Commun. Korean Math. Soc. **39** (2024), No. 3, pp. 595–609
https://doi.org/10.4134/CKMS.c230288
pISSN: 1225-1763 / eISSN: 2234-3024

ON DIFFERENTIAL IDENTITIES INVOLVING PARTITIONING IDEALS OF SEMIRINGS

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Abstract. In this article, we study a certain class of partitioning ideals known as Q-ideals, in semirings. Main objective is to investigate differential identities linking a semiring S to its prime Q-ideal I_Q, which ensure the commutativity and other features of S/I_Q.

1. Introduction and preliminaries

Semirings play a key role in the theory of automata [16], optimization theory [10], and theoretical computer science [11]. Idempotent analysis [15, 18] based on additive inverse semirings was proposed by a group of Russian mathematicians, and it has fascinating applications in quantum physics. MA-semirings due to Javed [13], are a class of semirings which properly contains the classes of distributive lattices and rings. In general, the notion of commutators satisfying Jacobian identities that is not sustainable in semirings, is a peculiarity of MA-semirings. The class of MA-semirings has a significant potential to accommodate the study of derivations satisfying different identities on semirings for probing commuting conditions. Now this class is well known and several research articles have been published (see [1, 2, 20]). Ideal theory has a wide variety of applications and intriguing features, and it has become a valuable concept in algebra and ring theory. For the various types of algebraic structures, various ideals such as Quasi-ideals [12], k-ideals [21], Q-ideals [7, 8], Jordan ideals [3], and Lie ideals [5] have been described and studied. Derivations satisfying certain identities on rings [6, 14, 19], as well as on ideals [3–5], leading to the commutativity of rings have become a burgeoning field of study and algebraists have made a significant contribution to it, and some of these problems have been generalized to semirings [1, 2, 13].

We now state some definitions and basic notions. A nonempty set S with two binary operations addition '$+$' and multiplication '$\cdot$' is said to be a semiring if the following axioms are satisfied: (i) $(S, +)$ is a commutative monoid with identity element '0' (ii) $(S, \cdot)$ is a semigroup such that $u0 = 0u = 0$ for all $u \in S$

Received October 28, 2023; Revised February 16, 2024; Accepted February 23, 2024.
2020 *Mathematics Subject Classification.* Primary 16Y60, 16W25, 16N60, 16U80.
Key words and phrases. Q-ideals, prime ideals, semirings, derivations.

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(iii) $u(v+w) = uv + uw$, $(v+w)u = vu + wu$ for all $u, v, w \in S$. A semiring S is said to be 2-torsion free if $2u = 0$ implies $u = 0$. A semiring S is said to be an additive inverse semiring if for each $u \in S$ there is a $u' \in S$ (called the pseudo inverse of u) such that $u + u' + u = u$ and $u' + u + u' = u'$. An additive inverse semiring S is said to be an MA-semiring if it satisfies $u + u' \in Z(S)$ for all $u \in S$, where $Z(S)$ is the center of S. In fact, every ring is an MA-semiring but converse may not hold in general. In the following we present some examples of MA-semirings which are not rings.

Example 1.1. Let $\mathbb{R}^0$ be the set of all non-negative real numbers and $a, b \in \mathbb{R}^0$. Define addition $\oplus$ and multiplication $\odot$ by $a \oplus b = \max\{a, b\}$ and $a \odot b = \min\{a, b\}$. Then $(\mathbb{R}^0, \oplus, \odot)$ forms an MA-semiring which is not a ring.

Example 1.2 ([22])**.** Let $(R, +, \cdot)$ be a ring and $I(R)$ be the collection of all ideals of R. Consider the set $S = R \times I(R)$ and let $u = (r_1, I), v = (r_2, J) \in S$. Define addition $\oplus$ and multiplication $\odot$ by $u \oplus v = (r_1 + r_2, I + J)$ and $u \odot v = (r_1 r_2, IJ)$. Then $(S, \oplus, \odot)$ forms an MA-semiring which is not a ring.

Example 1.3. Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z}_0^+$ be the set of all non-negative integers and $R = \mathbb{Z} \times \mathbb{Z}_0^+$. Define addition $\oplus$ and multiplication $\odot$ by $(u_1, v_1) \oplus (u_2, v_2) = (u_1 + u_2, v_1 \vee v_2)$ and $(u_1, v_1) \odot (u_2, v_2) = (u_1 \cdot u_2, v_1 \cdot v_2)$, where $v_1 \vee v_2 = \max\{v_1, v_2\}$. Then the triplet $(S, \oplus, \odot)$ forms an MA-semiring which is not a ring.

An ideal I of a semiring S is called prime if for $a, b \in S$, $aSb \subseteq I$ implies either $a \in I$ or $b \in I$. An ideal I of a semiring S is said to be a Q-ideal if there exists a partitioning subset Q of S such that $S = \bigcup\{q + I : q \in Q\}$ and if $q_1, q_2 \in Q$, then $(q_1 + I) \bigcap (q_2 + I) \neq \phi$ if and only if $q_1 = q_2$ (see [8]). An ideal I of a semiring S is said to be a k-ideal if $a + b \in I$ and $b \in I$, then $a \in I$. In fact every Q-ideal is a k-ideal but the converse may not be true in general; for details, see [7]. Throughout the sequel by I_Q, we mean a prime Q-ideal unless mentioned otherwise. An additive mapping $\varrho : S \longrightarrow S$ is a derivation if $\varrho(us) = \varrho(u)s + u\varrho(s)$. The commutator and the anti-commutator(or the Jordan product) of u, s in S are respectively defined as $[u, s] = us + s'u$ and $u \circ s = us + su$. Now we state some identities in MA-semirings which will be frequently used in the sequel. For all $u, s, w \in S$, we have $[u, us] = u[u, s]$, $[us, w] = u[s, w] + [u, w]s$, $[u, sw] = [u, s]w + s[u, w]$, $[u, s] + [s, u] = s(u + u') = u(s + s')$, $(us)' = u's = us'$, $[u, s]' = [u, s'] = [u', s]$, $u \circ (s + w) = u \circ s + u \circ w$, $u + s = 0$ implies $u = s'$, however the converse may not be true in general (see [13, 20] for ready reference).

