteq \mathrm{Sym}\Big(\Big\{\big(\pm (tW_1 + (1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_1\big), \\ \big(\pm (tW_1 + A_{11122}(1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_2\big), \\ \big(\pm (tW_1 + A_{11122}A_{11222}(1-t)W_2), \pm W_1, \pm W_1, \pm W_2, \pm W_2\big), \\ \big(\pm tW_1 + A_{22222}A_{12222}(1-t)W_2), \pm W_1, \pm W_2, \pm W_2, \pm W_2\big), \\ \big(\pm (tW_1 + A_{22222}A_{12222}(1-t)W_2), \pm W_2, \pm W_2, \pm W_2, \pm W_2\big) \\ : 0 \le t \le 1\Big\}\Big).$$

Case 1. $A_{11111} = |A_{i_1i_2i_3i_4i_5}| = 1$ for all $(i_1, i_2, i_3, i_4, i_5) \neq (1, 1, 1, 1, 1, 1)$.

We only give the proof of subcase 1.30 because the proofs of the other subcases are similar.

1.30. $A_{11111} = A_{11112} = -A_{22222} = -A_{11122} = -A_{11222} = -A_{12222} = 1$.

By $(\star\star)$ and Theorem 2.3,

$$\mathrm{Norm}(T) \\ = \mathrm{Sym}\Big(\Big\{\big(\pm (tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2), \pm(uW_1 + (1-u)W_2), \\ \pm W_2, \pm W_2\big), \big(\pm (tW_1 + (1-t)W_2), \pm W_1, \pm W_1, \pm W_1, \pm W_1\big) \\ : 0 \le t, s, u \le 1\Big\}\Big).$$

The proof of Case 2 follows from Theorem 2.2 and Case 1. Therefore, we complete the proof. □

Remark 2.5. (a) Since $\mathcal{L}_s({}^5\ell^2_{(\infty,0)}) = \mathcal{L}_s({}^5\ell^2_{\infty})$, Theorem 2.4 classifies the norming sets of $\mathcal{L}_s({}^5\ell^2_{\infty})$.
(b) By the fact that $\mathcal{L}_s({}^5\ell^2_{(\infty,\frac{\pi}{4})}) = \mathcal{L}_s({}^5\ell^2_1)$ and $\|(x, y)\|_{(\infty,\frac{\pi}{4})} = \frac{1}{\sqrt{2}}\|(x, y)\|_1$, Theorem 2.4 classifies the norming sets of $\mathcal{L}_s({}^5\ell^2_1)$.

We classify the norming set of $T \in \mathcal{L}_s({}^4\ell^2_{(\infty,\theta)})$.

Theorem 2.6. *Let* $0 \le \theta \le \frac{\pi}{4}$ *and*

$$W_1 = (-\sin\theta + \cos\theta,\ \sin\theta + \cos\theta) \text{ and } W_2 = (\sin\theta + \cos\theta,\ \sin\theta - \cos\theta).$$

Let

$$T\big((x_1^{(1)}, x_2^{(1)}), \ldots, (x_1^{(4)}, x_2^{(4)})\big) = \sum_{i_k=1,2,k=1,\ldots,4} a_{i_1i_2i_3i_4} x_{i_1}^{(1)} x_{i_2}^{(2)} x_{i_3}^{(3)} x_{i_4}^{(4)}$$

$$\in \mathcal{L}_s({}^4\ell^2_{(\infty,\theta)})$$

with $\|T\| = 1$.

Then the following assertions hold: Let $A_{i_1 i_2 i_3 i_4} = T(W_{i_1}, W_{i_2}, W_{i_3}, W_{i_4})$ *for* $i_k = 1, 2$ *and* $A_{1111} \geq 0$.

Case 1. $A_{1111} = |A_{i_1 i_2 i_3 i_4}| = 1$ *for every* $(i_1, i_2, i_3, i_4) \neq (1, 1, 1, 1)$.

