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ON NONNIL-m-FORMALLY NOETHERIAN RINGS

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Abstract. The purpose of this paper is to introduce a new class of rings containing the class of m-formally Noetherian rings and contained in the class of nonnil-SFT rings introduced and investigated by Benhissi and Dabbabi in 2023 [4]. Let A be a commutative ring with a unit. The ring A is said to be nonnil-m-formally Noetherian, where $m \geq 1$ is an integer, if for each increasing sequence of nonnil ideals $(I_n)_{n\geq 0}$ of A the (increasing) sequence $(\sum_{i_1+\cdots+i_m=n} I_{i_1}I_{i_2}\cdots I_{i_m})_{n\geq 0}$ is stationnary. We investigate the nonnil-m-formally Noetherian variant of some well known theorems on Noetherian and m-formally Noetherian rings. Also we study the transfer of this property to the trivial extension and the amalgamation algebra along an ideal. Among other results, it is shown that A is a nonnil-m-formally Noetherian ring if and only if the m-power of each nonnil radical ideal is finitely generated. Also, we prove that a flat overring of a nonnil-m-formally Noetherian ring is a nonnil-m-formally Noetherian. In addition, several characterizations are given. We establish some other results concerning m-formally Noetherian rings.

1. Introduction

In this paper, all rings are commutative with an identity element and the dimension of a ring means its Krull dimension. Let A be a ring. We shall denote by $Nil(A)$ the nilradical of A and $I \subset J$ means I is strictly contained in J for some sets I, J. In [11], Khalifa has introduced the concept of m-formally Noetherian ring. A ring A is called an m-formally Noetherian ring if the m-power of each ideal of A is finitely generated. The class of m-formally Noetherian rings contains the class of Noetherian rings and it is contained in the class of SFT rings. So each m-formally Noetherian ring has a Noetherian spectrum. Moreover, in a reduced ring the concepts of m-formally Noetherian and Noetherian coincide. For more results, see [11].

In [3], Badawi generalized the concept of Noetherian rings by introducing a new class of rings, namely, the nonnil-Noetherian rings. A ring A is called nonnil-Noetherian if each nonnil ideal of A is finitely generated. It is obvious that Noetherian rings are both m-formally Noetherian and nonnil-Noetherian

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but the converse is not true. This concept inspired several authors, for example see [4], [5] and [12]. Now, it is natural to investigate the relation between nonnil-Noetherian rings and m-formally Noetherian rings (see [11], Theorem 2.21). For instance, let $A = (K[X_i,\ i \geq 1])/\langle X_i^i,\ i \geq 1\rangle$, where K is a field and $X_1, X_2, \ldots$ is a countably family of indeterminates. Then A is nonnil-Noetherian but not m-formally Noetherian. Indeed, the m-power of the ideal of A generated by $\{\bar{X}_i,\ i \geq 1\}$ is not finitely generated for every $m \geq 1$ (see [11], Example 2.22).

The main purpose of this paper is to integrate the concepts of nonnil-Noetherian rings and m-formally Noetherian rings, and so to construct a new class of rings that contains the two previous classes. For this, we introduce the concept of nonnil-m-formally Noetherian rings as follows: Let A be a ring, I an ideal of A and $m \geq 1$ an integer. An ideal I of A is said to be a nonnil ideal if it is not contained in the nilradical of A (i.e. $I \nsubseteq Nil(A)$). The ring A is called nonnil-m-formally Noetherian if for each increasing sequence of nonnil-ideals $(I_n)_{n\geq 0}$ of A the (increasing) sequence $(\sum_{i_1+\cdots+i_m=n} I_{i_1}I_{i_2}\cdots I_{i_m})_{n\geq 0}$ is stationary. We start by showing that A is nonnil-m-formally Noetherian ring if and only if the m-power of each nonnil ideal of A is finitely generated. If $Nil(A) = 0$, then the notion of nonnil-m-formally Noetherian rings coincides with that of m-formally Noetherian rings. Clearly any nonnil-Noetherian ring is a nonnil-m-formally Noetherian ring.

Among other characterizations of nonnil-m-formally Noetherian rings, we show that if the ring A is in the class of the commutative ring with divided prime nilradical ideal (denoted $\mathcal{H}$), then A is a nonnil-m-formally Noetherian ring if and only if $A/Nil(A)$ is an m-formally Noetherian ring. Also, we study the transfer of this property to the trivial extension, the amalgamated algebra along an ideal and flat overrings. For instance, we prove that a flat overring of a nonnil-m-formally Noetherian ring is again a nonnil-m-formally Noetherian ring. On the other hand, we give a complete characterization of a decomposable ring to be nonnil-m-formally Noetherian and so we conclude a characterization of the product ring to be nonnil-m-formally Noetherian. Some other properties of m-formally Noetherian rings are also established.

