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TOEPLITZ DETERMINANTS FOR λ-PSEUDO-STARLIKE FUNCTIONS

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Abstract. In this article, by making use of the λ-pseudo-starlike functions, we introduce a certain family of normalized analytic functions in the open unit disk U and we establish coefficient estimates for the first four determinants of the Toeplitz matrices $T_2(2)$, $T_2(3)$, $T_3(2)$ and $T_3(1)$ for the functions belonging to this family. Further, some known and new results which follow as special cases of our results are also mentioned.

1. Introduction

Let $\mathcal{A}$ stand for the family of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$. Let S indicate the class of all functions in $\mathcal{A}$ which are univalent in U.

Babalola [2] defined the family $\mathcal{L}_\lambda$ of λ-pseudo-starlike functions as follows:

Definition ([2]). Let $f \in \mathcal{A}$. Suppose that $\lambda \geq 1$. A function f belongs to the family $\mathcal{L}_\lambda$ of λ-pseudo-starlike functions in U if and only if

$$Re\left\{\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}\right\} > 0, \quad (z \in U).$$

In particular, if $\lambda = 1$, we have S^* which in this context are called as starlike functions. This subclass was recently studied by [4], [5] and [9].

In the univalent function theory, an extensive focus has been given to estimate the bounds of Hankel matrices. Hankel matrices and determinants play an important role in several branches of mathematics and have many applications [11]. Toeplitz determinants are closely related to Hankel determinants. Hankel matrices have constant entries along the reverse diagonal, whereas Toeplitz

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matrices have constant entries along the diagonal. Recently, Ali et al. [1] introduced the symmetric Toeplitz determinant $T_q(n)$ for $f \in \mathcal{A}$, defined by

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix},$$

where $n \geq 1$, $q \geq 1$ and $a_1 = 1$. In particular,

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix},$$

and

$$T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}, \quad T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

The concept of Toeplitz matrices plays an important role in functional analysis, applied mathematics as well as in physics and technical sciences (for more details see [11]). Very recently, several authors established estimates of the Toeplitz determinant $|T_q(n)|$ for functions belonging to various families of univalent functions (see, for example, [1, 7, 8, 10]).

To derive the desired bounds in our study, we shall require the following lemmas. Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ with $p(0) = 1$ and $Re\{p(z)\} > 0$ $(z \in U)$.

Lemma 1.1 ([6]). *If the function $p \in \mathcal{P}$ is given by the series $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, then the sharp estimate $|p_k| \leq 2$ $(k = 1, 2, 3, \ldots)$ holds.*

Lemma 1.2 ([3]). *If the function $p \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + \left(4 - p_1^2\right) x,$$

$$4p_3 = p_1^3 + 2p_1\left(4 - p_1^2\right) x - p_1\left(4 - p_1^2\right) x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right) z$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

2. Main results

We begin this section by defining the family $\mathcal{L}_\lambda$ as follows:

Definition ([2]). A function $f \in \mathcal{A}$ is said to be in the family $\mathcal{L}_\lambda$ $(\lambda \geq 1)$ if it satisfies the condition:

$$Re\left\{\frac{z\left(f'(z)\right)^\lambda}{f(z)}\right\} > 0, \quad (z \in U).$$

Theorem 2.1. *Let $f \in \mathcal{L}_\lambda$ be given by* (1). *Then*

$$|a_2| \leq \frac{2}{B},$$

$$|a_3| \leq \frac{2}{C} + \frac{4|N|}{CB^2}$$

and

$$|a_4| \leq \frac{2}{A} + \frac{4|E|}{BCA} + \frac{8M}{3B^3CA},$$

where

$$\begin{aligned} A &= 4\lambda - 1,\ B = 2\lambda - 1,\ \ C = 3\lambda - 1, \\ E &= 6\lambda^2 - 11\lambda + 2, \quad N = 2\lambda^2 - 4\lambda + 1, \\ M &= 24\lambda^4 - 80\lambda^3 + 84\lambda^2 - 28\lambda + 3. \end{aligned} \tag{2}$$

Proof. For the function $f \in \mathcal{L}_\lambda$ given by (1), we know that there exists an analytic function $p \in \mathcal{P}$ in the unit disk U with $p(0) = 1$ and $Re\{p(z)\} > 0$ such that

$$z\left(f'(z)\right)^\lambda = f(z)p(z), \quad (z \in U). \tag{3}$$

where p has the following series representations:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots .$$

