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MULTI-JENSEN AND MULTI-EULER-LAGRANGE ADDITIVE MAPPINGS

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Abstract. In this work, an alternative fashion of the multi-Jensen is introduced. The structures of the multi-Jensen and the multi-Euler-Lagrange-Jensen mappings are described. In other words, the system of n equations defining each of the mentioned mappings is unified as a single equation. Furthermore, by applying a fixed point theorem, the Hyers-Ulam stability for the multi-Euler-Lagrange-Jensen mappings in the setting of Banach spaces is established. An appropriate counterexample is supplied to invalidate the results in the case of singularity for multiadditive mappings.

1. Introduction

The stability problem of the functional equation, initiated by the celebrated Ulam's question [33] about the stability of group homomorphisms (answered by Hyers [16], Aoki [1], Rassias [27] and Găvruţa [15] for additive and linear mappings) has been growing rapidly over the last decades and applied in sciences and engineering. Recall that a functional equation Γ is said to be *stable* if any function f satisfying the equation Γ approximately must be near to an exact solution of Γ.

It is well-known that among functional equations the additive (Cauchy) equation

$$\mathcal{A}(x+y)=\mathcal{A}(x)+\mathcal{A}(y) \tag{1.1}$$

and the Jensen functional equation

$$J\left(\frac{x+y}{2}\right)=\frac{J(x)+J(y)}{2} \tag{1.2}$$

play a significant role in many parts of mathematics. More information about them (in particular, about their solutions and stability) and their applications can be found for instance in [17, 19–21] and [30].

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Throughout this paper, $\mathbb{N}$ and $\mathbb{Q}$ are the sets of all positive integers and rationals, respectively, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$. Moreover, for the set X, we denote $\overbrace{X \times X \times \cdots \times X}^{n-\text{times}}$ by X^n.

Let V be a commutative group, W be a linear space over $\mathbb{Q}$, and $n \in \mathbb{N}$ with $n \geq 2$. A mapping $f : V^n \longrightarrow W$ is called

- *multiadditive* if it satisfies (1.1) in each variable;
- *multi-Jensen* if it satisfies (1.2) in each variable.

It is shown in [11] that a mapping f is multiadditive if and only if it satisfies

$$f(x_1 + x_2) = \sum_{j_1,\ldots,j_n \in \{1,2\}} f(x_{j_1 1}, \ldots, x_{j_n n}), \tag{1.3}$$

where $x_j = (x_{j1}, \ldots, x_{jn}) \in V^n$ with $j \in \{1, 2\}$. A lot of information about the structure of multiadditive mappings and their Ulam stabilities are available in [11, 12, 18] and [20, Sections 13.4 and 17.2].

The notion of multi-Jensen mappings with the connection to the generalized polynomials has been introduced by Prager and Schwaiger [25], where they obtained the general form of such mappings. In other words, the aim of this note was to study the stability of the multi-Jensen equation. Moreover, they represented a characterization of multi-Jensen mappings as an equation in [26, Lemma 1.1]. Next, the stability of multi-Jensen mappings in various normed spaces has been investigated by a number of authors; see for instance [9, 10] and [34]. Note that the multi-m-Jensen mappings (when $m \geq 2$) and their generalized form were studied in [22] and [32]. For some results on the characterization and stability of multi-Cauchy-Jensen, multi-Jensen-quadratic and multiadditive-quadratic, we refer to [2–7] and [31].

An alternative version of Jensen equation (1.2), namely the Jensen-type functional equation is as follows:

$$\mathcal{J}\left(\frac{x+y}{2}\right) + \mathcal{J}\left(\frac{x-y}{2}\right) = \mathcal{J}(x). \tag{1.4}$$

The Hyers-Ulam stability of homomorphisms in C^*-algebras for equation (1.2) was investigated in [24]. Additionally, the generalized case of the Jensen-type functional equation is given by

$$r\left[\mathcal{J}\left(\frac{x+y}{r}\right) + \mathcal{J}\left(\frac{x-y}{r}\right)\right] = 2\mathcal{J}(x), \tag{1.5}$$

where $r \in (1, \infty)$. The stability and superstability for J^*- derivations in J^*-algebras for (1.5) were studied in [13] (see also [23]). The equation

$$\mathfrak{J}(x+y) + \mathfrak{J}(x-y) = 2\mathfrak{J}(x) \tag{1.6}$$

is a special case of (1.5) when $r = 1$, where we focus on it in Sections 2 and 3. Recall that a mapping A is called *Euler-Lagrange additive* if it satisfies the equation

$$A(ax+by)+A(bx+ay)=(a+b)[A(x)+A(y)], \tag{1.7}$$

where $a,b\in\mathbb{R}\backslash\{0\}$ are fixed with $a+b\neq 0,\pm 1$. In fact, Rassias [28, 29] introduced and investigated the stability problem of Ulam for (1.7). Next, Xu extended the definition above to several variables mappings [35]. One can easily verified that the function $f(x)=cx$ is a common solution of equations (1.1), (1.5) and (1.7). In addition, equations (1.2), (1.4) and (1.6) are valid for the function $f(x)=cx+b$.

The rest of the current paper is organized as follows: In Section 2, we first define a new form of the multi-Jensen and also recall the multi-Euler-Lagrange additive mappings from [35]. We describe the structure of such mappings and indeed we prove that every multi-Jensen and multi-Euler-Lagrange additive mapping can be shown a single equation. Section 3 is devoted to the study of structure of multi-Euler-Lagrange-Jensen mappings. In other words, we reduce the system of n equations defining the multi-Euler-Lagrange-Jensen mappings to obtain a single equation. In Section 4, we prove the Hyers-Ulam stability for the multi-Euler-Lagrange-Jensen mappings in the setting of Banach spaces by applying a fixed point method. As an application of this result, we establish the (Hyers-Rassias) stability of multi-Euler-Lagrange mappings. In Section 5, by means of [20, Theorem 13.4.3], we present an example for the non-stability case on multiadditive mappings.

2. Characterization of multi-Jensen and multi-Euler-Lagrange additive mappings

Let S be a subset of $\mathbb{R}$. From now on, for any $l\in\mathbb{N}_0, n\in\mathbb{N}$, $t=(t_1,\dots,t_n)\in S^n$ and $x=(x_1,\dots,x_n)\in V^n$ we write $lx:=(lx_1,\dots,lx_n)$ and $tx:=(t_1x_1,\dots,t_nx_n)$. Throughout this paper, it is assumed that V and W are vector spaces over $\mathbb{R}$, $n\in\mathbb{N}$ and $x_i^{[n]}=(x_{1i},x_{2i},\dots,x_{ni})\in V^n$, where $i\in\{1,2\}$. We will write $x_i^{[n]}$ simply x_i (used in the last section) when no confusion can arise.

2.1. Multi-Jensen mappings

Motivated by equation (1.6), we bring a new definition of multi-Jensen mappings as follows.

