Commun. Korean Math. Soc. **39** (2024), No. 3, pp. 643–645
https://doi.org/10.4134/CKMS.c240008
pISSN: 1225-1763 / eISSN: 2234-3024

CORRIGENDUM ON "ORIENTED TRANSFORMATIONS ON A FINITE CHAIN: ANOTHER DESCRIPTION" [COMMUN. KOREAN MATH. SOC. 38 (2023), NO. 3, PP. 725–731]

Vítor H. Fernandes

Abstract. In this note, we aim to correct some of the results presented in [1]. Namely, the statements of Proposition 2.1, Corollary 2.2, Corollary 2.3, Theorem 2.4 and Theorem 2.6, concerning only the monoids $\mathcal{OP}_n$ and $\mathcal{POP}_n$, have to exclude transformations of rank two. All other results of [1], as well as those mentioned above but for the monoids $\mathcal{OR}_n$ and $\mathcal{POR}_n$, do not require correction.

We begin this note by briefly recalling some notions and by establishing some notations. Let n be a positive integer and let $\Omega_n = \{1 < \cdots < n\}$. Denote by $\mathcal{PT}_n$ the monoid (under composition) of all partial transformations on Ω_n, by $\mathcal{T}_n$ the submonoid of $\mathcal{PT}_n$ of all full transformations on Ω_n and by $\mathcal{I}_n$ the inverse submonoid of $\mathcal{PT}_n$ of all partial permutations on Ω_n. Let $s = (a_1, \ldots, a_t)$ be a sequence of t ($t \geqslant 0$) elements from the chain Ω_n. We say that s is *cyclic* [*anti-cyclic*] if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $a_i > a_{i+1}$ [$a_i < a_{i+1}$], where a_{t+1} denotes a_1. Given a partial transformation $\alpha \in \mathcal{PT}_n$ such that $\mathrm{Dom}(\alpha) = \{a_1 < \cdots < a_t\}$, with $t \geqslant 0$, we say that α is *preserves orientation* [*reverses orientation*] if the sequence of its images $(a_1\alpha, \ldots, a_t\alpha)$ is cyclic [anti-cyclic]. We denote by $\mathcal{POP}_n$ the submonoid of $\mathcal{PT}_n$ of all partial transformations that preserve orientation and by $\mathcal{POR}_n$ the submonoid of $\mathcal{PT}_n$ of all partial transformations that preserve or reverse orientation. Let $\mathcal{OP}_n = \mathcal{POP}_n \cap \mathcal{T}_n$, $\mathcal{OR}_n = \mathcal{POR}_n \cap \mathcal{T}_n$, $\mathcal{POPI}_n = \mathcal{POP}_n \cap \mathcal{I}_n$ and $\mathcal{PORI}_n = \mathcal{POR}_n \cap \mathcal{I}_n$.

Concerning the monoids $\mathcal{OP}_n$ and $\mathcal{POP}_n$, Corollary 2.2, Corollary 2.3, Theorem 2.4 and Theorem 2.6 of [1] depend on Proposition 2.1(1) of [1], which is,

Received January 9, 2024; Accepted March 11, 2024.
2020 *Mathematics Subject Classification.* Primary 20M20.
Key words and phrases. Orientation-preserving transformations, orientation-reversing transformations, oriented transformations.
This work is funded by national funds through the FCT - Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 (https://doi.org/10.54499/UIDB/00297/2020) and UIDP/00297/2020 (https://doi.org/10.54499/UIDP/00297/2020) (Center for Mathematics and Applications).

although in other words, a transcription of Theorem 3 of [2]. This last result, in turn, rewrites Proposition 1.1 of [3], which was presented without proof.

Although the statement of Theorem 3 of [2] does not mention any restriction on the rank of the transformation $\alpha \in \mathcal{T}_n$ taken, in the proof the authors considered that α has rank greater than two. In fact, there are transformations of rank two which do not preserve orientation and, simultaneously, which satisfy the condition (for $\mathcal{OP}_n$) of Theorem 3 of [2]. For example, $\alpha = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{smallmatrix}\right)$, with $n = 4$. On the other hand, clearly all transformations of rank one preserve orientation, so Theorem 3 of [2] is only valid for transformations of rank other than two. We should note, however, that the statement of Proposition 1.1 of [3] does not lack the same restriction.