One can find MA-semirings, in which well known properties of rings are not valid in general. For example if S is an MA-semiring and $s, t \in S$, then $st = ts$ does not admit $[s, t] = 0$; $[s, s] \neq 0$ if $s \neq 0$; if $s \in Z(S)$ and ϱ is derivation of S, then $\varrho(s)$ may not belong to $Z(S)$.

The following results are indeed useful to establish the main results of this paper.

Proposition 1.4. *The pseudo inverse of any element in an additive inverse semiring is unique.*

We often use the following Proposition in the sequel without mentioning it.

Proposition 1.5. *Let I_Q be a Q-ideal of an additive inverse semiring S and $t \in S$. Then $t \in I_Q$ if and only if $t + I_Q = I_Q$.*

Quotient MA-semirings can be defined canonically as the notion of quotient rings. Following lemma shows that for an MA-semiring S and its prime Q-ideal I_Q, the set $S/I_Q = \{t + I_Q : t \in S\}$ forms an MA-semiring known as a quotient MA-semiring.

Lemma 1.6 ([17])**.** *Let $(S, +, \cdot)$ be an MA-semiring and I_Q be a prime Q-ideal of S. Then the set $S/I_Q = \{t+I_Q : t \in S\}$ forms an MA-semiring with respect to the addition $\oplus$ and multiplication $\odot$ defined by*

(i) $(t + I_Q) \oplus (u + I_Q) = t + u + I_Q$
(ii) $(t + I_Q) \odot (u + I_Q) = t \cdot u + I_Q$

for all $t, u \in S$.

Following is an important result which is useful in proving the main results.

Lemma 1.7 ([17])**.** *Let I_Q be a prime Q-ideal of an MA-semiring S. If*

(i) $[t, u] \in I_Q$ *or*
(ii) $t \circ u \in I_Q$

for all $t, u \in S$, then S/I_Q is a commutative MA-semiring.

Proposition 1.8. *Let S be an additive inverse semiring and I_Q be its prime Q-ideal. Then S/I_Q is prime.*

Proof. Let $t + I_Q, u + I_Q \in S/I_Q$ and put

$$(t + I_Q) \odot (S/I_Q) \odot (u + I_Q) = I_Q.$$

Therefore

$$(t + I_Q) \odot (s + I_Q) \odot (u + I_Q) = I_Q$$

for all $s \in S$. By the definition of $\odot$ in S/I_Q, we have

$$tsu + I_Q = I_Q$$

for all $s \in S$, which further implies that $tsu \in I_Q$ for all $s \in S$ and therefore

$$tSu \subseteq I_Q.$$

By the primeness of I_Q, we have either $t \in I_Q$ or $u \in I_Q$ and hence $t + I_Q = I_Q$ or $u + I_Q = I_Q$. This shows that S/I_Q is prime. However, the primeness of S/I_Q does not imply the primeness of S in general. □

Proposition 1.9. *Let S be a 2-torsion free additive inverse semiring and I_Q be its prime ideal. Then S/I_Q is 2-torsion free.*

Proof. As S is 2-torsion free for $s \in S$, $2s = 0$ implies $s = 0$ (c.f. Section 1). This means that $2s + I_Q = 2(s + I_Q) = I_Q$ implies $s + I_Q = I_Q$. Hence S/I_Q is 2-torsion free. Observe that the 2-torsion freeness of S/I_Q does not imply the 2-torsion freeness of S in general. □

Mir et al. [9] proved some results on derivations satisfying certain identities on prime ideals of rings. The main objective of this paper is to prove these results for prime ideals of MA-semirings and investigate differential identities leading to the commutativity of quotient MA-semirings.

2. Main results

In the sequel, we use $(t + I_Q)(u + I_Q)$ and $(t + I_Q) + (u + I_Q)$ instead of $(t + I_Q) \odot (u + I_Q)$ and $(t + I_Q) \oplus (u + I_Q)$ (c.f. Lemma 1.6), respectively for the sake of convenience.

Lemma 1 of [9] is generalized as follows.

Theorem 2.1. *Let I_Q be a prime Q-ideal of an MA-semiring S and ϱ_1, ϱ_2 be derivations of S. If $\varrho_1(u)u + u'\varrho_2(u) \in I_Q$ for all $u \in S$, then one of the following holds*

(i) $(\varrho_1(S) \subseteq I_Q$ *and* $\varrho_2(S) \subseteq I_Q)$
(ii) S/I_Q *is commutative.*

Proof. Suppose that for all $u \in S$

$$\varrho_1(u)u + u'\varrho_2(u) \in I_Q. \tag{1}$$

In (1), substituting $u + s$ for u and using (1) again, we obtain

$$\varrho_1(u)s + \varrho_1(s)u + u'\varrho_2(s) + s'\varrho_2(u) \in I_Q. \tag{2}$$