2. Main results

Definition 2.1. Let A be a ring and $m \geq 1$ an integer. The ring A is said to be nonnil-m-formally Noetherian if for each increasing sequence of nonnil-ideals $(I_n)_{n\geq 0}$ of A, the (increasing) sequence $(\sum_{i_1+\cdots+i_m=n} I_{i_1}I_{i_2}\cdots I_{i_m})_{n\geq 0}$ is stationnary.

Clearly nonnil-Noetherian rings are nonnil-m-formally Noetherian rings.

Example 2.2. Let A be a ring and $m \geq 1$ an integer. If A is m-formally Noetherian, then A is nonnil-m-formally Noetherian but the converse is not true. Indeed, let A be a nonnil-Noetherian ring which is not m-formally Noetherian for every $m \geq 1$ (for example $K[X_n,\ n \geq 2]/\langle X_n^n,\ n \geq 2\rangle$). Since A is a nonnil-Noetherian, it is nonnil-m-formally Noetherian but A is not m-formally Noetherian for every $m \geq 1$.

Lemma 2.3 ([9], Lemma 1.2.). *Let A be a ring and I an ideal of A. If there exists an integer $n \geq 1$ such that I^n is finitely generated, then there exists a finitely generated ideal $J \subseteq I$ such that $I^n = J^n$.*

Theorem 2.4. *Let A be a ring and $m \geq 1$ an integer. The following conditions are equivalent*:

(1) *The ring A is nonnil-m-formally Noetherian.*
(2) *For each increasing sequence of nonnil ideals $(I_n)_{n\geq 0}$ of A, there exist $k_1, k_2, \dots, k_m \in \mathbb{N}$ such that for each $n_1, n_2, \dots, n_m \in \mathbb{N}$ satisfying $n_i \geq k_i$ for $i = 1, \dots, m$, we have $I_{n_1} \cdots I_{n_m} \subseteq I_{k_1} \cdots I_{k_m}$.*
(3) *For each increasing sequence of nonnil-ideals $(I_n)_{n\geq 0}$ of A, there exist $k_1, k_2, \dots, k_m \in \mathbb{N}$ such that for every integer $n \geq \max(k_1, \dots, k_m)$, $I_n^m \subseteq I_{k_1} \cdots I_{k_m}$.*
(4) *For each increasing sequence of nonnil ideals $(I_n)_{n\geq 0}$ of A, the (increasing) sequence $(I_n^m)_{n\geq 0}$ is stationnary.*
(5) *For each increasing sequence of finitely generated nonnil ideals $(F_n)_{n\geq 0}$ of A the (increasing) sequence $(\sum_{i_1+\cdots+i_m=n} F_{i_1} F_{i_2} \cdots F_{i_m})_{n\geq 0}$ is stationnary.*
(6) *For each increasing sequence of finitely generated nonnil ideals $(F_n)_{n\geq 0}$ of A, there exist $k_1, k_2, \dots, k_m \in \mathbb{N}$ such that for each $n_1, n_2, \dots, n_m \in \mathbb{N}$ satisfying $n_i \geq k_i$ for $i = 1, \dots, m$, we have $F_{n_1} \cdots F_{n_m} \subseteq F_{k_1} \cdots F_{k_m}$.*
(7) *For each increasing sequence of finitely generated nonnil ideals $(F_n)_{n\geq 0}$ of A, there exist $k_1, k_2, \dots, k_m \in \mathbb{N}$ such that for every integer $n \geq \max(k_1, \dots, k_m)$, $F_n^m \subseteq F_{k_1} \cdots F_{k_m}$.*
(8) *For each increasing sequence of finitely generated nonnil ideals $(F_n)_{n\geq 0}$ of A, the (increasing) sequence $(F_n^m)_{n\geq 0}$ is stationnary.*
(9) *For each nonnil ideal I of A, there exists a finitely generated ideal $F \subseteq I$ of A such that $I^m = F^m$.*
(10) *For each nonnil ideal I of A, I^m is finitely generated.*

Proof. For each $n \geq 0$, set $S_n = \sum_{i_1+\cdots+i_m=n} I_{i_1} I_{i_2} \cdots I_{i_m}$ and $S'_n = \sum_{i_1+\cdots+i_m=n} F_{i_1} F_{i_2} \cdots F_{i_m}$.

(1) $\Rightarrow$ (2) Let $k \geq 1$ be an integer such that for every $n \geq k$, $S_n = S_k$ (since A is nonnil-m-formally Noetherian). For each $1 \leq j \leq m$, set $k_j = \max\{i_j \in \mathbb{N} \mid \text{ there exist } i_1 + \cdots + i_m = k \text{ with } i_1 \leq \cdots \leq i_{j-1} \leq i_j \leq i_{j+1} \leq \cdots \leq i_m\}$. Now, let $i_1, \dots, i_m \in \mathbb{N}$ be such that $i_1 + \cdots + i_m = k$, with $i_1 \leq \cdots \leq i_m$. For every $1 \leq j \leq m$, we have $i_j \leq k_j$. Since $I_{i_1} \cdots I_{i_m} \subseteq I_{k_1} \cdots I_{k_m}$, so $S_k \subseteq I_{k_1} \cdots I_{k_m}$. Thus for every $n_1 \geq k_1, \dots, n_m \geq k_m$, $I_{n_1} \cdots I_{n_m} \subseteq S_{n_1+\cdots+n_m} = S_k \subseteq I_{k_1} \cdots I_{k_m}$.