1.1. $A_{i_1 i_2 i_3 i_4} = 1$ *for every* i_k

$$\begin{aligned}&\text{Norm}(T)\\ &= \Big\{\big(\pm(tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2), \pm(uW_1 + (1-u)W_2),\\ &\quad \pm(vW_1 + (1-v)W_2)\big) : 0 \leq t, s, u, v \leq 1\Big\}.\end{aligned}$$

1.2. $A_{1111} = -A_{2222} = A_{1112} = A_{1122} = A_{1222} = 1$

$$\begin{aligned}&\text{Norm}(T)\\ &= \text{Sym}\Big(\Big\{\big(\pm(tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2),\\ &\quad \pm(uW_1 + (1-u)W_2), \pm W_1\big), \big(\pm(tW_1 - (1-t)W_2), \pm W_2, \pm W_2, \pm W_2\big)\\ &\quad : 0 \leq t, s, u \leq 1\Big\}\Big).\end{aligned}$$

1.3. $A_{1111} = A_{2222} = -A_{1112} = A_{1122} = A_{1222} = 1$

$$\begin{aligned}&\text{Norm}(T)\\ &= \text{Sym}\Big(\Big\{\big(\pm(tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2), \pm W_1, \pm W_1, \big),\\ &\quad \big(\pm(tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2), \pm W_2, \pm W_2\big) : 0 \leq t, s \leq 1\Big\}\Big).\end{aligned}$$

1.4. $A_{1111} = A_{2222} = A_{1112} = -A_{1122} = A_{1222} = 1$

$$\begin{aligned}&\text{Norm}(T)\\ &= \text{Sym}\Big(\Big\{\big(\pm(tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2), \pm W_1, \pm W_2\big),\\ &\quad \big(\pm(tW_1 + (1-t)W_2), \pm W_1, \pm W_1, \pm W_1\big), \big(\pm(tW_1 + (1-t)W_2), \pm W_2,\\ &\quad \pm W_2, \pm W_2\big) : 0 \leq t, s \leq 1\Big\}\Big).\end{aligned}$$

1.5. $A_{1111} = A_{2222} = A_{1112} = A_{1122} = -A_{1222} = 1$

$$\begin{aligned}&\text{Norm}(T)\\ &= \text{Sym}\Big(\Big\{\big(\pm(tW_1 + (1-t)W_2), \pm(sW_1 + (1-s)W_2), \pm W_1, \pm W_1\big),\\ &\quad \big(\pm(tW_1 - (1-t)W_2), \pm(sW_1 - (1-s)W_2), \pm W_2, \pm W_2\big) : 0 \leq t, s \leq 1\Big\}\Big).\end{aligned}$$

1.6. $A_{1111} = -A_{2222} = -A_{1112} = A_{1122} = A_{1222} = 1$

$$\text{Norm}(T)$$

$$= \text{Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2), \pm(sW_1-(1-s)W_2), \pm W_1, \pm W_1\big), \big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_2, \pm W_2\big), \big(\pm(tW_1-(1-t)W_2), \pm W_2, \pm W_2, \pm W_2\big) : 0\le t,s\le 1\Big\}\Big).$$

1.7. $A_{1111} = -A_{2222} = A_{1112} = -A_{1122} = A_{1222} = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_1, \pm W_1\big), \big(\pm(tW_1-(1-t)W_2), \pm(sW_1-(1-s)W_2), \pm(uW_1-(1-u)W_2), \pm W_1\big) : 0\le t,s,u\le 1\Big\}\Big).$$

1.8. $A_{1111} = -A_{2222} = A_{1112} = A_{1122} = -A_{1222} = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2), \pm(sW_1+(1-s)W_2), \pm W_1, \pm W_1\big), \big(\pm(tW_1-(1-t)W_2), \pm(sW_1-(1-s)W_2), \pm W_2, \pm W_2\big) : 0\le t,s\le 1\Big\}\Big).$$

