Commun. Korean Math. Soc. **39** (2024), No. 3, pp. 563–574
https://doi.org/10.4134/CKMS.c230140
pISSN: 1225-1763 / eISSN: 2234-3024

THE CHOW RING OF A SEQUENCE OF POINT BLOW-UPS

Daniel Camazón Portela

Abstract. Given a sequence of point blow-ups of smooth $n-$dimensional projective varieties Z_i defined over an algebraically closed field k, $Z_s \xrightarrow{\pi_s} Z_{s-1} \xrightarrow{\pi_{s-1}} \cdots \xrightarrow{\pi_2} Z_1 \xrightarrow{\pi_1} Z_0$, with $Z_0 \cong \mathbb{P}^n$, we give two presentations of the Chow ring $A^\bullet(Z_s)$ of its sky. The first one uses the classes of the total transforms of the exceptional components as generators and the second one uses the classes of the strict transforms ones. We prove that the skies of two sequences of point blow-ups of the same length have isomorphic Chow rings. Finally we give a characterization of the final divisors of a sequence of point blow-ups in terms of some relations defined over the Chow group of zero-cycles $A_0(Z_s)$ of its sky.

1. Introduction

Sequences of blow-ups of smooth varieties along smooth centers are useful for general algebraic geometric purposes, in particular, for resolution and classification of singularities. We will assume, additionally, that the center of each blow up has normal crossings with the already created exceptional divisors by the precedent blow ups. This paper is devoted to explicitly compute the Chow ring of the variety obtained after a sequence of point blow-ups, with $Z_0 \cong \mathbb{P}^n$. In this case, the Chow ring is generated as a $\mathbb{Z}-$algebra by the exceptional divisors and the generic hyperplane.

The composition of the successive blow-ups of such sequences is projective (and therefore proper) birational morphisms $Z_s \to Z_0$, where Z_s and Z_0 are smooth algebraic varieties which are respectively called sky and ground. In this paper, we restrict to $Z_0 \cong \mathbb{P}^n$ and the case of sequences of point blow-ups. This special case appears in several geometric contexts, in particular, those related to the study of curves, hypersurfaces and vector fields (see [2], [3]).

Our main goal is to give an explicit presentation of the Chow ring of the sky of a sequence of point blow-ups as a finite type $\mathbb{Z}-$algebra. We give two presentations of the Chow ring of the sky of a sequence of point blow-ups $A^\bullet(Z_s)$ by considering respectively the total transforms and the strict transforms of

Received June 13, 2023; Revised April 6, 2024; Accepted May 28, 2024.
2020 *Mathematics Subject Classification.* Primary 14C15, 14C17, 14E05, 14N10.
Key words and phrases. Blow-ups, Chow ring, intersection theory.
This work was financially supported by PGC2018-096446-B-C21.

the irreducible components of the exceptional divisor as generators. We give polynomial generators of the ideal of relations for the total (respectively strict) exceptional components. As a corollary, we prove a surprising result: The Chow rings of the skies of two sequences of point blow-ups of the same length are isomorphic.

Moreover, an irreducible exceptional component E is called final if there are not blow-ups with center lying over the point producing E for the first time, see Definition 3.9. The presentation of the Chow ring in terms of the strict transforms of exceptional divisors allows us to characterize final components by polynomial conditions on the classes of the strict exceptional components, improving the conditions given in [1].

When the center C_j of the blow-up $\pi_j : Z_j \to Z_{j-1}$ has dimension bigger than 0 and it is regularly embedded in Z_{j-1}, then there exists a presentation of $A^\bullet(Z_j)$ if the restriction map $i_j^* : A^\bullet(Z_{j-1}) \to A^\bullet(C_j)$ is surjective (see [6]).

2. Some preliminaries on sequences of blow-ups

Fix an algebraically closed field k. Throughout this paper a variety will mean a reduced projective scheme over k.