Multiplying (2) by u from the right, we obtain

$$\varrho_1(u)su + \varrho_1(s)uu + u'\varrho_2(s)u + s'\varrho_2(u)u \in I_Q. \tag{3}$$

In (2), substituting su for s, we get

$$\varrho_1(u)su + \varrho_1(s)uu + s\varrho_1(u)u + u'\varrho_2(s)u + u's\varrho_2(u) + s'u\varrho_2(u) \in I_Q. \tag{4}$$

As $u = u + u' + u$ and $u + u' \in Z(S)$, we have

$$\begin{aligned}
&\varrho_1(u)su + \varrho_1(s)uu + s\varrho_1(u)u + u'\varrho_2(s)u + u's\varrho_2(u) + su'\varrho_2(u) \\
&= \varrho_1(u)su + \varrho_1(s)uu + s\varrho_1(u)u + u'\varrho_2(s)u + u's\varrho_2(u) + s(u' + u + u')\varrho_2(u) \\
&= \varrho_1(u)su + \varrho_1(s)uu + s\varrho_1(u)u + u'\varrho_2(s)u + u's\varrho_2(u) + su'\varrho_2(u) \\
&\quad + s\varrho_2(u)u + s\varrho_2(u)u' \\
&= (\varrho_1(u)s + \varrho_1(s)u + u'\varrho_2(s) + s'\varrho_2(u))u + s\varrho_1(u)u \\
&\quad + u's\varrho_2(u) + su'\varrho_2(u) + s\varrho_2(u)u,
\end{aligned}$$

$$\begin{aligned}
&(\varrho_1(u)s + \varrho_1(s)u + u'\varrho_2(s) + s'\varrho_2(u))u + s\varrho_1(u)u \\
&+ u's\varrho_2(u) + su'\varrho_2(u) + s\varrho_2(u)u \in I_Q.
\end{aligned} \tag{5}$$

As I_Q is a prime Q-ideal, therefore using (3) in (5), we obtain

$$s\varrho_1(u)u + u's\varrho_2(u) + su'\varrho_2(u) + s\varrho_2(u)u \in I_Q. \tag{6}$$

Multiplying (1) by s from the left, we obtain

$$s\varrho_1(u)u + su'\varrho_2(u) \in I_Q. \tag{7}$$

Using (7) in (6), we obtain $u's\varrho_2(u) + s\varrho_2(u)u \in I_Q$ and therefore

$$[s\varrho_2(u), u] \in I_Q. \tag{8}$$

In (8), substituting sz for s, we get $[sz\varrho_2(u), u] \in I_Q$. But by the MA-semiring identities $[sz\varrho_2(u), u] = s[z\varrho_2(u), u] + [s, u]z\varrho_2(u)$ and therefore $s[z\varrho_2(u), u] + [s, u]z\varrho_2(u) \in I_Q$. Using (8), we obtain $[s, u]S\varrho_2(u) \subseteq I_Q$. As I_Q is prime, therefore we have $[s, u] \in I_Q$ or $\varrho_2(u) \in I_Q$ for all $u, s \in S$. Consider the sets

$$S_1 = \{u \in S : [s, u] \in I_Q, \text{ for all } s \in S\}$$

and

$$S_2 = \{u \in S : \varrho_2(u) \in I_Q\}.$$

Our claim is that $S_1 \subseteq S_2$ or $S_1 \subseteq S_2$. Suppose that $u_1 \in S_1 \setminus S_2$ and $u_2 \in S_2 \setminus S_1$. We have $u_1 + u_2 \in S_1 + S_2 \subseteq S_1 \cup S_2$. Therefore either $u_1 + u_2 \in S_1$ or $u_1 + u_2 \in S_2$. If $u_1 + u_2 \in S_1$, then $[s, u_1] + [s, u_2] \in I_Q$ which further implies $[s, u_2] \in I_Q$ and therefore $u_2 \in S_1$, a contradiction. On the other hand if $u_1 + u_2 \in S_2$, then $\varrho_2(u_1) + \varrho_2(u_2) \in I_Q$, which further implies $\varrho_2(u_1) \in I_Q$ and therefore $u_1 \in S_2$, a contradiction. Therefore we conclude that either $S = S_1$ or $S_2 = S$ and either $[S, S] \subseteq I_Q$ or $\varrho_2(S) \subseteq I_Q$. In view of the Lemma 1.7, the assumption $[S, S] \subseteq I_Q$ implies that S/I_Q is commutative. On the other hand if $\varrho_2(S) \subseteq I_Q$, by the hypothesis, we obtain $\varrho_1(S) \subseteq I_Q$ and vice versa. □

Theorem 1 of [9] is generalized in the following result.

Theorem 2.2. *Let I_Q be a prime Q-ideal of an MA-semiring S and ϱ_1, ϱ_2 be two derivations of S. If any one of the following statements holds:*

(i) $[\varrho_1(u), \varrho_2(s)] + [u, s]' \in I_Q$

(ii) $\varrho_1(u)\varrho_2(s) + [u, s]' \in I_Q$

for all $s, u \in S$, then S/I_Q is commutative.