1.9. $A_{1111} = A_{2222} = -A_{1112} = -A_{1122} = A_{1222} = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2), \pm W_2, \pm W_2, \pm W_2\big), \big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_1, \pm W_2\big), \big(\pm(tW_1-(1-t)W_2), \pm W_1, \pm W_1, \pm W_1\big), \big(\pm(tW_1-(1-t)W_2), \pm W_1, \pm W_2, \pm W_2\big) : 0\le t\le 1\Big\}\Big).$$

1.10. $A_{1111} = A_{2222} = -A_{1112} = A_{1122} = -A_{1222} = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2), \pm(sW_1-(1-s)W_2), \pm(uW_1-(1-u)W_2), \pm(vW_1-(1-v)W_2)\big) : 0\le t,s,u,v\le 1\Big\}\Big).$$

1.11. $A_{1111} = A_{2222} = A_{1112} = -A_{1122} = -A_{1222} = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2), \pm W_2, \pm W_2, \pm W_2\big), \big(\pm(tW_1-(1-t)W_2), \pm W_1, \pm W_1, \pm W_2\big), \big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_1, \pm W_1\big), \big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_2, \pm W_2\big) : 0\le t,s\le 1\Big\}\Big).$$

1.12. $A_{1111} = -A_{2222} = -A_{1112} = -A_{1122} = A_{1222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2), \pm(sW_1-(1-s)W_2), \pm W_2, \pm W_2\big),\\ &\quad \big(\pm(tW_1+(1-t)W_2), \pm W_1, \pm W_1, \pm W_2\big), \big(\pm(tW_1-(1-t)W_2),\\ &\quad \pm W_1, \pm W_1, \pm W_1\big) : 0 \le t, s \le 1\Big\}\Big).\end{aligned}$$

1.13. $A_{1111} = -A_{2222} = -A_{1112} = A_{1122} = -A_{1222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2), \pm(sW_1-(1-s)W_2),\\ &\quad \pm(uW_1-(1-u)W_2), W_1\big), \big(\pm(tW_1+(1-t)W_2), \pm W_2, \pm W_2, \pm W_2\big)\\ &\quad : 0 \le t, s, u \le 1\Big\}\Big).\end{aligned}$$

1.14. $A_{1111} = A_{2222} = -A_{1112} = -A_{1122} = -A_{1222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2), \pm(sW_1+(1-s)W_2), \pm W_1, \pm W_2\big),\\ &\quad \big(\pm(tW_1-(1-t)W_2), \pm W_1, \pm W_1, \pm W_1\big), \big(\pm(tW_1-(1-t)W_2),\\ &\quad \pm W_2, \pm W_2, \pm W_2\big) : 0 \le t, s \le 1\Big\}\Big).\end{aligned}$$

1.15. $A_{1111} = -A_{2222} = A_{1112} = -A_{1122} = -A_{1222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2), \pm(sW_1+(1-s)W_2), \pm W_2, \pm W_2\big),\\ &\quad \big(\pm(tW_1-(1-t)W_2), \pm W_1, \pm W_1, \pm W_2\big), \big(\pm(tW_1+(1-t)W_2),\\ &\quad \pm W_1, \pm W_1, \pm W_1\big) : 0 \le t, s \le 1\Big\}\Big).\end{aligned}$$

1.16. $A_{1111} = -A_{2222} = -A_{1112} = -A_{1122} = -A_{1222} = 1$

$$\begin{aligned}&\mathrm{Norm}(T)\\ &= \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2), \pm(sW_1+(1-s)W_2),\\ &\quad \pm(uW_1+(1-u)W_2), W_1\big), \big(\pm(tW_1-(1-t)W_2), \pm W_1, \pm W_1, \pm W_1\big)\\ &\quad : 0 \le t, s, u \le 1\Big\}\Big).\end{aligned}$$

Case 2. $|A_{i_1\cdots i_4}| < 1$ *for some* $i_k \in \{1, 2\}$ $(k = 1, \ldots, 4)$.