(2) $\Rightarrow$ (3) Trivial. (3) $\Rightarrow$ (4) Let $k = \max(k_1, \dots, k_m)$. Then for every $n \geq k$, $I_n^m \subseteq I_{k_1} \cdots I_{k_m} \subseteq I_k^m$.

(4) $\Rightarrow$ (5) Let k be an integer such that $F_n^m = F_k^m$ for every $n \geq k$. Then for each $n \geq k$, $S'_{nm} \subseteq F_n^m = F_k^m \subseteq S'_{km} \subseteq S'_{nm}$.

(5) $\Rightarrow$ (6) By the same way as (1) $\Rightarrow$ (2).

(6) $\Rightarrow$ (7) By the same way as (2) $\Rightarrow$ (3).

(7) $\Rightarrow$ (8) By the same way as (3) $\Rightarrow$ (4).

(8) $\Rightarrow$ (9) Assume that there exists a nonnil ideal I such that for every finitely generated ideal $F \subseteq I$ of A, $I^m \neq F^m$. Then $I \neq (0)$. Take $a \in I \setminus (0)$. Since $I^m \neq \langle a \rangle^m$, there exist $a_{1,1}, \ldots, a_{1,m} \in I$ such that $a_{1,1} \cdots a_{1,m} \notin \langle a \rangle^m$. Again $I^m \neq \langle a, a_{1,1}, \ldots, a_{1,m} \rangle$, then there exist $a_{2,1}, \ldots, a_{2,m} \in I$ such that $a_{2,1}, \ldots, a_{2,m} \notin \langle a, a_{1,1}, \ldots, a_{1,m} \rangle$. By induction there exist m sequences $(a_{i,1})_{i \geq 0}, \ldots, (a_{i,m})_{i \geq 0}$ of elements of I such that $a_{k+1,1} \cdots a_{k+1,m} \notin F_k^m$ for each $k \geq 0$, where $F_k = \langle a, a_{1,1}, \ldots, a_{1,m}, \ldots, a_{k,1}, \ldots, a_{k,m} \rangle$, contradiction.

(9) $\Rightarrow$ (10) Clear. (10) $\Rightarrow$ (1) Let $(I_n)_{n \geq 0}$ be an increasing sequence of nonnil ideals of A, and $I = \bigcup_{n \geq 0} I_n$. Since the sequence $(I_n)_{n \geq 0}$ is increasing, then I is an ideal of A. It is clear that I is a nonnil ideal. Thus I^m is finitely generated. By Lemma 2.3, there exists a finitely generated ideal $F \subseteq I$ such that $I^m = F^m$. Let $k \geq 1$ be such that $F \subseteq I_k$ (since F is finitely generated). Hence for each $i_1, \ldots, i_m \in \mathbb{N}$, $I_{i_1} \ldots I_{i_m} \subseteq I^m = F^m \subseteq I_k^m \subseteq S_{km}$. □

Corollary 2.5. *Let $f : A \longrightarrow B$ be a surjective homomorphism of rings and $m \geq 1$ be an integer. If A is a nonnil-m-formally Noetherian ring, so is B.*

Proof. Let J be a nonnil ideal of B and $I = f^{-1}(J)$. Clearly I is a nonnil ideal, so there exists a finitely generated ideal $F \subseteq I$ such that $I^m = F^m$ by Theorem 2.4. Hence $J^m = f(I^m) = f(F^m) = f(F)^m$ with $f(F)$ a finitely generated ideal of B (since f is surjective). Thus J^m is finitely generated, and so B is nonnil-m-formally Noetherian. □

Example 2.6. Let A be a nonnil-m-formally Noetherian ring. Then for each ideal I of A, the ring A/I is nonnil-m-formally Noetherian.

Lemma 2.7 ([9], Lemmas 1.1 and 1.2)**.** *Let A be a ring, I an ideal of A and $m \geq 1$ an integer. If I^m is finitely generated, so is I^{m+1}.*

Corollary 2.8. *If a ring A is nonnil-m-formally Noetherian for some $m \geq 1$, then it is nonnil-$(m+1)$-formally Noetherian.*

Theorem 2.9. *Let $A \in \mathcal{H}$. The following statements are equivalent:*

(1) *A is a nonnil-m-formally Noetherian ring.*
(2) *$A/Nil(A)$ is an m-formally Noetherian ring.*

Proof. (1) $\Rightarrow$ (2) Let J be a nonzero ideal of $A/Nil(A)$. Then $J = I/Nil(A)$, where $Nil(A) \subset I$. Since I is a nonnil ideal of A, I^m is a finitely generated ideal of A. Thus, $J^m = (I/Nil(A))^m = (I^m + Nil(A))/Nil(A)$ is a finitely generated ideal of $A/Nil(A)$.