Definition 2.1. A sequence of blow-ups over k is defined as a sequence of morphisms

$$Z_s \xrightarrow{\pi_s} Z_{s-1} \xrightarrow{\pi_{s-1}} \cdots \xrightarrow{\pi_2} Z_1 \xrightarrow{\pi_1} Z_0,$$

where Z_0 is a smooth $n-$dimensional projective variety and

(1) The morphism $\pi_i : Z_i \to Z_{i-1}$ is the blow-up at center $C_i \subset Z_{i-1}$ for $i = 1, \dots, s$. We denote by E_i^i to be the exceptional hypersurface of π_i, and for $j < i$ $E_j^i \subset Z_i$ the strict transform of $E_j^j \subset Z_j$ in Z_i,
(2) $codim(C_{i+1}, Z_i) \geq 2$ for $i = 0, \dots, s-1$,
(3) the center C_i has simple normal crossings with $\{E_1^{i-1}, E_2^i, \dots, E_{i-1}^{i-1}\}$, for $i = 1, \dots, s$.

We will refer to Z_0 and Z_s as the ground and the sky of the sequence of blow-ups respectively.

Along the paper we fix a sequence of blow ups $(Z_0, \dots, Z_s, \pi)$ as in Definition 2.1 and we set $\pi_{j,i} : Z_j \to Z_i$, for $j > i$, to be the composition $\pi_{j,i} = \pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_{j-1} \circ \pi_j$.

The centers C_i, in general, can have any dimension. We extend the well-known notion of proximity for point blow-ups.

Definition 2.2. We say that C_j is proximate to C_i, and write $C_j \to C_i$ if and only if $C_j \subset E_i^{j-1}$.

Note that, if C_j is proximate to C_i then $j > i$.

Remark 2.3. For $j > i$ we denote by E_i^{j*} the total transform of E_i^i by the morphism $\pi_{j,i} : Z_j \to Z_i$. By an abuse of notation $E_i^{i*} = E_i^i$. Note that by definition of the total transform, one has

$$E_i^{k*} = E_i^k + \sum_{j>i} p_{ij} E_j^{k*}$$

where $p_{ij} = 1$ if $i < j \le k$ and C_j is proximate to C_i and $p_{ij} = 0$ in any other case.

We can now define two free $\mathbb{Z}-$modules with basis $\left\{E_i^{k*}\right\}_{i=1}^k$ and $\left\{E_i^k\right\}_{i=1}^k$ respectively, and construct the change of basis matrix B_k

$$B_k = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -p_{12} & 1 & \ddots & \vdots & \vdots \\ \vdots & -p_{23} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & \vdots \\ -p_{1k} & -p_{2k} & \cdots & -p_{k-1k} & 1 \end{pmatrix} \tag{1}$$

Note that its inverse B_k^{-1} has also integer entries.

Consider just one of the blow-ups conforming the sequence:

$$\begin{array}{ccc} E_{\alpha+1}^{\alpha+1} & \xrightarrow{j_{\alpha+1}} & Z_{\alpha+1} \\ {\scriptstyle g_{\alpha+1}}\downarrow & & \downarrow{\scriptstyle \pi_{\alpha+1}} \\ C_{\alpha+1} & \xrightarrow{i_{\alpha+1}} & Z_\alpha \end{array}$$

for $\alpha \in \{0, 1, \ldots, s-1\}$.

The following result shows us the multiplication rules of $A^\bullet(Z_{\alpha+1})$ for any blow-up at some smooth center $C_{\alpha+1}$.