Proof. (i) For all $s, u \in S$, we have

$$[\varrho_1(u), \varrho_2(s)] + [u, s]' \in I_Q. \tag{9}$$

In (9), substituting sr for s and then rearranging the terms, we obtain

$$\begin{aligned}&\varrho_2(s)[\varrho_1(u), r] + ([\varrho_1(u), \varrho_2(s)] + [u, s]')r \\ &\quad + s([\varrho_1(u), \varrho_2(r)] + [u, r]') + [\varrho_1(u), s]\varrho_2(r) \in I_Q.\end{aligned}$$

As I_Q is a prime Q-ideal, using (9) in the last identity, we obtain

$$\varrho_2(s)[\varrho_1(u), r] + [\varrho_1(u), s]\varrho_2(r) \in I_Q. \tag{10}$$

In (10), taking $s = \varrho_1(u)$, we get

$$\varrho_2(\varrho_1(u))[\varrho_1(u), r] + [\varrho_1(u), \varrho_1(u)]\varrho_2(r) \in I_Q. \tag{11}$$

Using MA-semiring identities, we have

$$[\varrho_1(u), \varrho_1(u)]\varrho_2(r) = 2[\varrho_1(u), \varrho_1(u)]\varrho_2(r),$$

therefore (11) becomes

$$\varrho_2(\varrho_1(u))[\varrho_1(u), r] + [\varrho_1(u), \varrho_1(u)]\varrho_2(r) + [\varrho_1(u), \varrho_1(u)]\varrho_2(r) \in I_Q.$$

As I_Q is a Q-ideal, using (11), we obtain $[\varrho_1(u), \varrho_1(u)]\varrho_2(r) \in I_Q$ and then again from (11), we get

$$\varrho_2(\varrho_1(u))[\varrho_1(u), r] \in I_Q. \tag{12}$$

In (12), substituting sr for r, we get

$$\varrho_2(\varrho_1(u))z[\varrho_1(u), r] + \varrho_2(\varrho_1(u))[\varrho_1(u), z]r \in I_Q.$$

Using (12), we get $\varrho_2(\varrho_1(u))S[\varrho_1(u), r] \subseteq I_Q$ and by the primeness of I_Q, we obtain $[\varrho_1(u), r] \in I_Q$ or $\varrho_2(\varrho_1(u)) \in I_Q$. Following the same arguments as above, we conclude that $[\varrho_1(S), S] \subseteq I_Q$ or $\varrho_2(\varrho_1(S)) \subseteq I_Q$. Suppose that $[\varrho_1(S), S] \subseteq I_Q$. Then

$$[\varrho_1(u), r] \in I_Q. \tag{13}$$

In (13), substituting ur for u, we obtain $[u\varrho_1(r), r] + [\varrho_1(u)r, r] \in I_Q$ and therefore $[u, r]\varrho_1(r) + u[\varrho_1(r), r] + [\varrho_1(u), r]r \in I_Q$. As I_Q is a Q-ideal, using (13), we get $[u, r]\varrho_1(r) \in I_Q$. Following the same process as above we obtain either S/I_Q is commutative or $\varrho_1(r) \in I_Q$. If $\varrho_1(r) \in I_Q$, then (9) gives $[u, s] \in I_Q$, which implies that S/I_Q is commutative. Secondly suppose that

$$\varrho_2(\varrho_1(z)) \in I_Q. \tag{14}$$

In (9), replacing s by $\varrho_1(s)$ and using (14) again, we get $[u, \varrho_1(s)] \in I_Q$. Then following similar lines as above we conclude that S/I_Q is commutative.

(ii) By the hypothesis for all $s, u \in S$, we have

$$\varrho_1(u)\varrho_2(s) + [u, s]' \in I_Q. \tag{15}$$

In (15), substituting sz for s, we get

$$\varrho_1(u)s\varrho_2(z) + (\varrho_1(u)\varrho_2(s) + [u, s]')z + s[u, z]' \in I_Q.$$

Using (15), we get

$$\varrho_1(u)s\varrho_2(z) + s[u, z]' \in I_Q. \tag{16}$$

In (16), substituting zu for z, we obtain $\varrho_1(u)s\varrho_2(z)u + \varrho_1(u)sz\varrho_2(u) + s[u, z]'u \in I_Q$ and using (16) again, we get $\varrho_1(u)sz\varrho_2(u) \in I_Q$ and therefore $\varrho_1(u)Sz\varrho_2(u) \subseteq I_Q$. By the primeness of I_Q, we have either $\varrho_1(u) \in I_Q$ or $z\varrho_2(u) \in I_Q$. Following the same process as above, we conclude that $\varrho_1(u) \in I_Q$ or $z\varrho_2(u) \in I_Q$ for all $u, z \in S$. If $\varrho_1(u) \in I_Q$, then (15) yields $[u, s] \in I_Q$, which shows that S/I_Q is commutative. Secondly if $z\varrho_2(u) \in I_Q$, then it is easy to verify that $\varrho_2(u) \in I_Q$, therefore the case becomes similar to the first case. □

Following result is an extended form of the Corollary 1 of [9].

Theorem 2.3. *Let ϱ_1, ϱ_2 be two derivations of a semiprime MA-semiring S. If any one of the following statements holds*:

(i) $[\varrho_1(u), \varrho_2(s)] + [u, s]' = 0$
(ii) $\varrho_1(u)\varrho_2(s) + [u, s]' = 0$

for all $s, u \in S$, then S is commutative.

Proof. (i) If either ϱ_1 or ϱ_2 is zero, then $[u, s] = 0$ and therefore S is commutative. So we consider the case when both ϱ_1 and ϱ_2 are nonzero. As S is semiprime, therefore there is a family $\mathbb{I}_{\mathbb{Q}}$ of prime ideals of S such that $\bigcap \mathbb{I}_{\mathbb{Q}} = \{0\}$. By the hypothesis, for all $s, u \in S$, we can write

$$[\varrho_1(u), \varrho_2(s)] + [u, s]' \in \bigcap \mathbb{I}_{\mathbb{Q}},$$

which further implies

$$[\varrho_1(u), \varrho_2(s)] + [u, s]' \in I_Q,$$

for all $I_Q \in \mathbb{I}_{\mathbb{Q}}$. Hence employing theorem 2.1, we obtain the required result.

