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(m,n)-CLOSED δ-PRIMARY IDEALS IN AMALGAMATION

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Abstract. Let R be a commutative ring with $1 \neq 0$. Let $Id(R)$ be the set of all ideals of R and let $\delta : Id(R) \longrightarrow Id(R)$ be a function. Then δ is called an expansion function of the ideals of R if whenever L, I, J are ideals of R with $J \subseteq I$, then $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$. Let δ be an expansion function of the ideals of R and $m \geq n > 0$ be positive integers. Then a proper ideal I of R is called an *(m,n)-closed δ-primary ideal* (resp., *weakly (m,n)-closed δ-primary ideal*) if $a^m \in I$ for some $a \in R$ implies $a^n \in \delta(I)$ (resp., if $0 \neq a^m \in I$ for some $a \in R$ implies $a^n \in \delta(I)$). Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B. This paper investigates the concept of (m,n)-closed δ-primary ideals in the amalgamation of A with B along J with respect to f denoted by $A \bowtie^f J$.

1. Introduction

We assume throughout the whole paper that all rings are commutative with $1 \neq 0$. The notion of an (m,n)-closed ideal was introduced and defined by Anderson and Badawi in [1], as follows: Let R be a ring, and m and n be two positive integers with $1 \leq n < m$. A proper ideal I of R is called an *(m,n)-closed ideal* of R if whenever $a^m \in I$ for some $a \in R$ implies $a^n \in I$. Later Anderson et al. introduced in [2] the concept of a weakly (m,n)-closed ideal. According to [2], a proper ideal I of R is called a weakly (m,n)-closed ideal of R if whenever $0 \neq a^m \in I$ for some $a \in R$ implies $a^n \in I$. In [3], the authors studied the notions of (m,n)-closed ideals in the trivial ring extension. Let A be a commutative ring and E be an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R := A(+)E$ whose underlying group is $A \times E$ with multiplication given by $(a,e)(a',e') = (aa', ae'+a'e)$. Trivial ring extension have been studied extensively. Considerable work, part of which is summarized in Huckaba's book [11], has been concerned with trivial ring extensions; these extensions have been useful for solving many open problems and conjectures. We recall that if I is a

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proper ideal of A, then $I(+)E$ is an ideal of $A(+)E$. And if F is a submodule of E such that $IE \subseteq F$, then $I(+)F$ is an ideal of $A(+)E$.

Let (A, B) be a pair of rings, $f : A \longrightarrow B$ be a ring homomorphism and J be an ideal of B. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) | a \in A, j \in J\}$$

called *the amalgamation of A and B along J with respect to f*, introduced and studied by D'Anna, Finocchiaro and Fontana in [4–6]. In particular, they have studied amalgamations in the frame of pull-backs which allowed them to establish numerous(prime) ideals and ring-theoretic basic properties for this new construction. This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontanna in [7, 8]. The interest of amalgamation resides, partly, in its stability to cover several basic constructions in commutative algebra, including pull-back and trivial ring extensions (also called Nagat's idealizations)(cf. [13, page 2])

In [12], Issoual et al. studied the concept of an (m, n)-closed ideal in amalgamated algebras along an ideal. In this paper, we will study a new type of concept of ideals called (m, n)-closed δ-primary and weakly (m, n)-closed δ-primary ideals, by continuing the study already made in the [9, 10].

Let $Id(A)$ be the set of all ideals of R. Zhao [14] introduced the concept of expansion of ideals of R. We recall from [14] that a function $\delta : Id(A) \longrightarrow Id(A)$ is called an expansion function of the ideals of R if whenever L, I, J are ideals of R with $J \subseteq I$, then $L \subseteq \delta(L)$ and $\delta(I) \subseteq \delta(I)$. Recall from [14] that a proper ideal of R is said to be δ-primary ideal of R if $a, b \in R$ with $ab \in I$ implies $a \in I$ or $b \in \delta(I)$, where δ is an expansion function of ideals of R. The concept of a δ-primary ideal was extended to the context of an (m, n)-closed-δ-primary ideal. Recall from [9] that a proper ideal I of R is said to be an (m, n)-closed-δ-primary ideal of R if whenever $a^m \in I$ for some $a \in R$, then $a^n \in \delta(I)$. In [10], the authors studied the concept of a weakly (m, n)-closed-δ-primary ideal. In this paper, we study the notion of a weakly (m, n)-closed-δ-primary ideal in the amalgamation of A with B along an ideal J with respect to f denoted by $A \bowtie^f J$.