By elementary calculations, we have

$$\begin{aligned} &z(1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots)^\lambda \\ &= (z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)\left(1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots\right) \end{aligned}$$

and

(4)

$$\begin{aligned} &z + 2\lambda a_2 z^2 + [3\lambda a_3 + 2\lambda(\lambda - 1)a_2^2]z^3 \\ &+ \left[4\lambda a_4 + 6\lambda(\lambda - 1)a_2 a_3 + \frac{4\lambda(\lambda - 1)(\lambda - 2)}{3}a_2^3\right]z^4 + \cdots \\ &= z + (p_1 + a_2)z^2 + (p_2 + p_1 a_2 + a_3)z^3 + (p_3 + p_2 a_2 + p_1 a_3 + a_4)z^4 + \cdots . \end{aligned}$$

By equating the coefficients in (4), we have the relations

$$a_2 = \frac{1}{B}p_2, \tag{5}$$

$$a_3 = \frac{1}{C}p_2 - \frac{N}{CB^2}p_1^2 \tag{6}$$

and

$$a_4 = \frac{1}{A}p_3 - \frac{E}{BCA}p_1 p_2 + \frac{M}{3B^3CA}p_1^3 \tag{7}$$

and by applying Lemma 1.1, we get

$$|a_2| \leq \frac{2}{B},$$

$$|a_3| \leq \frac{2}{C} + \frac{4|N|}{CB^2}$$

and

$$|a_4| \leq \frac{2}{A} + \frac{4|E|}{BCA} + \frac{8M}{3B^3CA},$$

where

$$A = 4\lambda - 1,\ B = 2\lambda - 1,\ \ C = 3\lambda - 1,$$
$$E = 6\lambda^2 - 11\lambda + 2,\quad N = 2\lambda^2 - 4\lambda + 1,$$
$$M = 24\lambda^4 - 80\lambda^3 + 84\lambda^2 - 28\lambda + 3.$$

□

Theorem 2.2. *Let* $f \in \mathcal{L}_\lambda$ *be given by* (1). *Then*

$$|T_2(2)| \leq \frac{4(B^4 - 4B^2N + 4N^2)}{B^4C^2} - \frac{4}{B^2}, \tag{8}$$

where

$$B = 2\lambda - 1,\ N = 2\lambda^2 - 4\lambda + 1.$$

Proof. In view of (5), (6) and (2), it easy to see that

$$\begin{aligned}|T_2(2)| &= \left|a_3^2 - a_2^2\right| \\ &= \left|\frac{p_2^2}{C^2} - \frac{2Np_1^2p_2}{C^2B^2} + \frac{N^2p_1^4}{C^2B^4} - \frac{p_1^2}{B^2}\right|.\end{aligned}$$

By applying Lemma 1.2 to express p_2 in terms p_1, it follows that

$$\begin{aligned}&\left|a_3^2 - a_2^2\right| \\ =&\left|\frac{(B^4 - 4B^2N + 4N^2)p_1^4}{4B^4C^2} - \frac{p_1^2}{B^2} + \frac{(B^2 - 2N)p_1^2x(4 - p_1^2)}{2B^2C^2} + \frac{x^2(4 - p_1^2)^2}{4C^2}\right|.\end{aligned}$$

For convenience of notation, we choose $p_1 = p$ and since the function p is in the family $\mathcal{P}$ simultaneously, we can suppose without loss of generality that $p \in [0, 2]$. Thus, by applying the triangle inequality with $P = 4 - p^2$, we deduce that

$$\begin{aligned}\left|a_3^2 - a_2^2\right| &\leq \left|\frac{(B^4 - 4B^2N + 4N^2)p^4}{4B^4C^2} - \frac{p^2}{B^2}\right| + \frac{(B^2 - 2N)c^2|x|P}{2B^2C^2} + \frac{|x|^2P^2}{4C^2} \\ &=: F(p, |x|).\end{aligned}$$

It is obvious that $F'(p, |x|) > 0$ on $[0, 1]$ and thus $F(p, |x|) \leq F(p, |1|)$.