(ii) Using similar arguments of first part, we obtain the required result. □

Theorem 2 of [9] is extended in the result that follows.

Theorem 2.4. *Let I_Q be a prime Q-ideal of an MA-semiring S and ϱ_1, ϱ_2 be two derivations of S. If any one the following statements holds:*

(i) *If $\varrho_1(u) \circ \varrho_2(s) + u' \circ s \in I_Q$*
(ii) *If $[\varrho_1(u), \varrho_2(s)] + u' \circ s \in I_Q$*

for all $s, u \in S$, then S/I_Q is commutative.

Proof. (i) By the hypothesis, for all $s, u \in S$, we have

$$\varrho_1(u) \circ \varrho_2(s) + u' \circ s \in I_Q. \tag{17}$$

In (17), substituting sz for s, we get

$$\varrho_1(u) \circ (s\varrho_2(z)) + \varrho_1(u) \circ (\varrho_2(s)z) + u' \circ (sz) \in I_Q,$$

which further gives

$$\varrho_1(u)s\varrho_2(z) + s\varrho_2(z)\varrho_1(u) + \varrho_1(u)\varrho_2(s)z + \varrho_2(s)z\varrho_1(u) + u'sz + szu' \in I_Q.$$

By the definition of MA-semiring, we can write

$$\begin{aligned} &s\varrho_1(u)\varrho_2(z) + s'\varrho_1(u)\varrho_2(z) + \varrho_1(u)s\varrho_2(z) + s\varrho_2(z)\varrho_1(u) + \varrho_1(u)\varrho_2(s)z \\ &+ \varrho_2(s)z\varrho_1(u) + \varrho_2(s)\varrho_1(u)z' + \varrho_2(s)\varrho_1(u)z + u'sz + su'z + suz + szu' \in I_Q \end{aligned}$$

and therefore

$$\begin{aligned} &s(\varrho_1(u) \circ \varrho_2(z)) + [\varrho_1(u), s]\varrho_2(z) + (\varrho_1(u) \circ \varrho_2(s))z \\ &+ \varrho_2(s)[z, \varrho_1(u)] + (u \circ s)'z + s[u, z] \in I_Q. \end{aligned}$$

As I_Q is Q-ideal, using (17), we get

$$s(\varrho_1(u) \circ \varrho_2(z)) + [\varrho_1(u), s]\varrho_2(z) + \varrho_2(s)[z, \varrho_1(u)] + s[u, z] \in I_Q. \tag{18}$$

Using MA-semiring identities, from (17), we can write

$$s(\varrho_1(u) \circ \varrho_2(z)) + [\varrho_1(u), s]\varrho_2(z) + \varrho_2(s)[z, \varrho_1(u)] + s(u \circ z)' + 2suz \in I_Q.$$

Using hypothesis, we obtain

$$[\varrho_1(u), s]\varrho_2(z) + \varrho_2(s)[z, \varrho_1(u)] + 2suz \in I_Q. \tag{19}$$

In (19), replacing z by $\varrho_1(u)$, we obtain

$$[\varrho_1(u), s]\varrho_2(\varrho_1(u)) + \varrho_2(s)[\varrho_1(u), \varrho_1(u)] + 2su\varrho_1(u) \in I_Q. \tag{20}$$

As S is an MA-semiring, we have $2[\varrho_1(u), \varrho_1(u)] = [\varrho_1(u), \varrho_1(u)]$ and therefore (20) becomes

$$\begin{aligned} &[\varrho_1(u), s]\varrho_2(\varrho_1(u)) + \varrho_2(s)[\varrho_1(u), \varrho_1(u)] \\ &+ \varrho_2(s)[\varrho_1(u), \varrho_1(u)] + 2su\varrho_1(u) \in I_Q. \end{aligned} \tag{21}$$

Using (20) in (21), we obtain $\varrho_2(s)[\varrho_1(u), \varrho_1(u)] \in I_Q$ and using it in (20), we obtain

$$[\varrho_1(u), s]\varrho_2(\varrho_1(u)) + 2su\varrho_1(u) \in I_Q. \tag{22}$$

In (22), writing sw in place of s, we obtain

$$s[\varrho_1(u), w]\varrho_2(\varrho_1(u)) + [\varrho_1(u), s]w\varrho_2(\varrho_1(u)) + 2swu\varrho_1(u) \in I_Q.$$

As I_Q is a Q-ideal, using (22), we obtain $[\varrho_1(u), s]S\varrho_2(\varrho_1(u)) \subseteq I_Q$. By the primeness of I_Q, we obtain either $[\varrho_1(u), s] \in I_Q$ or $\varrho_2(\varrho_1(u)) \in I_Q$. If $[\varrho_1(u), s] \in I_Q$, then by Theorem 2.1, S/I_Q is commutative. Secondly assume that $\varrho_2(\varrho_1(u)) \in I_Q$. In (19), replacing s by $\varrho_1(s)$, we obtain $\varrho_1(u) \circ \varrho_2(\varrho_1(s)) + u' \circ \varrho_(s) \in I_Q$ and using our supposition, we get $u' \circ \varrho_2(s) \in I_Q$, which further implies

$$u \circ \varrho_2(s) \in I_Q. \tag{23}$$

In (23), substituting $\varrho_1(u)$ for u and using (17) again, we obtain for all $s, u \in S$