(2) $\Rightarrow$ (1) Let I be a nonnil ideal of A. Since $Nil(A)$ is a divided ideal, $Nil(A) \subset I$. Then $I/Nil(A)$ is a nonzero ideal of $A/Nil(A)$, hence, $(I/Nil(A))^m$

is a finitely generated ideal of $A/Nil(A)$. Suppose that $I^m \subseteq Nil(A)$. Then $I^m \subseteq Nil(A) \subset I$. It follows that $\sqrt{I} = Nil(A)$, and so $I \subseteq Nil(A)$, absurd. Hence, $Nil(A) \subset I^m$, and $(I/Nil(A))^m = I^m/Nil(A)$. Therefore, $I^m/Nil(A) = (\overline{i_1}, \ldots, \overline{i_n})$ for some $i_1, \ldots, i_n \in I^m$. We will show that $I^m = (i_1, \ldots, i_n)$. Since $Nil(A)$ is a divided ideal, we have $Nil(A) \subset (i_1, \ldots, i_n)$. Indeed, suppose that $(i_1, \ldots, i_n) \subseteq Nil(A)$. Then $I^m/Nil(A) = (0)$, and $I^m = Nil(A)$, it follows that $Nil(A) = \sqrt{I^m} = \sqrt{I}$. Thus, $I \subseteq Nil(A)$, absurd. Now, let $x \in I^m \setminus Nil(A)$. There exist $w \in Nil(A)$ and $a_1, \ldots, a_n \in A$ such that $x + w = i_1 a_1 + \cdots + i_n a_n$. Since $x \notin Nil(A)$, clearly $Nil(A) \subset (x)$. Therefore $w = xy$ for some $y \in A$. On the other hand, $Nil(A)$ is prime ideal. It yields that $y \in Nil(A)$ and $1 + y$ is a unit of A. Thus, $x + w = x + xy = x(1 + y) \in (i_1, \ldots, i_n)$. Finally, $x \in (i_1, \ldots, i_n)$. Consequently, A is a nonnil-m-formally Noetherian ring. □

Using the previous theorem, ([11], Corollary 2.7) and ([3], Theorem 2.2), we have the following corollary.

Corollary 2.10. *Let $A \in \mathcal{H}$ and $m \geq 1$ an integer. The following statements are equivalent*:

1. *A is a nonnil-m-formally Noetherian ring.*
2. *$A/Nil(A)$ is an m-formally Noetherian ring.*
3. *$A/Nil(A)$ is a Noetherian ring.*
4. *A is a nonnil-Noetherian ring.*

Remark 2.11. In general (without the condition $A \in \mathcal{H}$), if the ring A is a nonnil-m-formally Noetherian ring, then $A/Nil(A)$ is an m-formally Noetherian, and so a Noetherian ring.

Theorem 2.12. *A ring A is nonnil-m-formally Noetherian if and only if the m-power of every nonnil radical ideal of A is finitely generated.*

Proof. Let I be a nonnil ideal of A. Let $\mathcal{P}$ be a prime ideal of A/I. Then some power of $\mathcal{P}$ is finitely generated. Indeed, $\mathcal{P} = P/I$ for some prime ideal P of A containing I. Then P is a nonnil radical ideal, thus P^m is finitely generated, and so is $\mathcal{P}^m$. Therefore, every prime ideal of A/I has a power, which is finitely generated. Using ([9], Proposition 1.18), $(A/I)/(\sqrt{I}/I) \simeq A/\sqrt{I}$ is a Noetherian ring and $(\sqrt{I}/I)^k = (0)$ for some $k \geq 1$. Thus $(\sqrt{I})^k \subseteq I$, and so $(\sqrt{I})^{km} \subseteq I^m \subseteq (\sqrt{I})^m$. Now, by ([11], Lemma 2.15) I^m is finitely generated (because $\sqrt{I}$ is nonnil). □

Corollary 2.13. *A nonnil-m-formally Noetherian ring such that the m-power of its nilradical is finitely generated is an m-formally Noetherian ring.*

Proposition 2.14. *If A is a nonnil-m-formally Noetherian ring, then the following assertions are equivalent*:

1. *A is a Noetherian ring.*
2. *$Nil(A)$ is finitely generated.*

3. *Each minimal prime ideal of A is finitely generated.*
4. *The A-module $Nil(A)/Nil(A)^2$ is finitely generated.*

Proof. If A is a nonnil-m-formally Noetherian ring, then $A/Nil(A)$ is Noetherian. Now, use ([9], Proposition 1.18). □

Remark 2.15. Since every nonnil-Noetherian ring is a nonnil-m-formally Noetherian ring, the above result remains true in the case of nonnil-Noetherian ring.

According to [4] a ring A is called a nonnil-SFT if each nonnil ideal I of A is SFT, i.e, there exist a finitely generated ideal $F \subseteq I$ of A and an integer $k \geq 1$ such that $x^k \in F$ for every $x \in I$. In the following, we will establish the relationship between the nonnil-SFT and the nonnil-m-formally Noetherian concepts.