Definition 2.1. A mapping $f:V^n\longrightarrow W$ is called multi-Jensen if it satisfies Jensen's equation (1.6) in each of its n arguments, that is,

$$\begin{aligned}&f(v_1,\dots,v_{i-1},v_i+v_i',v_{i+1},\dots,v_n)+f(v_1,\dots,v_{i-1},v_i-v_i',v_{i+1},\dots,v_n)\\&=2f(v_1,\dots,v_n).\end{aligned}$$

In the following result, we describe the multi-Jensen mappings as a single equation. Here and subsequently, the notation $\{-1,1\}^n$ means

$$\overbrace{\{-1,1\}\times\ldots\times\{-1,1\}}^{n-\text{times}}.$$

Theorem 2.2. *A mapping $f: V^n \longrightarrow W$ is multi-Jensen if and only if it satisfies the equation*

$$\sum_{q\in\{-1,1\}^n} f\left(x_1^{[n]}+qx_2^{[n]}\right)=2^n f\left(x_1^{[n]}\right), \tag{2.1}$$

for all $x_1^{[n]}, x_2^{[n]} \in V^n$.

Proof. Suppose that f is a multi-Jensen mapping. We proceed this implication by induction on n. For $n=1$, the result is trivial. Assume that (2.1) holds for $n=k$, that is,

$$\sum_{q\in\{-1,1\}^k} f\left(x_1^{[k]}+qx_2^{[k]},z\right)=2^k f\left(x_1^{[k]},z\right), \tag{2.2}$$

for all $x_1^{[k]}, x_2^{[k]} \in V^k$ and $z\in V$. Hence,

$$\begin{aligned}
&\sum_{q\in\{-1,1\}^{k+1}} f\left(x_1^{[k+1]}+qx_2^{[k+1]}\right)\\
&=\sum_{q\in\{-1,1\}^k}\sum_{t\in\{-1,1\}} f\left(x_1^{[k]}+qx_2^{[k]},x_{1,k+1}+tx_{2,k+1}\right)\\
&=2\sum_{q\in\{-1,1\}^k} f\left(x_1^{[k]}+qx_2^{[k]},x_{1,k+1}\right). \qquad (2.3)
\end{aligned}$$

It now follows the validity of (2.1) for $k+1$ from (2.2) and (2.3).

Conversely, assume that f fulfills (2.1). Fix $j\in\{1,\ldots,n\}$, put $x_{k2}=0$ for all $k\in\{1,\ldots,n\}\backslash\{j\}$. We have

$$\begin{aligned}
&2^{n-1}[f(x_{11},\ldots,x_{1,j-1},x_{1j}+x_{2j},x_{1,j+1},\ldots,x_{1n})\\
&\quad+f(x_{11},\ldots,x_{1,j-1},x_{1j}-x_{2j},x_{1,j+1},\ldots,x_{1n})]\\
&=2^n f(x_{11},\ldots,x_{1,j-1},x_{j1},x_{1,j+1},\ldots,x_{n1}). \qquad (2.4)
\end{aligned}$$

Relation (2.4) implies that f is Jensen in the jth variable. Since j is arbitrary, we obtain the desired result. □

2.2. Multi-Euler-Lagrange additive mappings

We start this subsection by a definition, which has been presented by Xu in [35].

Definition 2.3. A mapping $f : V^n \longrightarrow W$ is called multi-Euler-Lagrange additive if it satisfies the Euler-Lagrange additive equation (1.7) in each of their n arguments, namely,

$$\begin{aligned}&[f(v_1,\dots,v_{i-1},a_iv_i+b_iv_i',v_{i+1},\dots,v_n)\\&\quad+f(v_1,\dots,v_{i-1},b_iv_i+a_iv_i',\dots,v_n)]\\&=(a_i+b_i)\left[f(v_1,\dots,v_{i-1},v_i,\dots,v_n)+f(v_1,\dots,v_{i-1},v_i',\dots,v_n)\right],\end{aligned}$$

where $a_j, b_j \in \mathbb{R}\backslash\{0\}$ are fixed with $a_j + b_j \neq 0, \pm 1$.

In the sequel, consider $a_i^{[n]} = (a_{i1}, a_{i2}, \dots, a_{in}) \in \mathbb{R}^n\backslash\{(0,\dots,0)\}$ such that $a_{1j}+a_{2j} \neq 0$, where $i \in \{1,2\}$ and $j \in \{1,\dots,n\}$. We write $a_i^{[n]}$ simply a_i when there is no ambiguity. For $x_1, x_2 \in V^n$ and a_1, a_2 as in the above, consider the following notations:

$$B_j=\sum_{i=1}^{2}a_{ij}x_{ij}\ \text{and}\ B_j'=\sum_{i=1}^{2}a_{3-i,j}x_{ij}, \tag{2.5}$$

where $j \in \{1,\dots,n\}$. In continuation, we show that the equation

$$\sum_{\substack{\mathfrak{B}_j\in\{B_j,B_j'\}\\ j\in\{1,\dots,n\}}} f(\mathfrak{B}_1,\dots,\mathfrak{B}_n)=\prod_{j=1}^{n}(a_{1j}+a_{2j})\sum_{l_1,\cdots,l_n\in\{1,2\}}f(x_{l_11},\dots,x_{l_nn}) \tag{2.6}$$

holds for any multi-Euler-Lagrange additive mapping and vice versa. For this, we need the following definition.

Definition 2.4. We say a mapping $f : V^n \longrightarrow W$

(i) satisfies (has) the *linear condition* in the jth variable if

$$f(z_1,\dots,z_{j-1},a^*z_j,z_{j+1},\dots,z_n)=a^*f(z_1,\dots,z_{j-1},z_j,z_{j+1},\dots,z_n),$$

for all $z_1, \dots, z_n \in V^n$, where $a^* \in \{a_{1j}, a_{2j}, a_{1j} + a_{2j}\}$;

(ii) has *zero condition* if $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero.

Remark 2.5. It is clear that if a mapping $f : V^n \longrightarrow W$ satisfies the linear condition in the jth variable then it has zero condition in the same variable. Therefore, if f has the linear condition in each variable, then it has zero condition. We will use from this fact to prove the upcoming result.

Theorem 2.6. *For a mapping $f : V^n \longrightarrow W$, the following assertions are equivalent*:

(i) *f is multi-Euler-Lagrange additive*;

(ii) *f satisfies equation* (2.6) *and the linear condition in each variable.*

Proof. (i) ⇒ (ii) One can show that f satisfies the linear condition in each variable. We now proceed the proof of this implication by induction on n so that f satisfies equation (2.6). For $n=1$, it is trivial that f satisfies equation (1.7). Assume that (2.6) is valid for some positive integer $n>1$. Then

$$\begin{aligned}
&\sum_{\substack{\mathfrak{B}_j\in\{B_j,B'_j\}\\ j\in\{1,\ldots,n+1\}}} f\left(\mathfrak{B}_1,\ldots,\mathfrak{B}_{n+1}\right)\\
&= \sum_{\substack{\mathfrak{B}_j\in\{B_j,B'_j\}\\ j\in\{1,\ldots,n\}}} f\left(\mathfrak{B}_1,\ldots,\mathfrak{B}_n,B_n\right)+\sum_{\substack{\mathfrak{B}_j\in\{B_j,B'_j\}\\ j\in\{1,\ldots,n\}}} f\left(\mathfrak{B}_1,\ldots,\mathfrak{B}_n,B'_n\right)\\
&= (a_{1,n+1}+a_{2,n+1})\Big(\sum_{\substack{\mathfrak{B}_j\in\{B_j,B'_j\}\\ j\in\{1,\ldots,n\}}} f\left(\mathfrak{B}_1,\ldots,\mathfrak{B}_n,x_{1,n+1}\right)\\
&\quad+\sum_{\substack{\mathfrak{B}_j\in\{B_j,B'_j\}\\ j\in\{1,\ldots,n\}}} f\left(\mathfrak{B}_1,\ldots,\mathfrak{B}_n,x_{2,n+1}\right)\Big)\\
&= (a_{1,n+1}+a_{2,n+1})\prod_{j=1}^{n}(a_{1j}+a_{2j})\Big(\sum_{l_1,\cdots,l_n\in\{1,2\}} f(x_{l_11},\ldots,x_{l_nn},x_{1,n+1})\\
&\quad+\sum_{l_1,\ldots,l_n\in\{1,2\}} f(x_{l_11},\ldots,x_{l_nn},x_{2,n+1})\Big)\\
&= \prod_{j=1}^{n+1}(a_{1j}+a_{2j})\sum_{l_1,\ldots,l_{n+1}\in\{1,2\}} f(x_{l_11},\ldots,x_{l_{n+1}n+1}).
\end{aligned}$$