$$u \circ s \in I_Q. \tag{24}$$

In (24), substituting us for s, we obtain $u \circ (us) = uus + usu \in I_Q$. Using MA-semiring identities, we have $uus + usu = uus + u(s + s' + s)u = uus + usu + s'uu + suu = (u^2 \circ s) + [u, s]u$. Therefore

$$(u^2 \circ s) + [u, s]u \in I_Q. \tag{25}$$

As I_Q is a Q-ideal, using (24) we obtain

$$[u, s]u \in I_Q. \tag{26}$$

In (26), substituting sr for s and using (26) again, we obtain $[u, s]ru \in I_Q$. Replacing r by rs' we obtain

$$[u, s]rs'u \in I_Q. \tag{27}$$

Multiplying (26) by s from the right, we have

$$[u,s]rus \in I_Q. \tag{28}$$

Adding (27) and (28), we obtain $[u,s]r[u,s] \in I_Q$ and therefore $[u,s]S[u,s] \subseteq I_Q$. As I_Q is prime, therefore $[u,s] \in I_Q$, for all $s,u \in S$, which shows that S/I_Q is commutative.

(ii) By the hypothesis, for all $s,u \in S$, we have

$$[\varrho_1(u), \varrho_2(s)] + u' \circ s \in I_Q. \tag{29}$$

In (29), substituting sz for s, we get $[\varrho_1(u), \varrho_2(sz)] + u' \circ (sz) \in I_Q$ and therefore

$$[\varrho_1(u), s\varrho_2(z)] + [\varrho_1(u), \varrho_2(s)z] + u'sz + szu' \in I_Q,$$

which further implies

$$\begin{aligned} &s[\varrho_1(u), \varrho_2(z)] + [\varrho_1(u), s]\varrho_2(z) + [\varrho_1(u), \varrho_2(s)]z + \varrho_2(s)[\varrho_1(u), z] \\ &+ u'sz + sz(u' + u + u') \in I_Q. \end{aligned}$$

Using the definition of an MA-semiring, we can write

$$\begin{aligned} &s[\varrho_1(u), \varrho_2(z)] + [\varrho_1(u), s]\varrho_2(z) + [\varrho_1(u), \varrho_2(s)]z + \varrho_2(s)[\varrho_1(u), z] \\ &+ u'sz + szu' + suz + su'z \in I_Q. \end{aligned}$$

Rearranging the terms and using identities of an MA-semirings, we obtain

$$\begin{aligned} &s[\varrho_1(u), \varrho_2(z)] + [\varrho_1(u), s]\varrho_2(z) + [\varrho_1(u), \varrho_2(s)]z + \varrho_2(s)[\varrho_1(u), z] \\ &+ (u \circ s)'z + s[u, z] \in I_Q. \end{aligned}$$

Using (29), we obtain

$$s[\varrho_1(u), \varrho_2(z)] + [\varrho_1(u), s]\varrho_2(z) + \varrho_2(s)[\varrho_1(u), z] + s[u, z] \in I_Q,$$

and by the definition of an MA-semiring, it further implies

$$s([\varrho_1(u), \varrho_2(z)] + (u \circ z)') + [\varrho_1(u), s]\varrho_2(z) + \varrho_2(s)[\varrho_1(u), z] + 2suz \in I_Q. \tag{30}$$

Using (29) in (30), we get

$$[\varrho_1(u), s]\varrho_2(z) + \varrho_2(s)[\varrho_1(u), z] + 2suz \in I_Q. \tag{31}$$

In (31), substituting $\varrho_1(u)$ for z, we obtain

$$[\varrho_1(u), s]\varrho_2(\varrho_1(u)) + \varrho_2(s)[\varrho_1(u), \varrho_1(u)] + 2su\varrho_1(u) \in I_Q$$

and using the same arguments as above, we can write

$$[\varrho_1(u), s]\varrho_2(\varrho_1(u)) + 2su\varrho_1(u) \in I_Q. \tag{32}$$

In (32), substituting ws for s, we get

$$w[\varrho_1(u), s]\varrho_2(\varrho_1(u)) + [\varrho_1(u), w]s\varrho_2(\varrho_1(u)) + 2wsu\varrho_1(u) \in I_Q$$

and using (32) again, we obtain $[\varrho_1(u), w]S\varrho_2(\varrho_1(u)) \subseteq I_Q$, which further implies $[\varrho_1(u), w] \in I_Q$ or $\varrho_2(\varrho_1(u)) \in I_Q$ by the primeness of I_Q. Firstly, assume that $[\varrho_1(u), w] \in I_Q$. In view of (13), we can conclude that either S/I_Q is commutative or $\varrho_2(\varrho_1(u)) \in I_Q$. Secondly, assume that $\varrho_2(\varrho_1(u)) \in I_Q$. In (29), replacing s by $\varrho_1(s)$ and using the assumption, we get

$$u \circ \varrho_1(s) \in I_Q. \tag{33}$$

We see that (33) is the same as (23), therefore the remaining part follows as above. □

Following result is an extension of the Corollary 2 of [9].

Theorem 2.5. *Let S be a 2-torsion free MA-semiring which is either semiprime or unitary. Then there are no derivations ϱ_1, ϱ_2 satisfying one of the following*:

(i) $\varrho_1(u) \circ \varrho_2(s) + u' \circ s = 0$
(ii) $[\varrho_1(u), \varrho_2(s)] + u' \circ s = 0$

for all $s, u \in S$.