Proposition 2.4 ([4, Proposition 13.12]). *The Chow ring $A^\bullet(Z_{\alpha+1})$ is generated by $\pi_{\alpha+1}^* A^\bullet(Z_\alpha)$ and $j_{\alpha+1*}A^\bullet(E_{\alpha+1}^{\alpha+1})$ as an algebra, that is, by classes pulled back from Z_α and classes supported on $E_{\alpha+1}^{\alpha+1}$. The rules for multiplication are the following*:

$$\pi_{\alpha+1}^* x \cdot \pi_{\alpha+1}^* y = \pi_{\alpha+1}^*(x \cdot y) \text{ for } x, y \in A^\bullet(Z_\alpha) \tag{2}$$

$$\pi_{\alpha+1}^* x \cdot j_{\alpha+1*} t = j_{\alpha+1*}(t \cdot g_{\alpha+1}^* i_{\alpha+1}^* x) \text{ for } x \in A^\bullet(Z_\alpha), t \in A^\bullet(E_{\alpha+1}^{\alpha+1}) \tag{3}$$

$$j_{\alpha+1*} t \cdot j_{\alpha+1*} u = -j_{\alpha+1*}(t \cdot u \cdot \varsigma) \text{ for } t, u \in A^\bullet(E_{\alpha+1}^{\alpha+1}) \tag{4}$$

where $\varsigma = c_1(\mathcal{O}_{E_{\alpha+1}^{\alpha+1}}(1))$.

We identify $A^\bullet(Z_\alpha)$ as a subring of $A^\bullet(Z_{\alpha+1})$ by means of the ring homomorphism

$$\pi_{\alpha+1}^* : A^\bullet(Z_\alpha) \to A^\bullet(Z_{\alpha+1}).$$

Remark 2.5. We will denote by $e_i^{\alpha*}$ (resp. e_i^α) to be the class of $[E_i^{\alpha*}]$ (resp. $[E_i^\alpha]$) in $A^1(Z_\alpha)$ for $i = 1, \ldots, \alpha$.

In the next section we will give a presentation of the Chow ring $A^\bullet(Z_\alpha)$ as a $\mathbb{Z}-$algebra of finite type, in the particular case where $Z_0 \cong \mathbb{P}^n$.

3. Main results

Note that Proposition 2.4 does not give a presentation of $A^\bullet(Z_{\alpha+1})$ as a $A^\bullet(Z_\alpha)-$algebra, but only states the rules of multiplication.

If we could find generators of $A^\bullet(E_{\alpha+1}^{\alpha+1})$ as a $\mathbb{Z}-$algebra, $\{\gamma_1, \dots, \gamma_r\} \in A^\bullet(E_{\alpha+1}^{\alpha+1})$, then

$$A^\bullet(Z_{\alpha+1}) \cong A^\bullet(Z_\alpha)\left[j_{\alpha+1*}(\gamma_1), \dots, j_{\alpha+1*}(\gamma_r)\right]$$

would be a $A^\bullet(Z_\alpha)-$algebra of finite type. One would like to have a presentation

$$A^\bullet(Z_{\alpha+1}) \cong A^\bullet(Z_\alpha)\left[w_1, \dots, w_r\right]/\mathcal{J}$$

by sending w_i to $j_{\alpha+1*}(\gamma_i)$, with an explicit description of the ideal $\mathcal{J}$. The ideal $\mathcal{J}$ will be computed in Theorems 3.3 and 3.6. We will restrict ourselves to the case of sequences of point blow-ups, that is $C_\alpha = P_\alpha$, with the ground variety $Z_0 \cong \mathbb{P}^n$. By [4, Theorem 2.1.], $A^\bullet(Z_0) \cong \mathbb{Z}[u]/(u^{n+1})$, by sending u to h, where $h \in A^1(Z_0)$ is the rational equivalence class of any hyperplane $[H]$ in $\mathbb{P}^n$, and for all α $A^\bullet(E_\alpha^\alpha) \cong \mathbb{Z}[w]/(w^n)$ by sending w to ς_α, where $\varsigma_\alpha \in A^1(E_\alpha^\alpha)$ is the rational class of any hyperplane.

In the case of sequences of point blow-ups, we are able to give generators of the Chow ring $A^\bullet(Z_s)$ as a $\mathbb{Z}-$algebra.