(ii)⇒(i) Fix $j\in\{1,\ldots,n\}$. Putting $x_{2k}=0$ for all $k\in\{1,\ldots,n\}\backslash\{j\}$ in (2.6) and using Remark 2.5, we can show that the left side of (2.6) will be as follows:

$$\begin{aligned}
&f\left(a_{11}x_{11},\ldots,a_{1,j-1}x_{1,j-1},B_j,a_{1,j+1}x_{1,j+1},\ldots,a_{1n}x_{1n}\right)\\
&\quad+f\left(a_{21}x_{11},\ldots,a_{2,j-1}x_{1,j-1},B_j,a_{2,j+1}x_{1,j+1},\ldots,a_{2n}x_{1n}\right)\\
&\quad+f\left(a_{11}x_{11},\ldots,a_{1,j-1}x_{1,j-1},B'_j,a_{1,j+1}x_{1,j+1},\ldots,a_{1n}x_{1n}\right)\\
&\quad+f\left(a_{21}x_{11},\ldots,a_{2,j-1}x_{1,j-1},B'_j,a_{2,j+1}x_{1,j+1},\ldots,a_{2n}x_{1n}\right)\\
&= a_{11}a_{21}a_{12}a_{22}\cdots a_{1,j-1}a_{2,j-1}a_{1,j+1}a_{2,j+1}\cdots a_{1n}a_{2n}\\
&\quad\times\Big[f(x_{11},\ldots,x_{1,j-1},B_j,x_{1,j+1},\ldots,x_{1n})\\
&\quad+f(x_{11},\ldots,x_{1,j-1},B'_j,x_{1,j+1},\ldots,x_{1n})\Big].
\end{aligned}\tag{2.7}$$

On the other hand, by the replacements above, the right side of (2.6) is as

$$\begin{aligned}
&\prod_{\substack{k=1\\ k\neq j}}^{n-1}(a_{1k}+a_{2k})\Big[f(x_{11},\ldots,x_{1,j-1},x_{1j},x_{1,j+1},\ldots,x_{1n})\\
&\quad+f(x_{11},\ldots,x_{1,j-1},x_{2j},x_{1,j+1},\ldots,x_{1n})\Big].
\end{aligned}\tag{2.8}$$

It follows from (2.7) and (2.8) that f is Euler-Lagrange additive in the jth variable. Since j is arbitrary, we obtain the desired result, and this finishes the proof. □

3. Characterization of multi-Euler-Lagrange-Jensen mappings

Definition 3.1. Let V and W be linear spaces, $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$. A mapping $f : V^n \longrightarrow W$ is called k-Euler-Lagrange additive and $n-k$-Jensen (briefly, multi-Euler-Lagrange-Jensen) if f is Euler-Lagrange additive (in sense of Definition 2.3) in each of some k variables and is Jensen in each of the other variables (in sense of equation (1.6)).

In Definition 3.1, we assume for simplicity that f is Euler-Lagrange additive in each of the first k variables, but one can obtain analogous results without this assumption. Let us note that for $k = n$ ($k = 0$), the above definition leads to the so-called Euler-Lagrange additive (multi-Jensen) mappings, defined in the previous section.

In what follows, we assume that V and W are vector spaces over $\mathbb{Q}$. Moreover, we identify $x = (x_1, \dots, x_n) \in V^n$ with $(x^{[k]}, x^{[n-k]}) \in V^k \times V^{n-k}$, where $x^{[k]} := (x_1, \dots, x_k)$ and $x^{[n-k]} := (x_{k+1}, \dots, x_n)$, and we adopt the convention that $(x^{[n]}, x^{[0]}) := x^{[n]} := (x^{[0]}, x^{[n]})$. Put $x_i^{[k]} = (x_{i1}, \dots, x_{ik}) \in V^k$ and $x_i^{[n-k]} = (x_{i,k+1}, \dots, x_{in}) \in V^{n-k}$ where $i \in \{1, 2\}$.

In the upcoming result, we reduce the system of n equations defining the k-Euler-Lagrange additive and $n-k$-Jensen mapping to obtain a single functional equation.

Proposition 3.2. *Let $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$. If a mapping $f : V^n \longrightarrow W$ is a k-Euler-Lagrange additive and $n-k$-Jensen mapping, then f satisfies the equation*

$$\sum_{\substack{\mathfrak{B}_j \in \{B_j, B'_j\} \\ j \in \{1,\dots,k\}}} \sum_{q \in \{-1,1\}^{n-k}} f\left(\mathfrak{B}_1, \dots, \mathfrak{B}_k, x_1^{[n-k]} + q x_2^{[n-k]}\right)$$

$$= 2^{n-k} \prod_{j=1}^{k} (a_{1j} + a_{2j}) \sum_{l_1,\dots,l_k \in \{1,2\}} f\left(x_{l_1 1}, \dots, x_{l_k k}, x_1^{[n-k]}\right), \tag{3.1}$$

for all $x_1^{[n-k]}, x_2^{[n-k]} \in V^{n-k}$ where B_j and B'_j are defined as in (2.5).

Proof. Since for $k \in \{0, n\}$ our assertion follows from Theorem 2.2 and Theorem 2.6, we can assume that $k \in \{1, \dots, n-1\}$. For any $x^{[n-k]} \in V^{n-k}$, define the mapping $g_{x^{[n-k]}} : V^k \longrightarrow W$ by $g_{x^{[n-k]}}(x^{[k]}) := f(x^{[k]}, x^{[n-k]})$ for $x^{[k]} \in V^k$. By assumption, $g_{x^{[n-k]}}$ is k-Euler-Lagrange additive, and hence Theorem 2.6 implies that

$$\sum_{\substack{\mathfrak{B}_j \in \{B_j, B'_j\} \\ j \in \{1,\dots,k\}}} g_{x^{[n-k]}}\left(\mathfrak{B}_1, \dots, \mathfrak{B}_k\right)$$

$$= \prod_{j=1}^{k}(a_{1j}+a_{2j}) \sum_{l_1,\dots,l_k\in\{1,2\}} g_{x^{[n-k]}}(x_{l_1 1},\dots,x_{l_k k}).$$