Trivially when $p = 2$, we note that the expression $F(|x|)$ has a maximum value on $[0, 2]$. Consequently

$$|T_2(2)| = \left|a_3^2 - a_2^2\right| \leq \frac{4(B^4 - 4B^2N + 4N^2)}{B^4C^2} - \frac{4}{B^2}.$$

This concludes the proof. □

Theorem 2.3. *Let* $f \in \mathcal{L}_\lambda$ *be given by* (1). *Then*

$$|T_2(3)| = |a_4^2 - a_3^2| \leq \frac{4\Omega_1}{9A^2B^6C^2} - \frac{4(B^4 - 4B^2N + 4N^2)}{B^4C^2},$$

where

$$(9)\quad \begin{aligned} \Omega_1 &= 9B^6C^2 - 36B^5CE + 24B^3CM + 36B^4E^2 + 16M^2 - 48B^2EM,\\ \Omega_2 &= 3B^4C^2 - 9B^3CE + 4BCM + 6B^2E^2 - 4EM,\\ \Omega_3 &= 3B^3C - 6B^2E + 4M,\\ \Omega_4 &= B^2C^2 - 2BCE + E^2. \end{aligned}$$

Proof. Applying (6), (7), (2) and using Lemma 1.2, we have

$$\begin{aligned} |a_4^2 - a_3^2| = \Bigg| &\frac{(B^4 - 4B^2N + 4N^2)p_1^4}{4B^4C^2} + \frac{\Omega_1 p_1^6}{144A^2B^6C^2} - \frac{(B^2-2N)p_1^2x(4-p_1^2)}{2B^2C^2}\\ &+ \frac{\Omega_2 p_1^4(4-p_1^2)x}{12A^2B^4C^2} - \frac{\Omega_3 p_1^4(4-p_1^2)x^2}{24A^2B^3C} - \frac{x^2(4-p_1^2)^2}{4C^2}\\ &+ \frac{\Omega_4 p_1^2(4-p_1^2)^2x^2}{4A^2B^2C^2} - \frac{(BC-E)p_1^2(4-p_1^2)^2x^3}{4A^2BC} + \frac{p_1^2(4-p_1^2)^2x^4}{16A^2}\\ &+ \frac{\Omega_3 p_1^3(4-p_1^2)(1-|x|^2)z}{12A^2B^3C} + \frac{(BC-E)p_1(4-p_1^2)^2(1-|x|^2)xz}{2A^2BC}\\ &- \frac{p_1(4-p_1^2)^2(1-|x|^2)x^2z}{4A^2} + \frac{(4-p_1^2)^2(1-|x|^2)^2z^2}{4A^2}\Bigg|. \end{aligned}$$

We select $p_1 = p$ for ease of notation, and because the function p is in the family $\mathcal{P}$ at the same time, we may assume that $p \in [0,2]$ without losing generality. As a result, using the triangle inequality with $P = 4 - r^2$ and $Z = (1 - |x|^2)$, we may conclude

$$\begin{aligned} |a_4^2 - a_3^2| = &\left|\frac{\Omega_1 p^6}{144A^2B^6C^2} - \frac{(B^4 - 4B^2N + 4N^2)p^4}{4B^4C^2}\right| + \frac{(B^2-2N)p^2|x|P}{2B^2C^2}\\ &+ \frac{\Omega_2 p^4P|x|}{12A^2B^4C^2} + \frac{\Omega_3 p^4P|x|^2}{24A^2B^3C} + \frac{|x|^2P^2}{4C^2} + \frac{\Omega_4 p^2P^2|x|^2}{4A^2B^2C^2}\\ &+ \frac{(BC-E)p^2P^2|x|^3}{4A^2BC} + \frac{p^2P^2|x|^4}{16A^2} + \frac{\Omega_3 p^3PZ}{12A^2B^3C}\\ &+ \frac{(BC-E)p|x|P^2Z}{2A^2BC} + \frac{p|x|^2P^2Z}{4A^2} + \frac{P^2Z^2}{4A^2}\\ =:\ &F_1(p, |x|). \end{aligned}$$