Let $M = \{(i_1, \ldots, i_4) : |A_{i_1\cdots i_4}| < 1\}$ *and define* $S = \big(b_{i_1\cdots i_4}\big) \in \mathcal{L}_s({}^4\ell^2_{(\infty,\theta)})$ *by* $S(W_{i_1}, \ldots, W_{i_4}) = A_{i_1\cdots i_4}$ *if* $(i_1, \ldots, i_4) \notin M$ *and* $S(W_{i_1}, \ldots, W_{i_4}) = 1$ *if*

$(i_1, \ldots, i_4) \in M$. (*Notice that* S *is included in* Case 1.) *Then,*

$$\begin{aligned}&\operatorname{Norm}(T)\\ &= \bigcap_{(i_1,\ldots,i_4)\in M} \operatorname{Sym}\Big(\Big\{\big(t_1^{(1)}W_1 + t_2^{(1)}W_2, \ldots, t_1^{(4)}W_1 + t_2^{(4)}W_2\big) \in \operatorname{Norm}(S)\\ &\qquad : t_{i_1}^{(1)} \cdots t_{i_4}^{(4)} = 0\Big\}\Big).\end{aligned}$$

Proof. We will slightly modify the proof of Theorem 2.4.

We define $S_T \in \mathcal{L}_s({}^4\ell_1^2)$ by

$$\begin{aligned}S_T\Big((t_1^{(1)}, t_2^{(1)}), \ldots, (t_1^{(4)}, t_2^{(4)})\Big) &= T\Big(t_1^{(1)}W_1 + t_2^{(1)}W_2, \ldots, t_1^{(4)}W_1 + t_2^{(4)}W_2\Big)\\ &= \sum_{1\le k\le 4,\ i_k=1,2} A_{i_1\cdots i_4} t_{i_1}^{(1)} \cdots t_{i_4}^{(4)}.\end{aligned}$$

Notice that

$$\begin{aligned}\operatorname{Norm}(T) = \Big\{\Big(&t_1^{(1)}W_1 + t_2^{(1)}W_2, \ldots, t_1^{(4)}W_1 + t_2^{(4)}W_2\Big) :\\ &\Big((t_1^{(1)}, t_2^{(1)}), \ldots, (t_1^{(4)}, t_2^{(4)})\Big) \in \operatorname{Norm}(S_T)\Big\}.\end{aligned}$$

Notice that

$$(\star)\qquad \begin{aligned}&S_T\big((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\big)\\ &= x_1\Big\{x_2\big(x_3[A_{1111}x_4 + A_{1112}y_4] + y_3[A_{1112}x_4 + A_{1122}y_4]\big)\\ &\quad + y_2\big(x_3[A_{1112}x_4 + A_{1122}y_4] + y_3[A_{1122}x_4 + A_{1222}y_4]\big)\Big\}\\ &\quad + y_1\Big\{x_2\big(x_3[A_{1112}x_4 + A_{1122}y_4] + y_3[A_{1122}x_4 + A_{1222}y_4]\big)\\ &\quad + y_2\big(x_3[A_{1122}x_4 + A_{1222}y_4] + y_3[A_{1222}x_4 + A_{2222}y_4]\big)\Big\}.\end{aligned}$$

By $(\star)$ it follows that

$$\begin{aligned}&\operatorname{Norm}(S_T)\\ &\supseteq \operatorname{Sym}\Big(\Big\{\big(\pm(te_1 + A_{1112}(1-t)e_2), \pm e_1, \pm e_1, \pm e_1\big),\\ &\quad\big(\pm(te_1 + A_{1112}A_{1122}(1-t)e_2), \pm e_1, \pm e_1, \pm e_2\big),\\ &\quad\big(\pm(te_1 + A_{1122}A_{1222}(1-t)e_2), \pm e_1, \pm e_2, \pm e_2\big),\\ &\quad\big(\pm(te_1 + A_{2222}(1-t)e_2), \pm e_2, \pm e_2, \pm e_2\big) : 0 \le t \le 1\Big\}\Big).\end{aligned}$$