It now follows from the above equality that

$$\sum_{\substack{\mathfrak{B}_j\in\{B_j,B'_j\}\\ j\in\{1,\dots,k\}}} f\left(\mathfrak{B}_1,\dots,\mathfrak{B}_k,,x^{[n-k]}\right)$$

$$= \prod_{j=1}^{k}(a_{1j}+a_{2j}) \sum_{l_1,\dots,l_k\in\{1,2\}} f\left(x_{l_1 1},\dots,x_{l_k k},x^{[n-k]}\right), \tag{3.2}$$

for all $x^{[n-k]} \in V^{n-k}$. Similar to the above, for any $x^{[k]} \in V^k$, consider the mapping $h_{x^{[k]}} : V^{n-k} \longrightarrow W$ defined via $h_{x^{[k]}}(x^{[n-k]}) := f(x^{[k]}, x^{[n-k]})$ for $x^{[n-k]} \in V^{n-k}$. This mapping is $n-k$-Jensen, and hence Theorem 2.2 implies that

$$\sum_{q\in\{-1,1\}^{n-k}} h_{x^{[k]}}\left(x_1^{[n-k]}+qx_2^{[n-k]}\right) = 2^{n-k}h_{x^{[k]}}\left(x_1^{[n-k]}\right), \tag{3.3}$$

for all $x_1^{[n-k]}, x_2^{[n-k]} \in V^{n-k}$. By the definition of $h_{x^{[k]}}$, (3.3) is equivalent to

$$\sum_{q\in\{-1,1\}^{n-k}} f\left(x^{[k]},x_1^{[n-k]}+qx_2^{[n-k]}\right) = 2^{n-k}f\left(x^{[k]},x_1^{[n-k]}\right), \tag{3.4}$$

for all $x_1^{[n-k]}, x_2^{[n-k]} \in V^{n-k}$ and $x^{[k]} \in V^k$. Plugging (3.2) into (3.4), we obtain

$$\sum_{\substack{\mathfrak{B}_j\in\{B_j,B'_j\}\\ j\in\{1,\dots,k\}}} \sum_{q\in\{-1,1\}^{n-k}} f\left(\mathfrak{B}_1,\dots,\mathfrak{B}_k,x_1^{[n-k]}+qx_2^{[n-k]}\right)$$

$$= \prod_{j=1}^{k}(a_{1j}+a_{2j}) \sum_{l_1,\dots,l_k\in\{1,2\}} \sum_{q\in\{-1,1\}^{n-k}} f\left(x_{l_1 1},\dots,x_{l_k k},x_1^{[n-k]}+qx_2^{[n-k]}\right)$$

$$= 2^{n-k}\prod_{j=1}^{k}(a_{1j}+a_{2j}) \sum_{l_1,\dots,l_k\in\{1,2\}} f\left(x_{l_1 1},\dots,x_{l_k k},x_1^{[n-k]}\right),$$

which proves that f satisfies equation (3.1). □

Proposition 3.2 has a converse under some mild conditions as follows.

Proposition 3.3. *If a mapping $f : V^n \longrightarrow W$ satisfies* (3.1) *and linear condition in the first k variables, then it is a k-Euler-Lagrange additive and $n-k$-Jensen mapping.*

Proof. Putting $x_2^{[n-k]} = (0,\dots,0)$ in the left side of (3.1), we obtain

$$2^{n-k}\sum_{\substack{\mathfrak{B}_j\in\{B_j,B'_j\}\\ j\in\{1,\dots,k\}}} f\left(\mathfrak{B}_1,\dots,\mathfrak{B}_k,x_1^{[n-k]}\right)$$

$$= 2^{n-k} \prod_{j=1}^{k} (a_{1j} + a_{2j}) \sum_{l_1,\dots,l_k \in \{1,2\}} f\left(x_{l_1 1}, \dots, x_{l_k k}, x_1^{[n-k]}\right), \tag{3.5}$$

for all $x_1^{[n-k]} \in V^{n-k}$, where B_j, B'_j are defined in (2.5). By (3.5) and in view of Theorem 2.6, we see that f is Euler-Lagrange additive in each of the k first variables. Furthermore, by putting $x_1^{[k]} = x_2^{[k]}$ in (3.1) and using hypothesis, we get

$$\begin{aligned} &2^k \prod_{j=1}^{k} (a_{1j} + a_{2j}) \sum_{q \in \{-1,1\}^{n-k}} f\left(x_1^{[k]}, x_1^{[n-k]} + q x_2^{[n-k]}\right) \\ &= 2^{n-k} \prod_{j=1}^{k} (a_{1j} + a_{2j})\, 2^k f\left(x_1^{[k]}, x_1^{[n-k]}\right) \end{aligned}$$

for all $x_1^{[k]} \in V^k$ and $x_1^{[n-k]} \in V^{n-k}$, and thus the proof is complete by Theorem 2.2. □

4. Stability results

Let $a, b \in \mathbb{R}\backslash\{0\}$ be fixed with $a + b \neq 0, \pm 1$. If we put $a_{1j} = a$ and $a_{2j} = b$ in (3.1) for all $j \in \{1, \dots, k\}$, then this equation converts to the equation

$$\begin{aligned} &\sum_{t_1,\dots t_k \in \{(a,b),(b,a)\}} \sum_{q \in \{-1,1\}^{n-k}} f\left(\mathfrak{B}_1^{t_1}, \dots, \mathfrak{B}_k^{t_k}, x_1^{[n-k]} + q x_2^{[n-k]}\right) \\ &= 2^{n-k} m^k \sum_{l_1,\dots,l_k \in \{1,2\}} f\left(x_{l_1 1}, \dots, x_{l_k k}, x_1^{[n-k]}\right), \end{aligned} \tag{4.1}$$

where $m = a + b$, $\mathfrak{B}_j^{(a,b)} = a x_{1j} + b x_{2j}$, $\mathfrak{B}_j^{(b,a)} = b x_{1j} + a x_{2j}$ and $x_i^{[n-k]} = (x_{i,k+1}, \dots, x_{in}) \in V^{n-k}$ whereas $i \in \{1, 2\}$ and $j \in \{1, \dots, k\}$.

In this section, we prove the Găvruţa and Hyers-Ulam stabilities of equation (4.1) by the incoming fixed point theorem ([8, Theorem 1]) in Banach spaces. Throughout, for two sets X and Y, we denote the set of all mappings from X to Y by Y^X.

Theorem 4.1. *Let Y be a Banach space, $\mathcal{S}$ be a nonempty set, $j \in \mathbb{N}$, $g_1, \dots, g_j : \mathcal{S} \longrightarrow \mathcal{S}$ and $L_1, \dots, L_j : \mathcal{S} \longrightarrow \mathbb{R}_+$. Suppose that the hypotheses*

(H1) $\mathcal{T} : Y^{\mathcal{S}} \longrightarrow Y^{\mathcal{S}}$ *is an operator satisfying the inequality*

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^{j} L_i(x) \|\lambda(g_i(x)) - \mu(g_i(x))\|, \quad \lambda, \mu \in Y^{\mathcal{S}}, x \in \mathcal{S},$$

(H2) $\Lambda : \mathbb{R}_+^{\mathcal{S}} \longrightarrow \mathbb{R}_+^{\mathcal{S}}$ *is an operator defined through*

$$\Lambda\delta(x) := \sum_{i=1}^{j} L_i(x)\delta(g_i(x)) \qquad \delta \in \mathbb{R}_+^{\mathcal{S}}, x \in \mathcal{S}.$$

hold and a function $\theta : \mathcal{S} \longrightarrow \mathbb{R}_+$ *and a mapping* $\phi : \mathcal{S} \longrightarrow Y$ *fulfill the following two conditions*:

$$\|\mathcal{T}\phi(x) - \phi(x)\| \leq \theta(x), \quad \theta^*(x) := \sum_{l=0}^{\infty} \Lambda^l \theta(x) < \infty \qquad (x \in \mathcal{S}).$$

Then, there exists a unique fixed point ψ *of* $\mathcal{T}$ *such that*

$$\|\phi(x) - \psi(x)\| \leq \theta^*(x) \qquad (x \in \mathcal{S}).$$

Moreover, $\psi(x) = \lim_{l\to\infty} \mathcal{T}^l \phi(x)$ *for all* $x \in \mathcal{S}$.