Proposition 2.16. *Let A be a chained ring. The following statements are equivalent*:

1. *A is nonnil-m-formally Noetherian for some m.*
2. *A is an nonnil-SFT ring with Krull dimension ≤ 1.*

Proof. It is clear that $A \in \mathcal{H}$. Now, using ([11], Corollary 2.23) and ([4], Proposition 1.4), we have: A is a nonnil-m-formally Noetherian ring if and only if $A/Nil(A)$ is an m-formally Noetherian ring if and only if $A/Nil(A)$ is an SFT ring with Krull dimension ≤ 1 if and only if A is a nonnil-SFT ring with Krull dimension ≤ 1. □

Proposition 2.17. *If A is a nonnil-m-formally Noetherian ring, then the complete integral closure and the integral closure of A coincides.*

Proof. Let x be an element of the total quotient ring of A such that x is almost integral over A. Then there exist a regular element r of A such that $rx^n \in A$ for all nonzero integer n. Consider, for each n, $I_n = (r, rx, \ldots, rx^n)$. Since $r \in I_n$, I_n is a nonnil ideal of A. Clearly $I_n \subseteq I_{n+1}$. By hypotheses, $I_n^m = I_{n+1}^m$ for some integer n. Hence, $(rx^{n+1})^m \in I^m$. Since r is a regular element of A, we have $(x^{(n+1)m}) \in (1, x, \ldots, x^{nm})$. Consequently, x is integral over A. □

Recall that a ring A is said to have a Noetherian spectrum (or $Spec(A)$ is Noetherian) if each prime ideal is the radical of a finitely generated ideal, equivalently, each radical ideal is the radical of a finitely generated ideal, that is also equivalent to the fact that the ring A satisfies the ascending chain condition on the radical ideals. It is well-known that if A is a Noetherian ring, then $Spec(A)$ is Noetherian. We have the following similar result.

Proposition 2.18. *If A is a nonnil-m-formally Noetherian ring, then $Spec(A)$ is Noetherian.*

Proof. It suffices to show that each prime ideal is the radical of a finitely generated ideal. Let $P \in Spec(A)$. If $P = Nil(A)$, then $P = \sqrt{(0)}$. If $P \nsubseteq Nil(A)$, then by hypothesis, $P^m = I$ is a finitely generated ideal. Thus, $P = \sqrt{P^m} = \sqrt{I}$. □

Example 2.19. The converse of the previous proposition is false. Indeed, let V be a finite dimensional non-SFT valuation domain. Since V has only finite number of prime ideals, it has a Noetherian spectrum. But V is not SFT, hence it is not nonnil-SFT (because it is an integral domain). Thus, V is not nonnil-m-formally Noetherian.

Corollary 2.20. *If A is a nonnil-m-formally Noetherian ring, then each ideal of A has a finitely many minimal primes.*

Corollary 2.21. *If A is an m-formally Noetherian ring, then $Spec(A)$ is Noetherian.*

It is known that if $P \subset Q$ are prime ideals in a Noetherian ring such that there exists a prime ideal properly between them, then there are infinitely many. The following result is a generalization of the previous result.

Proposition 2.22. *Let $A \in \mathcal{H}$ be a nonnil-m-formally Noetherian ring, and let $P \subset Q$ be prime ideals in A such that there exists a prime ideal properly between them. Then there are infinitely many.*

Proof. Using ([3], Theorem 2.10) and Corollary 2.10. □

Let A be a ring and M an A-module. The set $A \times M$ endowed with the operations defined by:

$$(a, m) + (a', m') = (a + a', m + m') \text{ and } (a, m)(a', m') = (aa', am' + a'm)$$

with $a, a' \in A$ and $m, m' \in M$, is a commutative ring with identity, denoted by $A(+)M$. This ring, called the idealization of M in A, was first introduced by Nagata, and massively used in the construction of the counterexample. The reader is referred to [1, 10] for more information about the rings $A(+)M$.

Proposition 2.23. *Let A be a ring and M an A-module. The ring $A(+)M$ is nonnil-m-formally Noetherian if and only if A is nonnil-m-formally Noetherian ring and the A-module $I^{m-1}M/I^mM$ is finitely generated for every nonnil radical ideal I of A.*

Proof. ($\Rightarrow$) Since $(A(+)M)/((0)(+)M) \simeq A$, by Corollary 2.5, A is a nonnil-m-formally Noetherian ring. For the second part, let I be a nonnil radical ideal of A, then $(I(+)M)^m = I^m(+)I^{m-1}M$ is a nonnil radical ideal of $A(+)M$. Thus, $I^{m-1}M = H + I^mM$ for some finitely generated submodule H of M. Hence, $I^{m-1}M/I^mM$ is finitely generated.