2. On some (m, n)-closed δ-primary ideals of amalgamation $A \bowtie^f J$

To avoid unnecessary repetition, let us fix the notation for the rest of the paper. Let $f : A \longrightarrow B$ be a ring homomorphism and J be an ideal of B. All along this paper, $A \bowtie^f J$ will denote the amalgamation of A and B along J with respect to f. Let I be an ideal of A and K be an ideal of $f(A) + J$. Notice that $I \bowtie^f J := \{(i, f(i) + j) | i \in I, j \in J\}$ and $\overline{K}^f := \{(a, f(a) + j) | a \in A, j \in J, f(a) + j \in K\}$ are ideals of $A \bowtie^f J$. Our first result gives a necessary and sufficient condition for the ideals $I \bowtie^f J$ and $\overline{K}^f$ to be (m, n)-closed-δ-primary ideals of $A \bowtie^f J$, for all positive integers m and n, with $1 \leq n < m$ and an arbitrary expansion δ.

Now, let $\delta : Id(A) \to Id(A)$ and $\delta_1 : Id(f(A)+J) \to Id(f(A)+J)$ are expansion functions of $Id(A)$ and $Id(f(A)+J)$, respectively. We define a function $\delta_{\bowtie^f} : Id(A \bowtie^f J) \longrightarrow Id(A \bowtie^f J)$ such that

$$\delta_{\bowtie^f}(I \bowtie^f J) = \delta(I) \bowtie^f J$$

for every ideal I of A and

$$\delta_{\bowtie^f}(\overline{K}^f) = \{(a, f(a)+j) | a \in A, j \in J, f(a)+j \in \delta_1(K)\}$$

for every ideal K of $f(A)+J$. Then, $\delta_{\bowtie^f}$ is an expansion function of ideals of $A \bowtie^f J$.

Recall the notions of an (m,n)-closed δ-primary ideal defined in the introduction. According to [9], a proper ideal I of a commutative ring R is said to be an (m,n)-closed δ-primary ideal of R if $a^m \in I$ implies that $a^n \in \delta(I)$ for each $a \in A$, where m and n, with $m > n$ are positive integers.

Our first result gives a necessary and sufficient condition for the ideals $I \bowtie^f J$ and $\overline{K}^f$ to be (m,n)-closed δ-primary ideals of $A \bowtie^f J$ for all positive integers m and n, with $1 \leq n < m$ and an arbitrary expansion δ.

Theorem 2.1. *Under the above notations, the following statements hold.*

(1) *$I \bowtie^f J$ is an (m,n)-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$ if and only if I is an (m,n)-close δ-primary ideal of A.*

(2) *$\overline{K}^f$ is an (m,n)-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$ if and only if K is an (m,n)-closed δ_1-primary ideal of $f(A)+J$.*

Proof. (1) Suppose that I is an (m,n)-closed δ-primary ideal of A and let $(a, f(a)+j)^m \in I \bowtie^f J$ for some $(a, f(a)+j) \in A \bowtie^f J$. Then, $a^m \in I$. The fact that I is an (m,n)-closed δ-primary ideal of A, gives $a^n \in \delta(I)$, which implies $(a, f(a)+j)^n \in \delta(I) \bowtie^f J = \delta_{\bowtie^f}(I \bowtie^f J)$. Hence, $I \bowtie^f J$ is an (m,n)-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$. Conversely, suppose that $I \bowtie^f J$ is an (m,n)-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$. Let $a^m \in I$ for some $a \in A$. Then, $(a, f(a))^m \in I \bowtie^f J$ which implies $(a, f(a))^n \in \delta_{\bowtie^f}(I \bowtie^f J) = \delta(I) \bowtie^f J$. Then, $a^n \in \delta(I)$. Hence, I is an (m,n)-closed δ-primary ideal of A, as desired