Proof. Firstly assume that S is a 2-torsion free semiprime MA-semring and ϱ_1, ϱ_2 are derivations satisfying (i). Then there is a family $\mathbb{I}_{\mathbb{Q}}$ of prime ideals such that $\bigcap \mathbb{I}_{\mathbb{Q}} = \{0\}$. Therefore for all $I_Q \in \mathbb{I}_{\mathbb{Q}}$, we can write

$$\varrho_1(u) \circ \varrho_2(s) + u' \circ s \in I_Q$$

for all $u, s \in S$. From Theorem 2.2, we have $[u, s] \in I_Q$ for all $I_Q \in \mathbb{I}_{\mathbb{Q}}$ and therefore $[u, s] = 0$, which implies that S is commutative. Therefore from (i), we can write

$$2(\varrho_1(u)\varrho_2(s) + u's) = 0$$

and by the 2-torsion freeness of S, we have

$$\varrho_1(u)\varrho_2(s) + u's = 0. \tag{34}$$

In (34), substituting sw for s and using (34) again, we obtain $\varrho_1(u)s\varrho_2(w) = 0$ and therefore $\varrho_2(w)\varrho_1(u)S\varrho_2(w)\varrho_1(u) = \{0\}$ by the semiprimeness of S, we have $\varrho_2(w)\varrho_1(u) = 0$. Therefore (i) becomes $u \circ s = 0$ for all $u, s \in S$ and since S is commutative, by the 2-torsion freeness of S, we have $us = 0$ and therefore $uSu = \{0\}$. As S is semiprime, we have $u = 0$, which implies that $S = \{0\}$, a contradiction. Secondly, assume that S is unitary. Replacing s by 1 in (i), we have

$$\varrho_1(u)\varrho_2(1) + \varrho_2(1)\varrho_1(u) + 2u' = 0. \tag{35}$$

By the definition of derivation we have $\varrho_2(1) = \varrho_2(1 \cdot 1) = 1\varrho_2(1) + \varrho_2(1) \cdot 1 = 2\varrho_2(1)$. Therefore from (35), we get

$$\varrho_1(u)\varrho_2(1) + \varrho_1(u)\varrho_2(1) + \varrho_2(1)\varrho_1(u) + 2u' = 0.$$

Using (35) again, we get $\varrho_1(u)\varrho_2(1) = 0$. Similarly we can show that $\varrho_2(1)\varrho_1(u) = 0$. Therefore from (35) we obtain $2u = 0$ and by the 2-torsion freeness of S, we get $S = \{0\}$, which is a contradiction. Hence we conclude that there is no derivation satisfying (i). Similarly we can prove the result for hypothesis (ii). □

Following result is an extended form of the Proposition 1 of [9].

Theorem 2.6. *Let I_Q be a prime Q-ideal of an MA-semiring S and ϱ_1, ϱ_2 be derivations of S. If*

$$\varrho_1(u)\varrho_2(s) + u \circ s' \in I_Q \text{ for all } s, u \in S,$$

then S/I_Q is a 2-torsion commutative MA-semiring and $\varrho_1(S) \subseteq I_Q$ or $\varrho_2(S) \subseteq I_Q$.

Proof. By the hypothesis, we have

$$\varrho_1(u)\varrho_2(s) + u \circ s' \in I_Q. \tag{36}$$

In (36), substituting sr for s, we obtain

$$\varrho_1(u)\varrho_2(r)r + \varrho_1(u)s\varrho_2(r) + us'r + s'ru \in I_Q. \tag{37}$$

By MA-semiring identities, we have

$$\begin{aligned} us'r + s'ru &= us'r + (s' + s + s')ru \\ &= us'r + (s'ur + vur + s'ru) \\ &= (u \circ s')r + s[u, r]. \end{aligned}$$

Therefore (37) becomes

$$\varrho_1(u)\varrho_2(r)r + \varrho_1(u)s\varrho_2(r) + (u \circ s')r + s[u, r] \in I_Q. \tag{38}$$

Using (36) in (38), we obtain

$$\varrho_1(u)s\varrho_2(r) + s[u, r] \in I_Q. \tag{39}$$

In (39), replacing r by ru, we get $\varrho_1(u)s\varrho_2(r)u+\varrho_1(u)sr\varrho_2(u)+s[u, r]u \in I_Q$ and using (39) again, we obtain $\varrho_1(u)sr\varrho_2(u) \in I_Q$ and therefor $\varrho_1(u)Sr\varrho_2(u) \subseteq I_Q$. As I_Q is prime, therefore as above we have either $\varrho_1(u) \in I_Q$ or $r\varrho_2(u) \in I_Q$. If $r\varrho_2(u) \in I_Q$, then $\varrho_2(u) \in I_Q$. Therefore we conclude that either $\varrho_1(S) \subseteq I_Q$ or $\varrho_2(S) \subseteq I_Q$. In both cases, (39) becomes $s[u, r] \in I_Q$ and therefore $[u, r]S[u, r] \subseteq I_Q$. Using primeness of I_Q, we obtain $[u, r] \in I_Q$. Hence S/I_Q is commutative. We next show that S/I_Q is 2-torsion. For this, we suppose that S/I_Q is 2-torsion free. By the hypothesis, we have $x \circ y \in I_Q$, which further implies

$$\begin{aligned} I_Q &= x \circ y + I_Q \\ &= (x + I_Q) \circ (y + I_Q) \\ &= (x + I_Q)(y + I_Q) + (x + I_Q)(y + I_Q) \\ &= 2(x + I_Q)(y + I_Q) \end{aligned}$$

since S/I_Q is commutative. By our assumption, S/I_Q is 2-torsion free. By using the primeness of I_Q, we can find $x + I_Q = I_Q$ or $y + I_Q = I_Q$, implying that $S = I_Q$, which is a contradiction. Hence S/I_Q is 2-torsion. □

Following result provides a generalized form of the Theorem 3 of [9].