Thus,

$$(\star\star)\qquad \begin{aligned}&\operatorname{Norm}(T)\\ &\supseteq \operatorname{Sym}\Big(\Big\{\big(\pm(tW_1 + A_{1112}(1-t)W_2), \pm W_1, \pm W_1, \pm W_1\big),\\ &\quad\big(\pm(tW_1 + A_{1112}A_{1122}(1-t)W_2), \pm W_1, \pm W_1, \pm W_2\big),\end{aligned}$$

$$\big(\pm(tW_1+A_{1122}A_{1222}(1-t)W_2),\pm W_1,\pm W_2,\pm W_2\big),$$
$$\big(\pm(tW_1+A_{2222}(1-t)W_2),\pm W_2,\pm W_2,\pm W_2\big):0\le t\le 1\Big\}\Big).$$

Case 1. $A_{1111}=|A_{i_1i_2i_3i_4}|=1$ for every $(i_1,i_2,i_3,i_4)\neq(1,1,1,1)$.

We only give the proof of subcase 1.12 because the proofs of the other subcases are similar.

1.12. $A_{1111}=-A_{2222}=-A_{1112}=-A_{1122}=A_{1222}=1$.

By ($\star\star$) and Theorem 2.3,

$$\begin{aligned}\mathrm{Norm}(T)=\ \mathrm{Sym}\Big(\Big\{&\big(\pm(tW_1-(1-t)W_2),\pm(sW_1-(1-s)W_2),\pm W_2,\pm W_2\big),\\ &\big(\pm(tW_1+(1-t)W_2),\pm W_1,\pm W_1,\pm W_2\big),\big(\pm(tW_1-(1-t)W_2),\\ &\pm W_1,\pm W_1,\pm W_1\big):0\le t,s\le 1\Big\}\Big).\end{aligned}$$

The proof of Case 2 follows from Theorem 2.2 and Case 1.

This completes the proof. □

Remark 2.7. (a) Since $\mathcal{L}_s({}^4\ell^2_{(\infty,0)})=\mathcal{L}_s({}^4\ell^2_\infty)$, Theorem 2.6 classifies the norming sets of $\mathcal{L}_s({}^4\ell^2_\infty)$.
(b) By the fact that $\mathcal{L}_s({}^4\ell^2_{(\infty,\frac{\pi}{4})})=\mathcal{L}_s({}^4\ell^2_1)$ and $\|(x,y)\|_{(\infty,\frac{\pi}{4})}=\frac{1}{\sqrt{2}}\|(x,y)\|_1$, Theorem 2.6 classifies the norming sets of $\mathcal{L}_s({}^4\ell^2_1)$.

We classify the norming set of $T\in\mathcal{L}_s({}^3\ell^2_{(\infty,\theta)})$.

Theorem 2.8. *Let* $0\le\theta\le\frac{\pi}{4}$ *and*

$$W_1=(-\sin\theta+\cos\theta,\ \sin\theta+\cos\theta)\ \textit{and}\ W_2=(\sin\theta+\cos\theta,\ \sin\theta-\cos\theta).$$

Let

$$T\big((x_1^{(1)},x_2^{(1)}),(x_1^{(2)},x_2^{(2)}),(x_1^{(3)},x_2^{(3)})\big)=\sum_{i_k=1,2,k=1,2,3}a_{i_1i_2i_3}x_{i_1}^{(1)}x_{i_2}^{(2)}x_{i_3}^{(3)}\in\mathcal{L}_s({}^3\ell^2_{(\infty,\theta)})$$

with $\|T\|=1$. *Then the following assertions hold*: *Let* $A_{i_1i_2i_3}=T(W_{i_1},W_{i_2},W_{i_3})$ *for* $i_k=1,2$ *and* $A_{111}\ge 0$.