($\Leftarrow$) Let $\mathcal{I}$ be a nonnil radical ideal of $A(+)M$. Then $\mathcal{I} = I(+)M$ for some nonnil radical ideal I of A. Thus, I^m is a finitely generated ideal of A and the A-module $I^{m-1}M/I^mM$ is finitely generated. Then, $I^{m-1}M = H + I^mM$

for some finitely generated submodule H of M. Consequently, $(I(+)M)^m = I^m(+)I^{m-1}M$ is finitely generated ideals of $A(+)M$. $\square$

Corollary 2.24. *Let A be a ring and $m \geq 2$ an integer. Then $A(+)A$ is nonnil-m-formally Noetherian if and only if A is nonnil-$(m-1)$-formally Noetherian.*

Now, we deal with a subring of the product ring $A \times B$, where A and B are two rings, denoted by $A \bowtie^f J$. Let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. We consider:

$$A \bowtie^f J = \{(a, f(a) + j);\ a \in A \text{ and } j \in J\}.$$

This subring, called the amalgamation of A with B along J with respect to f, was introduced and studied by D'Anna, Finocchiaro, and Fontana in [6]. This construction is a generalization of the amalgamated duplication of a ring along an ideal that was introduced and studied in [7, 8] (the amalgamated duplication of a ring A along an ideal I is the amalgamation of A with A along I with respect to $f = id_A$). In the following, we will see when the ring $A \bowtie^f J$ is nonnil-m-formally Noetherian.

Theorem 2.25. *Let A and B be two rings, $J \neq \{0\}$ be an ideal of B and $f : A \longrightarrow B$ be a ring homomorphism. Assume that $A \bowtie^f J \in \mathcal{H}$. Then, the ring $A \bowtie^f J$ is nonnil-m-formally Noetherian if and only if the rings A and $f(A) + J$ are nonnil-m-formally Noetherian.*

Proof. " $\Rightarrow$ " It follows from Corollary 2.5, since $(A \bowtie^f J)/(\{0\} \times J) \simeq A$ and $(A \bowtie^f J)/(f^{-1}(J) \times \{0\}) \simeq f(A) + J$.

" $\Leftarrow$ " Consider $\bar{A} = A/\text{Nil}(A)$, $\bar{B} = B/\text{Nil}(B)$ and $\bar{J} = \pi(J)$, where $\pi : B \longrightarrow \bar{B}$ is the canonical epimorphism. We consider the map $\bar{f} : \bar{A} \longrightarrow \bar{B}$ defined by $\bar{f}(\bar{a}) = \overline{f(a)}$. It is clear that $\bar{f}$ is a ring homomorphism.

Now, let $\Psi : f(A) + J \longrightarrow \bar{f}(\bar{A}) + \bar{J}$ be the map defined by $\Psi(f(x) + j) = \bar{f}(\bar{x}) + \bar{j}$. By ([4], Remark 2.11), $f^{-1}(J) \subseteq Nil(A)$, then Ψ is well defined and is a ring homomorphism as the restriction of the canonical surjection from $B \longrightarrow \bar{B}$. Let $\bar{x} \in \bar{f}^{-1}(\bar{J})$. We have $\overline{f(x)} = \bar{f}(\bar{x}) \in \bar{J}$. Then there exists $j \in J$ such that $f(x) - j \in Nil(B)$. So $(f(x) - j)^k = 0$ for some $k \geq 1$. It follows that $f(x^k) \in J$. Thus $x \in \text{Nil}(A)$. It shows that $\bar{x} = \bar{0}$, and hence $\bar{f}(\bar{A}) \bigcap \bar{J} = \{\bar{0}\}$. Now, let $f(x) + j \in ker(\Psi)$. Then $\overline{f(x)} + \bar{j} = \bar{0}$. It yields that $\overline{f(x)} = \bar{0}$ and $\bar{j} = \bar{0}$, which implies that $f(x), j \in \text{Nil}(B)$. Hence $f(x) + j \in \text{Nil}(B) \cap (f(A) + J) = \text{Nil}(f(A)+J)$. Consequently, $ker(\Psi) \subseteq \text{Nil}(f(A)+J)$. The other inclusion is easy. Hence, $(f(A)+J)/\text{Nil}(f(A)+J) \simeq \bar{f}(\bar{A})+\bar{J}$. By Theorem 2.10, $\overline{A}$ and $\bar{f}(\bar{A}) + \bar{J}$ are Noetherian rings. Hence, using ([6], Proposition 5.6), $\bar{A} \bowtie^{\bar{f}} \bar{J}$ is a Noetherian ring. By ([13]. Remark 2.6), $(A \bowtie^f J)/\text{Nil}(A \bowtie^f J) \simeq \bar{A} \bowtie^{\bar{f}} \bar{J}$. Consequently, by Corollary 2.10, $A \bowtie^f J$ is a nonnil-m-formally Noetherian ring. $\square$

Example 2.26. Let A be a ring and I an ideal of A. If $A \bowtie I \in \mathcal{H}$, then:

$A \bowtie I$ is a nonnil-m-formally Noetherian if and only if so is A

In ([13], Corollary 2.3) it was shown that $A \in \mathcal{H}$ if and only if $A \bowtie \mathrm{Nil}(A) \in \mathcal{H}$. Then, we get immediately the following corollary.