Here and subsequently, for a mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathcal{D}f : V^n \times V^n \longrightarrow W$ by

$$\begin{aligned}\mathcal{D}f(x_1^{[n]}, x_2^{[n]}) := &\sum_{t_1,\ldots,t_n\in\{(a,b),(b,a)\}} f\left(\mathfrak{B}_1^{t_1}, \ldots, \mathfrak{B}_k^{t_k}, x_1^{[n-k]} + qx_2^{[n-k]}\right) \\ &- 2^{n-k}m^k \sum_{l_1,\ldots,l_k\in\{1,2\}} f\left(x_{l_1 1}, \ldots, x_{l_k k}, x_1^{[n-k]}\right).\end{aligned}$$

We recall the next lemma from [3] that is a fundamental tool in obtaining the stability results. For convenience, given an $m \in \mathbb{N}$, we write $S := \{0,1\}^d$, and S_i stands for the set of all elements of S having exactly i zeros, i.e.,

$$S_i : \{(s_1, \ldots, s_d) \in S : \mathrm{card}\{j : s_j = 0\} = i\}, \qquad i \in \{0, \ldots, d\}.$$

Lemma 4.2. *Let* $d \in \mathbb{N}$, $l \in \mathbb{N}_0$ *and* $\psi : S \longrightarrow \mathbb{R}$ *be a function. Then*

$$\sum_{v=0}^{d}\sum_{w=0}^{d}\sum_{s\in S_w}\sum_{t\in S_v}(2^l-1)^w\psi(st) = \sum_{i=0}^{d}\sum_{p\in S_i}(2^{l+1}-1)^i\psi(p).$$

We have the next stability result for equation (4.1). Note that in this theorem S stands for $\{0,1\}^{n-k}$ and $S_i \subseteq S$ for $i \in \{0, \ldots, n-k\}$.

Theorem 4.3. *Let* V *be a linear space and* W *be a Banach space. Suppose that* $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ *is a mapping satisfying the relations*

$$\begin{aligned}&\lim_{l\to\infty}\left(\frac{1}{2^{n-k}m^k}\right)^l \sum_{i=0}^{n-k}\sum_{p\in S_i}(2^l-1)^i \\ &\times \varphi\left(\left(m^l x_1^{[k]}, 2^l p x_1^{[n-k]}\right), \left(m^l x_2^{[k]}, 2^l p x_2^{[n-k]}\right)\right) \\ &= 0, \end{aligned} \tag{4.2}$$

for all $x_1^{[k]}, x_2^{[k]} \in V^n$ *and* $x_1^{[n-k]}, x_2^{[n-k]} \in V^{n-k}$ *and*

$$\Phi(x) = \frac{1}{2^n m^k} \sum_{l=0}^{n} \left(\frac{1}{2^{n-k} m^k} \right)^l \sum_{i=0}^{n-k} \sum_{p \in S_i} (2^l - 1)^i$$
$$\times \varphi \left(\left(m^l x^{[k]}, 2^l p x^{[n-k]} \right), \left(m^l x^{[k]}, 2^l p x^{[n-k]} \right) \right) < \infty, \tag{4.3}$$

for all $x = (x^{[k]}, x^{[n-k]}) \in V^n$. *Assume also* $f : V^n \longrightarrow W$ *is a mapping satisfying the inequality*

$$\left\| \mathcal{D}f \left(x_1^{[k]}, x_1^{[n-k]}, x_2^{[k]}, x_2^{[n-k]} \right) \right\| \leq \varphi \left(x_1^{[k]}, x_1^{[n-k]}, x_2^{[k]}, x_2^{[n-k]} \right), \tag{4.4}$$

for all $x_1^{[k]}, x_2^{[k]} \in V^n$ *and* $x_1^{[n-k]}, x_2^{[n-k]} \in V^{n-k}$. *Then, there exists a solution* $\mathcal{F} : V^n \longrightarrow W$ *of* (4.1) *such that*

$$\|f(x) - \mathcal{F}(x)\| \leq \Phi(x), \tag{4.5}$$

for all $x \in V^n$. *Moreover, if* $\mathcal{F}$ *has the linear condition in the first* k *variables, then it is a unique* k*-Euler-Lagrange additive and* $n-k$*-Jensen mapping.*

Proof. Putting $x_1^{[k]} = x_2^{[k]} := x^{[k]} \in V^k$ and $x_1^{[n-k]} = x_2^{[n-k]} := x^{[n-k]} \in V^{n-k}$ in (4.4), we have

$$\left\| 2^k \sum_{s \in S} f \left(m x^{[k]}, 2 s x^{[n-k]} \right) - 2^n m^k f(x) \right\| \leq \varphi(x, x),$$

for all $x := (x_1^{[k]}, x_1^{[n-k]}) = (x^{[k]}, x^{[n-k]}) \in V^n$ (and the rest of the proof if is necessary). Thus,

$$\left\| f(x) - \frac{1}{2^{n-k} m^k} \sum_{s \in S} f \left(m x^{[k]}, 2 s x^{[n-k]} \right) \right\| \leq \frac{1}{2^n m^k} \varphi(x, x). \tag{4.6}$$

Set $\theta(x) := \frac{1}{2^n m^k} \varphi(x, x)$ and $\mathcal{T}\theta(x) := \frac{1}{2^{n-k} m^k} \sum_{s \in S} f(m x^{[k]}, 2 s x^{[n-k]})$ where $\theta \in W^{V^n}$. Then, (4.6) can be rewritten as

$$\|f(x) - \mathcal{T}f(x)\| \leq \theta(x) \qquad (x \in V^n).$$

Define $\Lambda\eta(x) := \frac{1}{2^{n-k} m^k} \sum_{s \in S} \eta(m x^{[k]}, 2 s x^{[n-k]})$ for all $\eta \in \mathbb{R}_+^{V^n}$. We now see that Λ has the form presented in (H2) with $\mathcal{S} = V^n$, $g_i(x) = g_s(x) = (m x^{[k]}, 2 s x^{[n-k]})$ and $L_i(x) = \frac{1}{2^{n-k} m^k}$ for all i and $x \in V^n$. Once more, for each $\lambda, \mu \in W^{V^n}$, we obtain

$$\begin{aligned}
&\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \\
&= \left\| \frac{1}{2^{n-k} m^k} \left[\sum_{s \in S} (\lambda \left(m x^{[k]}, 2 s x^{[n-k]} \right) - \mu \left(m x^{[k]}, 2 s x^{[n-k]} \right) \right] \right\| \\
&\leq \frac{1}{2^{n-k} m^k} \sum_{s \in S} \left\| \lambda \left(m x^{[k]}, 2 s x^{[n-k]} \right) - \mu \left(m x^{[k]}, 2 s x^{[n-k]} \right) \right\|.
\end{aligned}$$