Case 1. $A_{111}=|A_{222}|=|A_{112}|=|A_{122}|=1$.

1.1. $A_{111}=A_{222}=A_{112}=A_{122}=1$

$$\begin{aligned}\mathrm{Norm}(T)=\Big\{&\big(\pm(tW_1+(1-t)W_2),\pm(sW_1+(1-s)W_2),\\ &\pm(uW_1+(1-u)W_2)\big):0\le t,s,u\le 1\Big\}.\end{aligned}$$

1.2. $A_{111}=-A_{222}=A_{112}=A_{122}=1$

$$\mathrm{Norm}(T)=\ \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2),\pm(sW_1+(1-s)W_2),\pm W_1\big),$$

$$\big(\pm(tW_1-(1-t)W_2),\pm W_2,\pm W_2\big):0\le t,s\le 1\Big\}\Big).$$

1.3. $A_{111}=A_{222}=-A_{112}=A_{122}=1$

$$\text{Norm}(T)=\text{ Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2),\pm(sW_1-(1-s)W_2),\pm W_1\big),$$
$$\big(\pm(tW_1+(1-t)W_2),\pm W_2,\pm W_2\big):0\le t,s\le 1\Big\}\Big).$$

1.4. $A_{111}=A_{222}=A_{112}=-A_{122}=1$

$$\text{Norm}(T)=\text{ Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2),\pm(sW_1-(1-s)W_2),\pm W_2\big),$$
$$\big(\pm(tW_1+(1-t)W_2),\pm W_1,\pm W_1\big):0\le t,s\le 1\Big\}\Big).$$

1.5. $A_{111}=-A_{222}=-A_{112}=A_{122}=1$

$$\text{Norm}(T)=\Big\{\big(\pm(tW_1-(1-t)W_2),\pm(sW_1-(1-s)W_2),$$
$$\pm(uW_1-(1-u)W_2)\big):0\le t,s,u\le 1\Big\}.$$

1.6. $A_{111}=A_{222}=-A_{112}=-A_{122}=1$

$$\text{Norm}(T)=\text{ Sym}\Big(\Big\{\big(\pm(tW_1-(1-t)W_2),\pm W_1,\pm W_1\big),$$
$$\big(\pm(tW_1-(1-t)W_2),\pm W_2,\pm W_2\big),$$
$$\big(\pm(tW_1+(1-t)W_2),\pm W_1,\pm W_2\big):0\le t\le 1\Big\}\Big).$$

1.7. $A_{111}=-A_{222}=A_{112}=-A_{122}=1$

$$\text{Norm}(T)=\text{ Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2),\pm W_1,\pm W_1\big),$$
$$\big(\pm(tW_1+(1-t)W_2),\pm W_2,\pm W_2\big),$$
$$\big(\pm(tW_1-(1-t)W_2),\pm W_1,\pm W_2\big):0\le t\le 1\Big\}\Big).$$

1.8. $A_{111}=-A_{222}=-A_{112}=-A_{122}=1$

$$\text{Norm}(T)=\text{ Sym}\Big(\Big\{\big(\pm(tW_1+(1-t)W_2),\pm(sW_1+(1-s)W_2),\pm W_2\big),$$
$$\big(\pm(tW_1-(1-t)W_2),\pm W_1,\pm W_1\big):0\le t,s\le 1\Big\}\Big).$$

Case 2. $|A_{i_1i_2i_3}|<1$ *for some* $i_k\in\{1,2\}$ $(k=1,2,3)$.