Corollary 2.27. *Let $A \in \mathcal{H}$ be a ring. Then the ring $A \bowtie \mathrm{Nil}(A)$ is nonnil-m-formally Noetherian if and only if the ring A is nonnil-m-formally Noetherian.*

A ring A is called decomposable if it can be written in the form $A = A_1 \oplus A_2$ where A_1 and A_2 are two nonzero rings. The decomposition of A is not unique and for each decomposition $A = A_1 \oplus A_2$, we define the two following projections, $\pi_1 : A \longrightarrow A_1$ and $\pi_2 : A \longrightarrow A_2$ by $\pi_1(x) = x_1$ and $\pi_2(x) = x_2$ for each $x = x_1 + x_2 \in A$. It is clear that $A_1 = \pi_1(A)$, $A_2 = \pi_2(A)$ and $A = \pi_1(A) \oplus \pi_2(A)$. Therefore, we can describe the set of rings of the decomposition of A by their associated projections, i.e, a family $\{\pi_i \mid i \in \Lambda\}$ of epimorphisms from A in $\pi_i(A)$ with $\pi_i(A) \neq \{0\}$ for every $i \in \Lambda$, and for each $i \in \Lambda$, there exists $j \in \Lambda$ such that $A = \pi_i(A) \oplus \pi_j(A)$.

Theorem 2.28. *Let A be a decomposable ring and $\{\pi_i \mid i \in \Lambda\}$ the set of canonical epimorphisms from A to each component of a decomposition of A. The following statements are equivalent*:

(1) *The ring A is m-formally Noetherian for some $m \geq 1$.*
(2) *The ring A is nonnil-m-formally Noetherian for some $m \geq 1$.*
(3) *For each $i \in \Lambda$, the ring $\pi_i(A)$ is m-formally Noetherian for some $m \geq 1$.*
(4) *There exists $m \geq 1$ such that the following condition holds. If $e \in A \setminus \{0,1\}$ is an idempotent element, then the m-power of each ideal of A contained in $\langle e \rangle$ is finitely generated.*

Proof. (1) $\Rightarrow$ (2) Trivial. (2) $\Rightarrow$ (3) Let $i \in \Lambda$. Then $A = \pi_i(A) \oplus \pi_j(A)$ for some $j \in \Lambda$. Let I be an ideal of $\pi_i(A)$. We have $I \oplus \pi_j(A)$ is a nonnil ideal of A. It follows that the m power of $I \oplus \pi_j(A)$ is finitely generated. Thus there exists a finitely generated ideal $F \subseteq I \oplus \pi_j(A)$ of A such that $(I \oplus \pi_j(A))^m = F$. Therefore $I^m = \pi_i(F)$, with $\pi_i(F)$ is a finitely generated ideal of $\pi_i(A)$. Thus $\pi_i(A)$ is m-formally Noetherian.

(3) $\Rightarrow$ (4) Let $e \in A \setminus \{0,1\}$ be an idempotent element and I be an ideal of A contained in $\langle e \rangle$. We have $A = \langle e \rangle \oplus \langle 1-e \rangle$. By hypothesis, the ring $\langle e \rangle$ is m-formally Noetherian. Thus there exists a finitely generated ideal $F \subseteq I$ of $\langle e \rangle$ such that $I^m = F$. Since the ideal $F \oplus \{0\}$ of A is finitely generated, the ideal I^m of A is finitely generated, which complete the proof.

(4) $\Rightarrow$ (1) Let I be an ideal of A. Since A is decomposable, $A = \langle e \rangle \oplus \langle 1-e \rangle$ for some idempotent element $e \in A \setminus \{0,1\}$. It yields that $I = I_e \oplus I_{1-e}$ where I_e and I_{1-e} are two ideals of $\langle e \rangle$ and $\langle 1-e \rangle$ respectively. There exist $i, j \in \Lambda$ such that $\langle e \rangle = \pi_i(A)$ and $\langle 1-e \rangle = \pi_j(A)$. Consequently, there exist finitely generated ideals $E \subseteq I_e$ and $F \subseteq I_{1-e}$ of A such that $I_e^m = E$ and $I_{1-e}^m = F$. Hence $I^{2m} = (I_e \oplus I_{1-e})^{2m} = \sum_{i=0}^{2m} C_{2m}^i I_e^i I_{1-e}^{2m-i} = \sum_{i=0}^{m} C_{2m}^r I_e^i I_{1-e}^{2m-i} + \sum_{i=m+1}^{2m} C_{2m}^i I_e^i I_{1-e}^{2m-i} = E \oplus F$. Thus I^{2m} is a finitely generated ideal of A, and so A is $2m$-formally Noetherian. □

Corollary 2.29. *Let $\{A_i\}_{i\in\Lambda}$ be a family of rings with cardinality at least 2. We consider the product ring $A = \prod_{i\in\Lambda} A_i$. The following conditions are equivalent:*