(2) Suppose that $\overline{K}^f$ is an (m,n)-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$. We claim that K is an (m,n)-closed δ_1-primary ideal of $f(A)+J$. Indeed, let $(f(a)+j)^m \in K$ with $f(a)+j \in f(A)+J$. Thus, $(a, f(a)+j)^m \in \overline{K}^f$. Since $\overline{K}^f$ is an (m,n)-closed $\delta_{\bowtie^f}$-primary ideal, so $(a, f(a)+j)^n \in \delta_{\bowtie^f}(\overline{K}^f) = \{(a, f(a)+j) |\ a \in A, j \in J,\ f(a)+j \in \delta_1(K)\}$. Therefore, $(f(a)+j)^n \in \delta_1(K)$. Hence, K is an (m,n)-closed δ_1-primary ideal of $f(A)+J$. Conversely, assume that K is an (m,n)-closed δ_1-primary ideal of $f(A)+J$. Let $(a, f(a)+j)^m \in \overline{K}^f$ with $(a, f(a)+j) \in A \bowtie^f J$. Obviously, $f(a)+j \in f(A)+j$ and $(f(a)+j)^m \in K$, which is an (m,n)-closed δ_1-primary ideal. So, $(f(a)+j)^n \in \delta_1(K)$, which implies $(a, f(a)+j)^n \in \delta_{\bowtie^f}(\overline{K}^f)$. Hence, $\overline{K}^f$ is an (m,n)-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$, as desired. $\square$

Let I be a proper ideal of A. The (amalgamated) duplication of A along I is a special amalgamation given by

$$A \bowtie I := A \bowtie^{id_A} I = \{(a, a+i) | a \in A, i \in I\}.$$

Let J and K be two proper ideals of A, we recall that $J \bowtie I := \{(a, a+i) | a \in A, i \in I\}$ and $\overline{K} := \{(a, a+i) | a \in A, i \in I, a+i \in K\}$ are ideals of $A \bowtie I$.

Now, let $\delta : Id(A) \to Id(A)$ be an expansion function of $Id(A)$. We define a function $\delta_{\bowtie} : Id(A \bowtie I) \longrightarrow Id(A \bowtie I)$ such that

$$\delta_{\bowtie}(J \bowtie I) = \delta(J) \bowtie^f I$$

for every ideal J of A and

$$\delta_{\bowtie}(\overline{K}) = \{(a, a+i) |\ a \in A, i \in I, a+i \in \delta(K)\}$$

for every ideal K of A. Then, $\delta_{\bowtie}$ is an expansion function of ideals of $A \bowtie I$.

The next corollary is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Let A be ring and I be an ideal of A. Consider K an ideal of A. Then the following statements hold*:

(1) *$J \bowtie I$ is an (m, n)-closed $\delta_{\bowtie}$-primary ideal of $A \bowtie I$ if and only if I is an (m, n)-closed δ-primary ideal of A.*
(2) *$\overline{K}$ is an (m, n)-closed $\delta_{\bowtie}$-primary ideal of $A \bowtie I$ if and only if K is an (m, n)-closed δ-primary ideal of A.*

Example 2.3. The Nagat's idealization can be interpreted as a particular case of the general amalgamation construction. Let A be a ring, M be an A-module and $B = A(+)M$ be the trivial extension of A by M. Let I be an ideal of A, and set $J = 0(+)M$. Consider the ring homomorphism $f : A \longrightarrow B$ defined by $f(a) = (a, 0)$. The, the ring $A \bowtie^f J$ coincides with the amalgamation $A \bowtie^f J$. Then $I(+)M$ is an (m, n)-closed δ_f-primary ideal of $A(+)M$ if and only if I is an (m, n)-closed δ-primary ideal of A.

3. On some weakly (m, n)-closed δ-primary ideal of $A \bowtie^f J$

Recall the notions of a weakly (m, n)-closed δ-primary ideal defined in the introduction. According to [10], a proper ideal I of a commutative ring R is said to be a *weakly (m, n)-closed δ-primary ideal* of R if $0 \neq a^m \in I$ implies that $a^n \in \delta(I)$ for each $a \in A$, where m and n, with $m > n$ are positive integers.

Definition. Let R be a commutative ring and I be a proper ideal of R. Let $a \in R$. We say that a is a *δ-(m, n)-unbreakable element* of I if $a^m = 0$ and $a^n \notin \delta(I)$.

Remark 3.1. If I is a weakly (m, n)-closed δ-primary ideal without δ-(m, n)-unbreakable element, then I is an (m, n)-closed δ-primary ideal of R.