Let $M=\{(i_1,i_2,i_3):|A_{i_1i_2i_3}|<1\}$ *and define* $S=\big(b_{i_1i_2i_3}\big)\in\mathcal{L}_s({}^3\ell^2_{(\infty,\theta)})$ *by* $S(W_{i_1},W_{i_2},W_{i_3})=A_{i_1i_2i_3}$ *if* $(i_1,i_2,i_3)\notin M$ *and* $S(W_{i_1},W_{i_2},W_{i_3})=1$ *if* $(i_1,i_2,i_3)\in M$. (*Notice that* S *is included in* Case 1.) *Then,*

$$\text{Norm}(T)$$
$$=\bigcap_{(i_1,\ldots,i_3)\in M}\text{Sym}\Big(\Big\{\big(t_1^{(1)}W_1+t_2^{(1)}W_2,\ldots,t_1^{(3)}W_1+t_2^{(3)}W_2\big)\in\text{Norm}(S)$$

$$: \ t_{i_1}^{(1)} \cdots t_{i_3}^{(3)} = 0\Big\}\Big).$$

Proof. We will slightly modify the proof of Theorem 2.4. We define $S_T \in \mathcal{L}_s({}^3\ell_1^2)$ by

$$\begin{aligned} S_T\Big((t_1^{(1)}, t_2^{(1)}), \ldots, (t_1^{(3)}, t_2^{(3)})\Big) &= T\Big(t_1^{(1)}W_1 + t_2^{(1)}W_2, \ldots, t_1^{(3)}W_1 + t_2^{(3)}W_2\Big) \\ &= \sum_{1\le k\le 3,\ i_k=1,2} A_{i_1\cdots i_3} t_{i_1}^{(1)} \cdots t_{i_3}^{(3)}. \end{aligned}$$

Notice that

$$\begin{aligned} \mathrm{Norm}(T) = \Big\{ &\Big(t_1^{(1)}W_1 + t_2^{(1)}W_2, \ldots, t_1^{(3)}W_1 + t_2^{(3)}W_2\Big) \\ &: \Big((t_1^{(1)}, t_2^{(1)}), \ldots, (t_1^{(3)}, t_2^{(3)})\Big) \in \mathrm{Norm}(S_T)\Big\}. \end{aligned}$$

Notice that

$$\begin{aligned} (\star) \qquad & S_T\big((x_1, y_1), (x_2, y_2), (x_3, y_3)\big) \\ &= x_1\Big\{x_2[A_{111}x_4 + A_{112}y_4] + y_2[A_{112}x_3 + A_{122}y_3]\Big\} \\ &\quad + y_1\Big\{x_2[A_{112}x_3 + A_{122}y_3] + y_2[A_{122}x_3 + A_{222}y_3]\Big\}. \end{aligned}$$

By $(\star)$, it follows that

$$\begin{aligned} &\mathrm{Norm}(S_T) \\ &\supseteq \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 + A_{112}(1-t)e_2), \pm e_1, \pm e_1\big), \\ &\quad \big(\pm(te_1 + A_{112}A_{122}(1-t)e_2), \pm e_1, \pm e_2\big), \\ &\quad \big(\pm(te_1 + A_{122}A_{222}(1-t)e_2), \pm e_2, \pm e_2\big) : 0 \le t \le 1\Big\}\Big). \end{aligned}$$

Thus,

$$\begin{aligned} (\star\star) \qquad &\mathrm{Norm}(T) \\ &\supseteq \mathrm{Sym}\Big(\Big\{\big(\pm(tW_1 + A_{112}(1-t)W_2), \pm W_1, \pm W_1\big), \\ &\quad \big(\pm(tW_1 + A_{112}A_{122}(1-t)W_2), \pm W_1, \pm W_2\big), \\ &\quad \big(\pm(tW_1 + A_{122}A_{222}(1-t)W_2), \pm W_2, \pm W_2\big) : 0 \le t \le 1\Big\}\Big). \end{aligned}$$

Case 1. $A_{111} = |A_{222}| = |A_{112}| = |A_{122}| = 1$.

We only give the proof of subcase 1.6 because the proofs of the other subcases are similar.

1.6. $A_{111} = A_{222} = -A_{112} = -A_{122} = 1$.