(1) *The set Λ is finite and for each $i \in \Lambda$, the ring A_i is m-formally Noetherian for some $m \geq 1$.*
(2) *The ring A is m-formally Noetherian for some $m \geq 1$.*
(3) *The ring A is nonnil-m-formally Noetherian for some $m \geq 1$.*

Proof. (1) $\Rightarrow$ (2) By the previous theorem. (2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) By the previous theorem, for each $i \in \Lambda$ the ring A_i is m-formally Noetherian. Assume that $|\Lambda| = \infty$. Consider the ideal I of A of all elements with finite support. Since I^m is finitely generated, there exists $n \geq 1$ such that $I^m = \langle e_{i_1}, \ldots, e_{i_n}\rangle$ where $e_i = (\delta_{i,j})_{j\in\Lambda}$, for each $i \in \Lambda$. Let $r \in \Lambda\setminus\{i_1, \ldots, i_n\}$. Then $e_r = e_r^m \in I^m = \langle e_{i_1}, \ldots, e_{i_n}\rangle$. Consequently, $\mathrm{supp}(e_r) \subseteq \{i_1, \ldots, i_n\}$, which is impossible. Hence Λ is finite. □

Example 2.30. Let A be a ring and $m \geq 1$ an integer. Then the product ring $A \times A$ is nonnil-m-formally Noetherian if and only if A is m-formally Noetherian.

Remark 2.31. Let $A_1, \ldots, A_n$ be a finite number of rings. Assume that each A_i is an m_i-formally Noetherian ring for some $m_i \geq 1$. Then $A_1 \times \cdots \times A_n$ is m-formally Noetherian, where $m = \max\{m_1, \ldots, m_n\}$.

Let A be a ring. It is shown in [2] that each flat overring B of A has the form $B = \{x \in T \mid \text{there exists } I \in S \text{ such that } xI \subseteq A\}$, where T is the total quotient ring of A and S is a multiplicative system of ideals of A. Moreover, we can choose S such that $IB = B$ for each $I \in S$. Also Arnold and Brewer in ([2]. Theorem 1.3) have proved that for each prime ideal Q of B, there exists a prime ideal P of A such that $Q = P_S = \{x \in T \mid \text{there exists } I \in S \text{ such that } xI \subseteq P\}$. By [4], each flat overring of a nonnil-SFT ring is also a nonnil-SFT ring. It turns out that a flat overring of a nonnil-m-formally Noetherian ring has a Noetherian spectrum. Now, assume that A is nonnil-m-formally Noetherian. Thus by the previous discussion each radical ideal of B is a finite intersection of prime ideals. In this case, one can easily see that for each radical ideal Q of B there exists an ideal L of A such that $Q = L_S$. In the following theorem we study the stability of the nonnil-m-formally Noetherian concept via flat overrings.

Theorem 2.32. *Let A be a nonnil-m-formally Noetherian ring. Then each flat overring of A is nonnil-m-formally Noetherian.*

Proof. Let B be a flat overring of A. Then there exists a multiplicative system of ideals S of A such that

$$B = \{x \in T \mid \text{ there exists } I \in S \text{ such that } xI \subseteq A\},$$

where T is the total quotient ring of A. As we said previously, we may assume that $IB = B$ for each $I \in S$. Let Q be a nonnil radical ideal of B. Thus $L = Q \cap A \nsubseteq Nil(A)$. Indeed, if $L \subseteq Nil(A)$, then for each $x \in Q$ there exists $I \in S$ such that $xI \subseteq L$. It yields that $xB = xIB \subseteq LB$. It follows that $x \in LB \subseteq Nil(B)$. Thus $Q \subseteq Nil(B)$, absurd. As A is nonnil-m-formally Noetherian, there exists a finitely generated ideal $F \subseteq L$ of A such that $L^m = F$. Let $x_1, \ldots, x_m \in Q$. For each $i = 1, \ldots, m$ there exists $I_i \in S$ such that $x_i I_i \subseteq L$. Set $I = I_1 I_2 \cdots I_m$. Then $I \in S$ and $x_1 x_2 \cdots x_m I = (x_1 I_1) \cdots (x_m I_m) \subseteq L^m = F$. It yields that $(x_1 \cdots x_m)B = (x_1 \cdots x_m I)B \subseteq FB$. Hence $x_1 \cdots x_m \in FB$, so $Q^m \subseteq FB = L^m B \subseteq Q^m$. Thus $Q^m = FB$ with FB is a finitely generated ideal of B. Thus B is nonnil-m-formally Noetherian. □

An immediate corollary of the above theorem can be found by observing the fact that a localization of a ring is a flat overring of the ring.

Corollary 2.33. *Let A be a ring and $S \subset A$ a multiplicative set. If A is nonnil-m-formally Noetherian, so is A_S.*

Remark 2.34. Let A be a ring and $m \geq 1$ an integer. A similar proof as the proof of the previous theorem, one can see that if A is an m-formally Noetherian ring, so is each flat overring of A.

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