Using elementary calculus to differentiate $F_1(p, |x|)$ w.r.t. $|x|$, we have

$$\begin{aligned} &\frac{\partial F_1(p,|x|)}{\partial |x|}\\ &= \frac{(B^2-2N)p^2(4-p^2)}{2B^2C^2} + \frac{\Omega_2 p^4(4-p^2)}{12A^2B^4C^2} - \frac{2\Omega_3 p^3(4-p^2)|x|}{12A^2B^3C}\\ &\quad + \frac{\Omega_3 p^4(4-p^2)|x|}{12A^2B^3C} + \frac{\Omega_4 p^2(4-p^2)^2|x|}{2A^2B^2C^2} - \frac{p(BC-E)(4-p^2)^2|x|^2}{A^2BC} \end{aligned}$$

$$+\frac{3(BC-E)p^2(4-p^2)^2|x|^2}{4A^2BC}-\frac{p(4-p^2)^2|x|^3}{2A^2}+\frac{p^2(4-p^2)^2|x|^3}{4A^2}$$
$$+\frac{(BC-E)p(4-p^2)^2(1-|x|^2)}{2A^2BC}-\frac{|x|(4-p^2)^2(1-|x|^2)}{A^2}$$
$$+\frac{p|x|(4-p^2)^2(1-|x|^2)}{2A^2}.$$

It is shown that $(\partial F_1(p,|x|)/\partial|x|)\geq 0$ for $|x|\in[0,1]$ and fixed $p\in[0,2]$. As a result, $F_1(p,|x|)$ is an increasing function of $|x|$. So, $F_1(p,|x|)\leq F_1(p,|1|)$. Therefore,

$$|a_4^2-a_3^2|\leq\left|\frac{\Omega_1p^6}{144A^2B^6C^2}-\frac{(B^4-4B^2N+4N^2)p^4}{4B^4C^2}\right|+\frac{(B^2-2N)p^2(4-p^2)}{2B^2C^2}$$
$$+\frac{(4-p^2)^2}{4C^2}+\frac{(2\Omega_2+\Omega_3BC)p^4(4-p^2)}{24A^2B^4C^2}$$
$$+\frac{(4\Omega_4+4(BC-E)BC+B^2C^2)p^2(4-p^2)^2}{16A^2B^2C^2}.$$

Now, on $[0,2]$ at $p=2$, we have

$$|a_4^2-a_3^2|\leq\frac{4\Omega_1}{9A^2B^6C^2}-\frac{4(B^4-4B^2N+4N^2)}{B^4C^2}.$$ □

Theorem 2.4. *Let $f\in\mathcal{L}_\lambda$ be given by* (1). *Then*

$$\begin{aligned}|T_3(2)|&=|(a_2-a_4)(a_2^2-2a_3^2+a_2a_4)|\\&\leq\left[\frac{2}{B}-\frac{8(3B^2C-6BE+4M)p^3}{12AB^2C}\right]\left[\frac{4}{B^2}-\frac{4\Omega_5}{3B^4C^2A}\right],\end{aligned}\tag{10}$$

where

$$\begin{aligned}\Omega_5&=6A(B^4-4B^2N+4N^2)-3B^2C(BC-2E)-4CM,\\\Omega_6&=2A(B^2-2N)-BC^2+CE.\end{aligned}\tag{11}$$

Proof. From (5), (7), (2) and applying Lemma 1.2, we have

$$|a_2-a_4|=\left|\frac{p_1}{B}-\frac{p_1^3}{4A}-\frac{p_1(4-p_1^2)x}{2A}+\frac{p_1(4-p_1^2)x^2}{4A}-\frac{(4-p_1^2)(1-|x|^2)z}{2A}\right.$$
$$\left.+\frac{Ep_1^3}{2BCA}+\frac{Ep_1(4-p_1^2)x}{2BCA}-\frac{Mp_1^3}{3B^2CA}\right|.$$

Applying triangle inequality and $p_1=p$, we have

$$|a_2-a_4|\leq\left|\frac{p}{B}-\frac{(3B^2C-6BE+4M)p^3}{12AB^2C}\right|+\frac{p(BC-E)|x|P}{BC}$$
$$+\frac{p|x|^2P}{4A}+\frac{PZ}{2A}+\frac{Ep|x|P}{2BCA}.$$

Using the same methods as Theorem 2.2 and 2.3, we have

$$|a_2 - a_4| \leq \frac{2}{B} - \frac{8(3B^2C - 6BE + 4M)p^3}{12AB^2C}. \tag{12}$$