By $(\star\star)$ and Theorem 2.3,

$$\begin{aligned} \mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{&\big(\pm(tW_1 - (1-t)W_2), \pm W_1, \pm W_1\big), \\ &\big(\pm(tW_1 - (1-t)W_2), \pm W_2, \pm W_2\big), \end{aligned}$$

$$\Big(\pm (tW_1 + (1-t)W_2), \pm W_1, \pm W_2 \Big) : 0 \le t \le 1 \Big\} \Big).$$

The proof of Case 2 follows from Theorem 2.2 and Case 1. This completes the proof. □

Remark 2.9. (a) Since $\mathcal{L}_s({}^3\ell^2_{(\infty,0)}) = \mathcal{L}_s({}^3\ell^2_\infty)$, Theorem 2.8 classifies the norming sets of $\mathcal{L}_s({}^3\ell^2_\infty)$.
(b) By the fact that $\mathcal{L}_s({}^3\ell^2_{(\infty,\frac{\pi}{4})}) = \mathcal{L}_s({}^3\ell^2_1)$ and $\|(x,y)\|_{(\infty,\frac{\pi}{4})} = \frac{1}{\sqrt{2}}\|(x,y)\|_1$, Theorem 2.8 classifies the norming sets of $\mathcal{L}_s({}^3\ell^2_1)$, which is the results of [9].

References

[1] R. M. Aron, C. Finet, and E. M. Werner, *Some remarks on norm-attaining n-linear forms*, Function Spaces (Edwardsville, IL, 1994), 19–28, Lecture Notes in Pure and Appl. Math., 172, Dekker, New York, 1995.

[2] E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. **67** (1961), 97–98. `https://doi.org/10.1090/S0002-9904-1961-10514-4`

[3] Y. S. Choi and S. G. Kim, *Norm or numerical radius attaining multilinear mappings and polynomials*, J. London Math. Soc. (2) **54** (1996), no. 1, 135–147. `https://doi.org/10.1112/jlms/54.1.135`

[4] S. Dineen, *Complex Analysis on Infinite-Dimensional Spaces*, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 1999. `https://doi.org/10.1007/978-1-4471-0869-6`

[5] M. Jiménez-Sevilla and R. Payá, *Norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces*, Studia Math. **127** (1998), no. 2, 99–112. `https://doi.org/10.4064/sm-127-2-99-112`

[6] S. G. Kim, *The norming set of a bilinear form on l^2_∞*, Comment. Math. **60** (2020), no. 1-2, 37–63.

[7] S. G. Kim, *The norming set of a polynomial in $\mathcal{P}({}^2l^2_\infty)$*, Honam Math. J. **42** (2020), no. 3, 569–576. `https://doi.org/10.5831/HMJ.2020.42.3.569`

[8] S. G. Kim, *The norming set of a symmetric bilinear form on the plane with the supremum norm*, Mat. Stud. **55** (2021), no. 2, 171–180. `https://doi.org/10.30970/ms.55.2.171-180`

[9] S. G. Kim, *The norming set of a symmetric 3-linear form on the plane with the l_1-norm*, N. Z. J. Math. **51** (2021), 95–108. `https://doi.org/10.53733/177`

[10] S. G. Kim, *The norming sets of $\mathcal{L}({}^2l^2_1)$ and $\mathcal{L}_s({}^2l^3_1)$*, Bull. Transilv. Univ. Braşov Ser. III. Math. Comput. Sci. **2(64)** (2022), no. 2, 125–149. `https://doi.org/10.31926/but.mif.2022.2.64.2.10`

[11] S. G. Kim, *The norming sets of $\mathcal{L}({}^2\mathbb{R}^2_{h(w)})$*, Acta Sci. Math. (Szeged) **89** (2023), no. 1-2, 61–79. `https://doi.org/10.1007/s44146-023-00078-7`

Sung Guen Kim
Department of Mathematics
Kyungpook National University
Daegu 41566, Korea
Email address: `sgk317@knu.ac.kr`