Our goal is to study the necessary and sufficient conditions for $I \bowtie^f J$ and $\overline{K}$ to be weakly (m, n)-closed δ-primary ideals of $A \bowtie^f J$ that are not (m, n)-closed δ-primary ideals.

Theorem 3.2. *Under the above notations in Section 2, the following statements are equivalent*:

(1) *$I \bowtie^f J$ is a weakly (m,n)-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$ that is not (m,n)-closed $\delta_{\bowtie^f}$-primary.*

(2) *I is a weakly (m,n)-closed δ-primary ideal of A that is not (m,n)-closed δ-primary and for every δ-(m,n)-unbreakable-zero element a of I, we have $(f(a)+j)^m = 0$ for every $j \in J$.*

Proof. (1) $\Rightarrow$ (2). Assume that $I \bowtie^f J$ is a weakly (m,n)-closed δ-primary ideal of $A \bowtie^f J$. Let $0 \neq a^m \in I$. Then, $0 \neq (a, f(a))^m \in I \bowtie^f J$ which implies $(a, f(a))^n \in \delta_{\bowtie^f}(I \bowtie^f J) = \delta(I) \bowtie^f J$. Thus, $a^n \in \delta(I)$ and consequently I is a weakly (m,n)-closed δ-primary ideal of A. On the other hand, by the Theorem 2.1, we have I is not an (m,n)-closed δ-primary ideal of A. So, there exists a δ-(m,n)-unbreakable-zero element a of I. We will show that $(f(a)+j)^m = 0$ for every j in J. By the way of contradiction, suppose that $0 \neq (f(a)+j)^m$ for some $j \in J$. Then, $0 \neq (a, f(a)+j)^m \in I \bowtie^f J$. As $I \bowtie^f J$ is a weakly (m,n)-closed $\delta_{\bowtie^f}$-primary, we get $(a, f(a)+j)^n \in \delta_{\bowtie^f}(I \bowtie^f J)$, which implies $a^n \in \delta(I)$. As desired contradiction.

(2) $\Rightarrow$ (1). Let $0 \neq (a, f(a)+j)^m \in I \bowtie^f J$ for some $a \in A$ and $j \in J$. Then, $a^m \in I$. If $0 \neq a^m$, as I is a weakly (m,n)-closed δ-primary ideal, we get $a^n \in \delta(I)$. Hence, $(a, f(a)+j)^n \in \delta_{\bowtie^f}(I \bowtie^f J)$. Now assume that $a^m = 0$, necessarily $a^n \in \delta(I)$. Suppose on the contrary that $a^n \notin \delta(I)$, then a is a δ-(m,n)-unbreakable-zero element of I. So, by assumption we have $(f(a)+j)^m = 0$. This implies $(a, f(a)+j)^m = 0$, which is a contradiction. Hence, $I \bowtie^f J$ is a weakly (m,n)-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$. $\square$

Corollary 3.3. *Suppose that $char(f(A)+J) = m$ and $J^m = 0$. Then the following statements are equivalent*:

(1) *$I \bowtie^f J$ is a weakly (m,n)-closed δ-primary ideal of $A \bowtie^f J$ that is not (m,n)-closed δ-primary.*

(2) *I is a weakly (m,n)-closed δ-primary ideal of A that is not (m,n)-closed δ-primary.*

Proof. It suffices to show that $(f(a)+j)^m = 0$ for each (m,n)-δ-unbreakable element a of I. Indeed, for every $j \in J$, we have

$$(f(a)+j)^m = f(a^m) + \sum_{k=1}^{m-1} \binom{m}{k} f(a^{m-k}) j^k + j^m.$$

As $J^m = 0$, we got $j^m = 0$ for every $j \in J$. On the other hand, we have $\binom{m}{k} = 0$ since $char(f(A)+j) = m$ and m divides $\binom{m}{k}$. By using the Binomial theorem, it follows that $(f(a)+j)^m = 0$. Now, the result follows from the Theorem 3.2. $\square$

Corollary 3.4. *Let A be a commutative ring, and I be a proper ideal of A. Let K be a proper ideal of A. Then the following statements are equivalent*:

(1) $K \bowtie I$ *is a weakly* (m,n)*-closed* $\delta_{\bowtie}$*-primary ideal of* $A \bowtie I$ *which is not* (m,n)*-closed* $\delta_{\bowtie}$*-primary.*
(2) K *is a weakly* (m,n)*-closed* δ*-primary ideal of* A *which is not* (m,n)*-closed* δ*-primary and* $(a+i)^m = 0$ *for every* δ*-*(m,n)*-unbreakable element* a *of* K *and every element* $i \in I$.