It follows the relation above that the hypothesis (H1) holds. An induction argument on l shows that for any $l \in \mathbb{N}_0$ and $x \in V^n$

$$\Lambda^l\theta(x) := \left(\frac{1}{2^{n-k}m^k}\right)^l \sum_{i=0}^{n-k}(2^l-1)^i \sum_{p\in S_i} \theta\left(m^l x^{[k]}, 2^l p x^{[n-k]}\right). \tag{4.7}$$

Fix an $x \in V^n$. It is convenient to adapt the convention that $0^0 = 1$ and so (4.7) is trivially valid for $l = 0$. Next, assume that (4.7) holds for a $l \in \mathbb{N}_0$. Applying Lemma 4.2 for $d = n-k$ and $\psi(s) := \theta\left(m^{l+1}x^{[k]}, 2^{l+1}sx^{[n-k]}\right)$ for $s \in S$, we obtain

$$\begin{aligned}
&\Lambda^{l+1}\theta(x)\\
&= \Lambda(\Lambda^l\theta)(x) = \frac{1}{2^{n-k}m^k}\sum_{v=0}^{n-k}\sum_{t\in S_v}(\Lambda^l\theta)\left(mx^{[k]}, 2tx^{[n-k]}\right)\\
&= \left(\frac{1}{2^{n-k}m^k}\right)^{l+1}\sum_{v=0}^{n-k}\sum_{t\in S_v}\sum_{w=0}^{n-k}(2^l-1)^w\sum_{s\in S_w}\theta\left(m^{l+1}x^{[k]}, 2^{l+1}stx^{[n-k]}\right)\\
&= \left(\frac{1}{2^{n-k}m^k}\right)^{l+1}\sum_{v=0}^{n-k}\sum_{w=0}^{n-k}\sum_{s\in S_w}\sum_{t\in S_v}(2^l-1)^w\theta\left(m^{l+1}x^{[k]}, 2^{l+1}stx^{[n-k]}\right)\\
&= \left(\frac{1}{2^{n-k}m^k}\right)^{l+1}\sum_{i=0}^{n-k}\sum_{p\in S_i}(2^{l+1}-1)^i\theta\left(m^{l+1}x^{[k]}, 2^{l+1}px^{[n-k]}\right).
\end{aligned}$$

Therefore, (4.7) holds for any $l \in \mathbb{N}_0$ and $x \in V^n$. It now follows from (4.3) and (4.7) that all assumptions of Theorem 4.1 are fulfilled. Hence, there exists a mapping $\mathcal{F} : V^n \longrightarrow W$ such that

$$\mathcal{F}(x) = \lim_{l\to\infty}(\mathcal{T}^l f)(x) = \frac{1}{2^{n-k}m^k}\sum_{s\in S}\mathcal{F}\left(mx^{[k]}, 2sx^{[n-k]}\right) \qquad (x \in V^n),$$

and moreover (4.5) holds. We wish to show that

$$\begin{aligned}
&\left\|\mathcal{D}(\mathcal{T}^l f)\left(x_1^{[k]}, x_1^{[n-k]}, x_2^{[k]}, x_2^{[n-k]}\right)\right\|\\
&\leq \left(\frac{1}{2^{n-k}m^k}\right)^l \sum_{i=0}^{n-k}\sum_{p\in S_i}(2^l-1)^i\\
&\quad\times \varphi\left(\left(m^l x_1^{[k]}, 2^l p x_1^{[n-k]}\right), \left(m^l x_2^{[k]}, 2^l p x_2^{[n-k]}\right)\right),
\end{aligned} \tag{4.8}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}_0$. We argue by induction on l. Clearly, (4.8) is true for $l = 0$ by (4.4). Assume that (4.8) is valid for an $l \in \mathbb{N}_0$. Then

$$\left\|\mathcal{D}(\mathcal{T}^{l+1} f)\left(x_1^{[k]}, x_1^{[n-k]}, x_2^{[k]}, x_2^{[n-k]}\right)\right\|$$

$$
\begin{aligned}
&= \frac{1}{2^{n-k}m^k}\left\|\sum_{s\in S}\mathcal{D}(\mathcal{T}^l f)\left(mx_1^{[k]},2x_1^{[n-k]},mx_2^{[k]},2x_2^{[n-k]}\right)\right\| \\
&\le \left(\frac{1}{2^{n-k}m^k}\right)^{l+1}\sum_{s\in S}\sum_{i=0}^{n}\sum_{t\in S_i}(2^l-1)^i \\
&\quad \times\varphi\left(\left(m^{l+1}x_1^{[k]},2^{l+1}stx_1^{[n-k]}\right),\left(m^{l+1}x_2^{[k]},2^{l+1}stx_2^{[n-k]}\right)\right) \\
&= \left(\frac{1}{2^{n-k}m^k}\right)^{l+1}\sum_{i=0}^{n}\sum_{p\in S_i}(2^{l+1}-1)^i \\
&\quad \times\varphi\left(\left(m^{l+1}x_1^{[k]},2^{l+1}px_1^{[n-k]}\right),\left(m^{l+1}x_2^{[k]},2^{l+1}px_2^{[n-k]}\right)\right),
\end{aligned}
$$

for all $x_1^{[k]},x_2^{[k]}\in V^n$ and $x_1^{[n-k]},x_2^{[n-k]}\in V^{n-k}$. We note that the last equality follows from Lemma 4.2 with $d:=n-k$ and

$$
\psi(s):=\phi\left(\left(m^{l+1}x_1^{[k]},2^{l+1}sx_1^{[n-k]}\right),\left(m^{l+1}x_2^{[k]},2^{l+1}sx_2^{[n-k]}\right)\right),\qquad (s\in S).
$$

Taking $l\to\infty$ in (4.8) and using (4.2), we have $\mathcal{DF}(x_1^{[k]},x_1^{[n-k]},x_2^{[k]},x_2^{[n-k]})=0$ for all $x_1^{[k]},x_2^{[k]}\in V^k$ and $x_1^{[n-k]},x_2^{[n-k]}\in V^{n-k}$. Hence, (4.1) holds for $\mathcal{F}$. If $\mathcal{F}$ has the linear condition in the first k variables, then it is a k-Euler-Lagrange additive and $n-k$-Jensen mapping by Proposition 3.3. Lastly, let $\mathfrak{F}:V^n\longrightarrow W$ be another Euler-Lagrange-Jensen mapping satisfying (4.1) and (4.5). Fix $x=(x^{[k]},x^{[n-k]})\in V^n$ and $j\in\mathbb{N}$. By (4.3), we obtain

$$
\begin{aligned}
&\|\mathcal{F}(x)-\mathfrak{F}(x)\| \\
&=\left\|\left(\frac{1}{2^{n-k}m^k}\right)^j\mathcal{F}\left(m^jx^{[k]},2^jx^{[n-k]}\right)-\left(\frac{1}{2^{n-k}m^k}\right)^j\mathfrak{F}\left(m^jx^{[k]},2^jx^{[n-k]}\right)\right\| \\
&\le\left(\frac{1}{2^{n-k}m^k}\right)^j\left(\left\|\mathcal{F}\left(m^jx^{[k]},2^jx^{[n-k]}\right)-f\left(m^jx^{[k]},2^jx^{[n-k]}\right)\right\|\right. \\
&\quad\left.+\left\|\mathfrak{F}\left(m^jx^{[k]},2^jx^{[n-k]}\right)-f\left(m^jx^{[k]},2^jx^{[n-k]}\right)\right\|\right) \\
&\le 2\left(\frac{1}{2^{n-k}m^k}\right)^j\Phi\left(m^jx^{[k]},2^jx^{[n-k]}\right) \\
&\le\frac{2}{2^nm^k}\sum_{l=j}^{n}\left(\frac{1}{2^{n-k}m^k}\right)^l\sum_{i=0}^{n-k}\sum_{p\in S_i}(2^l-1)^i \\
&\quad\times\varphi\left(\left(m^lx^{[k]},2^lpx^{[n-k]}\right),\left(m^lx^{[k]},2^lpx^{[n-k]}\right)\right).
\end{aligned}
$$