As a consequence of the foregoing, for the monoids $\mathcal{OP}_n$ and $\mathcal{POP}_n$, the statements of the mentioned results of [1] must comply with the same restriction as Theorem 3 of [2], i.e. they should be stated as follows.

Proposition 2.1. (1) *Let $\alpha \in \mathcal{T}_n$ be such that $|\operatorname{Im}(\alpha)| \neq 2$. Then, $\alpha \in \mathcal{OP}_n$ if and only if, for every triple (a_1, a_2, a_3) of elements of Ω_n, (a_1, a_2, a_3) and $(a_1\alpha, a_2\alpha, a_3\alpha)$ are both cyclic or both anti-cyclic.*

Corollary 2.2. *Let $\alpha \in \mathcal{T}_n$ be such that $|\operatorname{Im}(\alpha)| \neq 2$. Then, $\alpha \in \mathcal{OP}_n$ if and only if, for every cyclic triple (a_1, a_2, a_3) of elements of Ω_n, $(a_1\alpha, a_2\alpha, a_3\alpha)$ is also cyclic.*

Corollary 2.3. *Let $\alpha \in \mathcal{T}_n$ be such that $|\operatorname{Im}(\alpha)| \neq 2$. Then, $\alpha \in \mathcal{OP}_n$ if and only if, for every non-decreasing triple (a_1, a_2, a_3) of elements of Ω_n, the triple $(a_1\alpha, a_2\alpha, a_3\alpha)$ is cyclic.*

Recall that the *width* of a partial transformation is the number of elements in its domain.

Theorem 2.4. *Let $\alpha \in \mathcal{T}_n$ be such that $|\operatorname{Im}(\alpha)| \neq 2$. Then, $\alpha \in \mathcal{OP}_n$ if and only if every restriction of α of width three belongs to $\mathcal{POP}_n$.*

Theorem 2.6. *Let $\alpha \in \mathcal{PT}_n$ be such that $|\operatorname{Im}(\alpha)| \neq 2$. Then, $\alpha \in \mathcal{POP}_n$ if and only if every restriction of α of width three belongs to $\mathcal{POP}_n$.*

Now, let $\alpha \in \mathcal{PT}_n$ be such that $|\operatorname{Im}(\alpha)| = 2$. Suppose that $\operatorname{Dom}(\alpha) = \{i_1 < i_2 < \cdots < i_k\}$ for some $2 \leqslant k \leqslant n$. Then, it is clear that $\alpha \in \mathcal{POP}_n$ if and only if α admits as kernel classes $\{i_r, i_{r+1}, \ldots, i_s\}$ and $\{i_1, \ldots, i_{r-1}, i_{s+1}, \ldots, i_k\}$ for some $1 \leqslant r \leqslant s \leqslant k$ (and $s - r + 1 < k$).

We finish this note by observing that Theorem 2.7 of [1] does not require the same restriction (even for $\mathcal{POPI}_n$), since all partial permuations of rank two preserve orientation.

References

[1] V. H. Fernandes, *Oriented transformations on a finite chain: another description*, Commun. Korean Math. Soc. **38** (2023), no. 3, 725–731. https://doi.org/10.4134/CKMS.c220272

[2] P. M. Higgins and A. Vernitski, *Orientation-preserving and orientation-reversing mappings: a new description*, Semigroup Forum **104** (2022), no. 2, 509–514. https://doi.org/10.1007/s00233-022-10256-8

[3] I. Levi and J. D. Mitchell, *On rank properties of endomorphisms of finite circular orders*, Comm. Algebra **34** (2006), no. 4, 1237–1250. https://doi.org/10.1080/00927870500454091

Vítor H. Fernandes
Center for Mathematics and Applications (NOVA Math) and
Department of Mathematics
Faculdade de Ciências e Tecnologia
Universidade Nova de Lisboa
Monte da Caparica, 2829-516 Caparica, Portugal
Email address: vhf@fct.unl.pt