Proof. Take $A = B$ and $f = id_A$ in Theorem 3.2, where id_A is the identity map $id_A : A \to A$. □

Corollary 3.5. *Let* I *be a proper ideal of* A *and* $m \geq n > 0$ *be two positive integers. Then the following are equivalent*:

(1) $I \bowtie^f J$ *is a weakly* (m,n)*-closed ideal of* $A \bowtie^f J$ *that is not* (m,n)*-closed.*
(2) I *is a weakly* (m,n)*-closed ideal of* A *that is not* (m,n)*-closed and for every* (m,n)*-unbreakable element* a *of* I*, we have* $(f(a)+j)^m = 0$ *for each* $j \in J$.

Proof. It suffices to check $\delta = id_{Id(A)}$ in the Theorem 3.2. □

Corollary 3.6. *Let* A *be a commutative ring,* M *be an* A*-module. Let* I *be a proper ideal of* A*. Then the following statements are equivalent*:

(1) $I(+)M$ *is a weakly* (m,n)*-closed* $\delta_{(+)}$*-primary ideal of* $A(+)M$ *that is not* (m,n)*-closed* $\delta_{(+)}$*-primary.*
(2) I *is a weakly* (m,n)*-closed* δ*-primary ideal of* A *that is not* (m,n)*-clo-sed* δ*-primary and* $m(a^{m-1})M = 0$ *for every* δ*-*(m,n)*-unbreakable element* a *of* I.

Proof. Let $f : A \to B$ be the canonical homomorphism defined by $f(a) = (a,0)$ for every $a \in A$ and $J := 0 \propto M$. It not difficult to check that $A \propto E$ is naturally isomorphic to $A \bowtie^f J$, and the ideal $I \bowtie^f J$ is canonically isomorphic to $I \propto M$. By Theorem 3.2, we have $I(+)M$ is a weakly (m,n)-closed $\delta_{(+)}$-primary ideal of $A(+)M$ which is not (m,n)-closed $\delta_{(+)}$-primary if and only if I is a weakly (m,n)-closed δ-primary ideal of A which is not (m,n)-closed δ-primary and for every (m,n)-unbreakable element a of I, we have $(f(a)+j)^m = 0$ for every $j \in J = (0)(+)M$. Now, if $x \in M$, then $((a,0)+(0,x))^m = (a^m, ma^{m-1}x) = 0$, thus $m(a^{m-1}x) = 0$. Hence, $m(a^{m-1}M) = 0$. □

Remark 3.7. If I is a weakly (m,n)-closed δ-primary ideal of A, then $I \bowtie^f B$ need not to be a weakly (m,n)-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$. Let $A = Z_8$. Then, $I = \{0\}$ is clearly a weakly $(3,1)$-closed $\delta_{\sqrt{I}}$-primary ideal that not $(3,1)$-closed $\delta_{\sqrt{I}}$-primary, since $2^3 \in \{0\}$ but $2^2 \notin \{0\}$. Let M be an A-module and set $J := 0(+)M$. Let $f : A \hookrightarrow A(+)M$ be the canonical homomorphism defined by $f(a) = (a,0)$ for every $a \in A$. It is clear to see that $0 \bowtie^f J$ is isomorphic to $0(+)M$. Since $3(2^2)(0(+)M) \neq 0$ by the Corollary 3.6, we conclude that $0 \bowtie^f J$ is not a weakly $(3,1)$-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$.

The next result provides the necessary and sufficient conditions for $\overline{K}$ to be weakly (m,n)-closed δ-primary ideals of $A \bowtie^f J$, which are not (m,n)-closed δ- primary ideals.