Consequently, letting $j\to\infty$ and using the fact that series (4.3) is convergent for all $x\in V^n$, we have $\mathcal{F}(x)=\mathfrak{F}(x)$ for all $x\in V^n$, and therefore the proof is completed. □

We recall that the binomial coefficient for all $s, r \in \mathbb{N}_0$ with $s \geq r$ is defined and denoted by $\binom{s}{r} := \frac{s!}{r!(s-r)!}$. In the next corollary, we show that the functional equation (4.1) is Hyers stable when $\|\mathcal{D}f(x_1^{[n]}, x_2^{[n]})\|$ is controlled by a positive small real number δ.

Corollary 4.4. *Given $\delta > 0$. Let V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\left\|\mathcal{D}f\left(x_1^{[k]}, x_1^{[n-k]}, x_2^{[k]}, x_2^{[n-k]}\right)\right\| \leq \delta,$$

for all $x_1^{[k]}, x_2^{[k]} \in V^n$ and $x_1^{[n-k]}, x_2^{[n-k]} \in V^{n-k}$, then there exists a unique solution $\mathcal{F} : V^n \longrightarrow W$ of (4.1) such that

$$\|f(x) - \mathcal{F}(x)\| \leq \frac{\delta}{2^n(m^k - 1)},$$

for all $x \in V^n$.

Proof. Considering the constant function $\varphi(x_1^{[k]}, x_1^{[n-k]}, x_2^{[k]}, x_2^{[n-k]}) = \delta$ for all $x_1^{[k]}, x_2^{[k]} \in V^n$, $x_1^{[n-k]}, x_2^{[n-k]} \in V^{n-k}$, and applying Theorem 4.3, we have

$$\begin{aligned}
\Phi(x) = & \sum_{l=0}^{n} \left(\frac{1}{2^{n-k}m^k}\right)^l \sum_{i=0}^{n-k} \sum_{p \in S_i} (2^l - 1)^i \\
& \times \varphi\left(\left(m^l x^{[k]}, 2^l p x^{[n-k]}\right), \left(m^l x^{[k]}, 2^l p x^{[n-k]}\right)\right) \\
= & \frac{\delta}{2^n m^k} \sum_{l=0}^{\infty} \left(\frac{1}{2^{n-k}m^k}\right)^l \sum_{i=0}^{n-k} \binom{n}{i} (2^l - 1)^i \times 1^{n-i} \\
= & \frac{\delta}{2^n m^k} \sum_{l=0}^{\infty} \left(\frac{1}{2^{n-k}m^k}\right)^l 2^{(n-k)l} \\
= & \frac{\delta}{2^n m^k} \sum_{l=0}^{\infty} \left(\frac{1}{m^k}\right)^l \\
= & \frac{\delta}{2^n(m^k - 1)}.
\end{aligned}$$

for all $x = (x^{[k]}, x^{[n-k]}) \in V^n$. □

A special case of (4.1) is the following equation when $k = n$.

$$\sum_{t_1, \ldots t_n \in \{(a,b),(b,a)\}} f\left(\mathfrak{B}_1^{t_1}, \ldots, \mathfrak{B}_n^{t_n}\right) = m^n \sum_{l_1, \ldots, l_n \in \{1,2\}} f(x_{l_1 1}, \ldots, x_{l_n n}). \tag{4.9}$$

Putting $k = n$ in Theorem 4.3, we obtain the below result on the Hyers stability of multi-Euler-Lagrange additive equation (4.9).

Corollary 4.5. *Given $\delta > 0$. Let V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\left\| \sum_{t_1,\ldots,t_n \in \{(a,b),(b,a)\}} f\left(\mathfrak{B}_1^{t_1}, \ldots, \mathfrak{B}_n^{t_n}\right) - m^n \sum_{l_1,\ldots,l_n \in \{1,2\}} f(x_{l_1 1}, \ldots, x_{l_n n}) \right\| \leq \delta,$$

then there exists a solution $\mathcal{A} : V^n \longrightarrow W$ of (4.9) such that

$$\|f(x) - \mathcal{A}(x)\| \leq \frac{\delta}{2^n(m^n - 1)}$$

for all $x \in V^n$. In particular, if $\mathcal{F}$ has the linear condition in all variables, then it is a unique multi-Euler-Lagrange additive.

Corollary 4.6. *Let $\alpha > 0$ with $\alpha \neq n$. Let also V be a normed space and W be a Banach space. Suppose that $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\left\| \sum_{t_1,\ldots,t_n \in \{(a,b),(b,a)\}} f\left(\mathfrak{B}_1^{t_1}, \ldots, \mathfrak{B}_n^{t_n}\right) - m^n \sum_{l_1,\ldots,l_n \in \{1,2\}} f(x_{l_1 1}, \ldots, x_{l_n n}) \right\|$$
$$\leq \sum_{i=1}^{2} \sum_{j=1}^{n} \|x_{ij}\|^\alpha.$$

Then, there exists a solution $\mathcal{A} : V^n \longrightarrow W$ of (4.9) such that

$$\|f(x) - \mathcal{A}(x)\| \leq \begin{cases} \frac{1}{2^{n-1}(m^n - m^\alpha)} \sum_{j=1}^n \|x_{1j}\|^\alpha & \alpha < n, \\ \\ \frac{m^\alpha}{2^{n-1}(m^\alpha - m^n)} \sum_{j=1}^n \|x_{1j}\|^\alpha & \alpha > n, \end{cases}$$

for all $x := x_1^{[n]} \in V^n$. Furthermore, if $\mathcal{F}$ has the linear condition in all variables, then it is a unique multi-Euler-Lagrange additive.

5. Non-stability example

In this section, we present a non-stability example for multiadditive mappings on $\mathbb{R}^n$. For doing this, we need the following result, indicated in [20, Theorem 13.4.3].

Theorem 5.1. *Let $g : \mathbb{R}^{p^N} \longrightarrow \mathbb{R}$ be a continuous p-additive function. Then, there exist constants $c_{j_1 \cdots j_p} \in \mathbb{R}$, $j_1, \ldots, j_p = 1, \ldots, N$, such that*

$$g(x_1, \ldots, x_p) = \sum_{j_1=1}^{N} \cdots \sum_{j_p=1}^{N} c_{j_1 \ldots j_p} x_{1 j_1} \ldots x_{p j_p},$$

for all $x_i = (x_{i1}, \ldots, x_{iN})$ and $i = 1, \ldots, p$.

Remark 5.2. In the proof of Theorem 5.1 only the continuity of g with respect to each variable separately was used. Therefore, the result is true if and only if f is supposed separately continuous with respect to each variable. On the other hand, in virtue of the proof of Theorem 5.1, if the continuity condition of g is removed, then the theorem remains valid for a function $g:\mathbb{Q}^p\longrightarrow\mathbb{Q}$ in the case $N=1$. We use this fact to make a non-stable example.