Theorem 2.7. *Let I_Q be a prime Q-ideal of an MA-semiring S and ϱ_1, ϱ_2 be derivations of S. If*

$$\varrho_1(u) \circ \varrho_2(s) + [u,s]' \in I_Q \ for\ all\ u, s \in S,$$

then S/I_Q is commutative. Moreover if S/I_Q is 2-torsion free, then $\varrho_1(S) \subseteq I_Q$ or $\varrho_2(S) \subseteq I_Q$.

Proof. For all $s, u \in S$, we have

$$\varrho_1(u) \circ \varrho_2(s) + [u,s]' \in I_Q. \tag{40}$$

In (40), substituting sr for s, we get

$$\varrho_1(u)s\varrho_2(r) + s\varrho_2(r)\varrho_1(u) + \varrho_1(u)\varrho_2(s)r + \varrho_2(s)r\varrho_1(u) + s[u,r]' + [u,s]'r \in I_Q.$$

As $r + r' + r = r$ and $r + r' \in Z(S)$ for all $r \in S$, from the last identity, we can write

$$\begin{aligned} &\varrho_1(u)s\varrho_2(r) + s'\varrho_1(u)\varrho_2(r) + s\varrho_1(u)\varrho_2(r) + s\varrho_2(r)\varrho_1(u) + \varrho_1(u)\varrho_2(s)r \\ &+ \varrho_2(s)r\varrho_1(u) + \varrho_2(s)\varrho_1(u)r' + \varrho_2(s)\varrho_1(u)r + s[u,r]' + [u,s]'r \in I_Q. \end{aligned}$$

Rearranging the terms, we get

$$\begin{aligned} &[\varrho_1(u), s]\varrho_2(r) + s(\varrho_1(u) \circ \varrho_2(r)) + (\varrho_1(u) \circ \varrho_2(s))r \\ &+ \varrho_2(s)[r, \varrho_1(u)] + s[u,r]' + [u,s]'r \in I_Q. \end{aligned}$$

As I_Q is a Q-ideal, using (40), we get

$$[\varrho_1(u), s]\varrho_2(r) + \varrho_2(s)[r, \varrho_1(u)] \in I_Q. \tag{41}$$

In (41), substituting $\varrho_1(u)$ for s, we get

$$[\varrho_1(u), \varrho_1(u)]\varrho_2(r) + \varrho_2(\varrho_1(u))[r, \varrho_1(u)] \in I_Q.$$

Using the same arguments as above, we obtain

$$\varrho_2(\varrho_1(u))[r, \varrho_1(u)] \in I_Q,$$

and hence either $\varrho_2(\varrho_1(u)) \in I_Q$ or $[r, \varrho_1(u)] \in I_Q$. Assume that $\varrho_2(\varrho_1(u)) \in I_Q$. In (40), replacing s by $\varrho_1(s)$, we obtain $\varrho_1(u) \circ \varrho_2(\varrho_1(s)) + [u, \varrho_1(s)]' \in I_Q$. As $\varrho_2(\varrho_1(u)) \in I_Q$, therefore we obtain

$$[\varrho_1(s), u] \in I_Q. \tag{42}$$

We observe that (42) is the same as (13) of the proof of Theorem 2.2, therefore S/I_Q is commutative. We next assume that S/I_Q is 2-torsion free. Since S/I_Q is commutative, therefore from (40), we can write $\varrho_1(u)S\varrho_2(s) \subseteq I_Q$. By the primeness of I_Q, we have $\varrho_1(S) \subseteq I_Q$ or $\varrho_2(S) \subseteq I_Q$. □

Remark 2.8. If S/I_Q is commutative, then it has no zero divisors. For $x + I_Q, y + I_Q \in S/I_Q$, let $I_Q = (x + I_Q)(y + I_Q)$, then for any $s + I_Q \in S/I_Q$, we have $I_Q = (x + I_Q)(s + I_Q)(y + I_Q) = xsy + I_Q$. By the Proposition 1.5, we can write $xSy \subseteq I_Q$. By the primeness of I_Q, we have $x \in I_Q$ or $y \in I_Q$ and hence $x + I_Q = I_Q$ or $y + I_Q = I_Q$.

In view of Theorem 2.7 and using the similar arguments of Theorem 2.5, we can obtain the following result, which is a generalized version of Corollary 3 of [9].

Theorem 2.9. *Let S be a 2-torsion free MA-semiring, which is either semiprime or with unity. There are no derivations ϱ_1 and ϱ_2 satisfying one of the following conditions*:

(i) $\varrho_1(u) \circ \varrho_2(s) + (u \circ s)' = 0$
(ii) $\varrho_1(u) \circ \varrho_2(s) + [u, s]' = 0$

for all $s, u \in S$.

3. Conclusion

This study has mainly dealt with the semirings and their partitioning ideals known as Q-ideals, linked by the several identities of derivations. In this regard, we have investigated some differential identities involving a semiring S and its prime Q-ideal I_Q, leading to the commutativity of the quotient semiring S/I_Q. We have also explored some other features of the quotient semirings. We can get some interesting outcomes for the prime semirings by taking $I_Q = \{0\}$ in the main section, and this shows that proving results for the aforementioned quotient semirings is more generalized than looking into the results of prime semirings. The results of this article for semiprime ideals would be a fascinating open topic for researchers.

Acknowledgements. The authors wish to express their sincere thanks to the referee for many useful comments.

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