Theorem 3.8. *The following statements are equivalent.*

(1) *$\overline{K}^f$ is a weakly (m,n)-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$ which is not (m,n)-closed $\delta_{\bowtie^f}$-primary.*

(2) *K is a weakly (m,n)-closed δ_1-primary ideal of $f(A)+J$ which is not (m,n)-closed δ_1-primary and $a^m = 0$ for every δ-(m,n)-unbreakable-zero $f(a)+j$ of K.*

Proof. (1) $\Rightarrow$ (2). Suppose $\overline{K}^f$ is a weakly (m,n)-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$. We claim that K is a weakly (m,n)-closed δ_1-primary ideal of $f(A)+J$. Indeed, let $0 \neq (f(a)+j)^m \in K$ with $f(a)+j \in f(A)+J$. Then, $0 \neq (a, f(a)+j)^m \in \overline{K}^f$. Since $\overline{K}^f$ is a weakly (m,n)-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$, we have $(a, f(a)+j)^n \in \overline{K}^f$. Therefore, $f((a)+j)^n \in K$. Hence, K is a weakly (m,n)-closed δ_1-primary ideal of $f(A)+J$. By Theorem 2.1(2), K is not an (m,n)-closed δ_1-primary ideal of $f(A)+J$. Now, let $f(a)+j \in f(A)+J$ be an (m,n)-unbreakable-zero element of K. We claim that $a^m = 0$. Indeed if $0 \neq a^m$, we get $0 \neq (a, f(a)+j)^m \in \overline{K}^f$, then $(a, f(a)+j)^n \in \overline{K}^f$, which implies $(f(a)+j)^n \in K$. This is a contradiction. Hence, $a^m = 0$.

(2) $\Rightarrow$ (1). Suppose that K is a weakly (m,n)-closed δ_1-primary ideal of $f(A)+J$. Let $0 \neq (a, f(a)+j)^m \in \overline{K}^f$ for some $(a, f(a)+j) \in A \bowtie^f J$. Then, obviously, $f(a)+j \in f(A)+J$ and $(f(a)+j)^m \in K$. If $0 \neq (f(a)+j)^m$, as K is a weakly (m,n)-closed δ_1-primary ideal of $f(A)+J$, we get $(f(a)+j)^n \in K$, which implies $(a, f(a)+j)^n \in \overline{K}^f$. Now, if $(f(a)+j)^m = 0$. We claim that $(f(a)+j)^n \in K$, for if not, we get $f(a)+j$ is (m,n)-unbreakable-zero. Then by assumption, we have $a^m = 0$ and thus $(a, f(a)+j)^m = 0$. This is a contradiction. Thus, $(f(a)+j)^n \in K$. Therefore, $(a, f(a)+j)^n \in \overline{K}^f$. Hence, $\overline{K}^f$ is a weakly (m,n)-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$. On the other hand, by Theorem 2.1, we have $\overline{K}^f$ is not an (m,n)-closed $\delta_{\bowtie^f}$-primary ideal of $A \bowtie^f J$. $\square$

Let I be a proper ideal of A. The (amalgamated) duplication of A along I is a special amalgamation given by

$$A \bowtie I := A \bowtie^{id_A} I = \{(a, a+i) | a \in A, i \in I\}.$$

If K is an ideal of A, then $\overline{K} := \{(a, a+i) | a \in A, i \in I, a+i \in K\}$ is an ideal of $A \bowtie I$.

The next corollary is an immediate consequence of Theorem 3.2.

Corollary 3.9. *Let A be a ring and I be a proper ideal of A. Let K be a proper ideal of A. Then the following are equivalent*:

(1) *$\overline{K} := \{(a, a+i) | a \in A, i \in I, a+i \in K\}$ is a weakly (m,n)-closed $\delta_{\bowtie}$-primary ideal of $A \bowtie I$ that is not an (m,n)-closed δ-primary ideal.*

(2) *K is a weakly (m,n)-closed δ-primary ideal of A that is not an (m,n)-closed δ-primary ideal and $(a-i)^m = 0$ for every δ-(m,n) unbreakable element a of K and for each $i \in I$.*

Proof. (1) $\Longrightarrow$ (2). It follows from the Theorem 3.8 with $A = B$ and $f = id_A$. Now, if $a \in A$ is a δ-(m,n)-unbreakable element of K, that is $a^m = 0$ and $a^n \notin \delta(K)$, then $(a-i,a) \notin \delta_{\bowtie}(\overline{K})$. This necessarily implies that $(a-i)^m = 0$ for every element $i \in I$. This completes the proof of ((1) $\Longrightarrow$ (2)).

(2) $\Longrightarrow$ (1). It follows from the Theorem 3.8. □

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