Note that for $a=b=1$, every multi-Euler-Lagrange additive is multiadditive and so two equations (1.3) and (4.9) coincide. We close the paper with a counterexample such that the hypothesis $\alpha\neq n$ is necessary and can not be removed in Corollary 4.6 for multiadditive mappings. Here, we remind the idea of the proof idea is taken from [14].

Example 5.3. Let $\delta>0$ and $n\in\mathbb{N}$. Put $\mu=\frac{2^n-1}{2^{2n}(2^n+1)}\delta$. Define the function $\psi:\mathbb{Q}^n\longrightarrow\mathbb{Q}$ through

$$\psi(r_1,\dots,r_n)=\begin{cases}\mu\prod_{j=1}^n r_j & \text{for all } r_j \text{ with } |r_j|<1,\\ \mu & \text{otherwise.}\end{cases}$$

Moreover, define the function $f:\mathbb{Q}^n\longrightarrow\mathbb{Q}$ via

$$f(r_1,\dots,r_n)=\sum_{l=0}^{\infty}\frac{\psi(2^l r_1,\dots,2^l r_n)}{2^{nl}},\qquad (r_j\in\mathbb{Q}).$$

Obviously, ψ is bounded by μ. Indeed, for each $(r_1,\dots,r_n)\in\mathbb{Q}^n$, we have $|f(r_1,\dots,r_n)|\leq\frac{2^n}{2^n-1}\mu$. Put $x_i=(x_{i1},\dots,x_{in})$, where $i\in\{1,2\}$. We claim that

$$|\mathbf{D}f\,(x_1,x_2)|\leq\delta\sum_{i=1}^{2}\sum_{j=1}^{n}|x_{ij}|^n, \tag{5.1}$$

for all $x_1,x_2\in\mathbb{Q}^n$, where

$$\mathbf{D}f\,(x_1,x_2):=f\,(x_1+x_2)-\sum_{j_1,\dots,j_n\in\{1,2\}}f(x_{j_11},\dots,x_{j_nn}),$$

in which $x_j=(x_{j1},\dots,x_{jn})\in\mathbb{Q}^n$ with $j\in\{1,2\}$. It is clear that (5.1) holds for $x_1=x_2=0$. Let $x_1,x_2\in\mathbb{Q}^n$ with

$$\sum_{i=1}^{2}\sum_{j=1}^{n}|x_{ij}|^n<\frac{1}{2^n}. \tag{5.2}$$

Thus, there exists a positive integer N such that

$$\frac{1}{2^{n(N+1)}}<\sum_{i=1}^{2}\sum_{j=1}^{n}|x_{ij}|^n<\frac{1}{2^{nN}} \tag{5.3}$$

and hence $|x_{ij}|^n<\sum_{i=1}^2\sum_{j=1}^n|x_{ij}|^n<\frac{1}{2^{nN}}$. The last relation implies that $2^N|x_{ij}|<1$ for all $i\in\{1,2\}$ and $j\in\{1,\dots,n\}$. Therefore, $2^{N-1}|x_{ij}|<$

1. If $y_1, y_2 \in \{x_{ij} | \, i \in \{1,2\}, \; j \in \{1,\dots,n\}\}$, then $2^{N-1}|\, y_1 \pm y_2| < 1$. Since ψ is multiadditive function on $(-1,1)^n$, $\mathbf{D}\psi\left(2^l x_1, 2^l x_2\right) = 0$ for all $l \in \{0,1,2,\dots,N-1\}$. We conclude from the last equality and (5.3) that

$$\begin{aligned}\frac{\left|\mathbf{D}f\left(2^l x_1, 2^l x_2\right)\right|}{\sum_{i=1}^2 \sum_{j=1}^n |x_{ij}|^n} &\le \sum_{l=N}^{\infty} \frac{\left|\mathbf{D}\psi\left(2^l x_1, 2^l x_2\right)\right|}{2^{nl}\sum_{i=1}^2 \sum_{j=1}^n |x_{ij}|^n}\\ &\le \sum_{l=0}^{\infty} \frac{\mu(2^n+1)}{2^{n(l+N)}\sum_{i=1}^2 \sum_{j=1}^n |x_{ij}|^n}\\ &\le \mu 2^n(2^n+1)\sum_{l=0}^{\infty}\frac{1}{2^{nl}}\\ &= \mu(2^n+1)\frac{2^{2n}}{2^n-1} = \delta,\end{aligned}$$

for all $x_1, x_2 \in \mathbb{Q}^n$ and thus (5.1) is true when (5.2) happens. If

$$\sum_{i=1}^{2}\sum_{j=1}^{n}|x_{ij}|^n \ge \frac{1}{2^n},$$

then

$$\frac{\left|\mathbf{D}f\left(2^l x_1, 2^l x_2\right)\right|}{\sum_{i=1}^2 \sum_{j=1}^n |x_{ij}|^n} \le 2^n \frac{2^n}{2^n-1}\mu(2^n+1) = \delta.$$

Therefore, f satisfies (5.1) for all $x_1, x_2 \in \mathbb{Q}^n$. Now, suppose the assertion is false, there exist a number $\lambda \in [0,\infty)$ and a multiadditive function $\mathcal{A} : \mathbb{Q}^n \longrightarrow \mathbb{Q}$ such that

$$|f(r_1,\dots,r_n) - \mathcal{A}(r_1,\dots,r_n)| \le \lambda \sum_{j=1}^{n}|r_j|^n,$$

for all $(r_1,\dots,r_n) \in \mathbb{Q}^n$. Without loss of generality, one can take a number $b \in [0,\infty)$ so that

$$\lambda \sum_{j=1}^{n}|r_j|^n \le b\prod_{j=1}^{n}|r_j|.$$

Hence, $|f(r_1,\dots,r_n) - \mathcal{A}(r_1,\dots,r_n)| < b\prod_{j=1}^n |r_j|$ for all $(r_1,\dots,r_n) \in \mathbb{Q}^n$. It follows now from Lemma 5.2 that there is a constant $c \in \mathbb{R}$ such that $\mathcal{A}(r_1,\dots,r_n) = c\prod_{j=1}^n r_j$ for all $(r_1,\dots,r_n) \in \mathbb{Q}^n$ and therefore

$$|f(r_1,\dots,r_n)| \le (|c|+b)\prod_{j=1}^{n}|r_j|, \tag{5.4}$$

for all $(r_1,\dots,r_n) \in \mathbb{Q}^n$. On the other hand, one can choose $N \in \mathbb{N}$ such that $(N+1)\mu > |c|+b$. If $r = (r_1,\dots,r_n) \in \mathbb{Q}^n$ such that $r_j \in \left(0, \frac{1}{2^N}\right)$ for all

$j \in \{1, \ldots, n\}$, then $2^l r_j \in (0,1)$ for all $l = 0, 1, \ldots, N$. Hence

$$|f(r_1, \ldots, r_n)| = \left| \sum_{l=0}^{\infty} \frac{\psi\left(2^l r_1, \ldots, 2^l r_2\right)}{2^{nl}} \right| = \left| \sum_{l=0}^{N} \frac{\mu 2^{nl} \prod_{j=1}^{n} r_j}{2^{nl}} \right|$$
$$= (N+1)\mu \prod_{j=1}^{n} |r_j| > (|c| + b) \prod_{j=1}^{n} |r_j|,$$

that leads us to a contradiction with (5.4).

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