Also, using (5), (6), (7) and (2), applying Lemma 1.2 and taking $p_1 = p \in [0,2]$, we have

$$\begin{aligned}|a_2^2 - 2a_3^2 + a_2a_4| \leq \left|\frac{p^2}{B^2} - \frac{\Omega_5 p^4}{12B^4C^2A}\right| + \frac{\Omega_6 p^2(4-p^2)|x|}{2B^2C^2A} + \frac{p^2(4-p^2)|x|^2}{4BA} \\ + \frac{(4-p^2)^2|x|^2}{2C^2} + \frac{p(4-p^2)(1-|x|^2)}{2BA} := F_2(p,|x|).\end{aligned}$$

On the closed area $[0,2]\times[0,1]$, we need to find the maximum value of $F_2(p,|x|)$. Assume that a maximum of $[0,2]\times[0,1]$ exists at an interior point $(p_0,|x|)$. After that, by differentiating $F_2(p,|x|)$ w.r.t $|x|$, we have

$$\frac{\partial F_2(p,|x|)}{\partial |x|} = \frac{\Omega_6 p^2(4-p^2)}{2B^2C^2A} + \frac{p^2(4-p^2)|x|}{2BA} + \frac{(4-p^2)^2|x|}{C^2} - \frac{p(4-p^2)|x|)}{BA}.$$

If $p = 0$,

$$F_2(0,|x|) = \frac{8}{C^2}|x|^2 \leq \frac{8}{C^2}.$$

If $p = 2$,

$$F_2(2,|x|) = \frac{4}{B^2} - \frac{4\Omega_5}{3B^4C^2A}.$$

If $|x| = 0$,

$$F_2(p,0) = \left|\frac{p^2}{B^2} - \frac{\Omega_5 p^4}{12B^4C^2A}\right| + \frac{p(4-p^2)}{2BA},$$

which has the highest possible value

$$\frac{4}{B^2} - \frac{4\Omega_5}{3B^4C^2A}$$

on $[0,2]$. Also, if $|x| = 1$, we have

$$F_2(p,1) = \left|\frac{p^2}{B^2} - \frac{\Omega_5 p^4}{12B^4C^2A}\right| + \frac{(2\Omega_6 + BC^2)p^2(4-p^2)}{4B^2C^2A} + \frac{(4-p^2)^2}{2C^2},$$

which has the highest possible value

$$\frac{4}{B^2} - \frac{4\Omega_5}{3B^4C^2A}$$

on $[0,2]$. So,

$$\begin{aligned}|T_3(2)| &= |(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)| \\ &\leq \left[\frac{2}{B} - \frac{8(3B^2C - 6BE + 4M)p^3}{12AB^2C}\right]\left[\frac{4}{B^2} - \frac{4\Omega_5}{3B^4C^2A}\right].\end{aligned}$$

□

Theorem 2.5. *Let $f \in \mathcal{L}_\lambda$ be given by* (1)*. Then*

$$|T_3(1)| = |1 + 2a_2^2(a_3 - 1) - a_3^2| \leq 1 + \frac{4\Omega_7}{B^4C^2} - \frac{8}{B^2}, \tag{13}$$

where

$$\Omega_7 = 4B^3C - 8CN - B^4 + 4B^2N - 4N^2. \tag{14}$$

Proof. From (5), (6), and (2), applying Lemma 1.2 and some calculations, we have

$$|T_3(1)| = \left| 1 + \frac{p_1^4}{BC} + \frac{p_1^2 x(4-p_1^2)}{BC} - \frac{2Np_1^4}{B^4C} - \frac{2p_1^2}{B^2} - \frac{(B^4 - 4B^2N + 4N^2)p_1^4}{4B^4C^2} \right.$$
$$\left. - \frac{(B^2 - 2N)p_1^2 x(4-p_1^2)}{2B^2C^2} - \frac{x^2(4-p_1^2)^2}{4C^2} \right|.$$

We select $p_1 = p$ for ease of notation, and because the function p is in the family $\mathcal{P}$ at the same time, we may assume that $p \in [0, 2]$ without losing generality. As a result, using the triangle inequality with $p = 4 - r^2$, we have

$$|T_3(1)| \leq \left| 1 + \frac{\Omega_7 p^4}{4B^4C^2} - \frac{2p^2}{B^2} \right| + \frac{(B^2 - 2N)p^2(4-p^2)}{2B^2C^2} + \frac{(4-p^2)^2}{4C^2}.$$

Hence, at $p = 2$, we have

$$|T_3(1)| \leq 1 + \frac{4\Omega_7}{B^4C^2} - \frac{8}{B^2}.$$ □

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