Lemma 3.1. *The Chow ring of the sky $A^\bullet(Z_s)$ is generated by $\{h^{s*}, \{e_i^{s*}\}_{i=1}^s\}$ as a $\mathbb{Z}-$algebra.*

Proof. This follows by induction on α. It is clear that $A^\bullet(Z_0)$ is generated by $\{h\}$. Let us suppose that $A^\bullet(Z_\alpha)$ is generated by $\{h^{\alpha*}, \{e_i^{\alpha*}\}_{i=1}^\alpha\}$. Now by Proposition 2.4 and due to the fact that $E_{\alpha+1}^{\alpha+1} \cong \mathbb{P}^{n-1}$, that is $A^\bullet(E_{\alpha+1}^{\alpha+1}) \cong \mathbb{Z}[t]/(t^n)$, by sending t to $\varsigma_{\alpha+1}$, with $\varsigma_{\alpha+1}$ the rational equivalence class of any hyperplane in $\mathbb{P}^{n-1}$, and $e_{\alpha+1}^{\alpha+1*} \cdot e_{\alpha+1}^{\alpha+1*} = -j_{\alpha+1*}(\varsigma_{\alpha+1})$ by (4) then $A^\bullet(Z_{\alpha+1})$ is generated by $\{h^{\alpha+1*}, \{e_i^{\alpha+1*}\}_{i=1}^{\alpha+1}\}$ as a $\mathbb{Z}-$algebra. $\square$

Remark 3.2. It makes sense then to define the augmented free $\mathbb{Z}-$modules with basis $\{H^{k*}, \{E_i^{k*}\}_{i=1}^k\}$ and $\{\widetilde{H}^k, \{E_i^k\}_{i=1}^k\}$ and the augmented change of basis matrix B_k^*

$$B_k^* = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \vdots & \vdots \\ 0 & -p_{12} & 1 & \ddots & \vdots & \vdots \\ \vdots & -p_{13} & -p_{23} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & \vdots \\ 0 & -p_{1k} & -p_{2k} & \cdots & -p_{k-1k} & 1 \end{pmatrix} \tag{5}$$

and its inverse B_k^{*-1}.

Theorem 3.3. *The Chow ring of the sky $A^\bullet(Z_s)$ is isomorphic to*

$$A^\bullet(Z_s) \cong \mathbb{Z}[x_0, x_1, \ldots, x_s] / (\{x_i \cdot x_j\}_{\substack{i,j=0 \\ i \neq j}}^s, \{(-1)^n (x_i)^n + (x_0)^n\}_{i=1}^s), \tag{6}$$

by sending x_0 to the class h^{s*} and x_i to the class e_i^{s*} for $i = 1, \ldots, s$.

Proof. By Lemma 3.1 there exists a surjective morphism

$$\phi : \mathbb{Z}[x_0, x_1, \ldots, x_s] \to A^\bullet(Z_s),$$

such that $\phi(x_0) = h^{s*}$ and $\phi(x_i) = e_i^{s*}$ for $i = 1, \ldots$. Firstly we will prove that

$$\mathcal{J} := \langle \{x_i \cdot x_j\}_{\substack{i,j=0 \\ i \neq j}}^s, \{(-1)^n (x_i)^n + (x_0)^n\}_{i=1}^s \rangle \subset Ker(\phi).$$

To begin with, let us express the classes of the basis $\{E_i^{\alpha+1*}\}_{i=1}^{\alpha+1}$ in terms of the classes of the basis $\{E_i^{\alpha+1}\}_{i=1}^{\alpha+1}$, that is, since

$$e_i^{\alpha*} = e_i^\alpha + \sum_{j=i+1}^{\alpha} b_{j,i} e_j^\alpha,$$

then

$$e_i^{\alpha+1*} = e_i^{\alpha+1} + \sum_{j=i+1}^{\alpha} b_{j,i} e_j^{\alpha+1} + (\sum_{j=i}^{\alpha} p_{j\alpha+1} b_{j,i}) e_{\alpha+1}^{\alpha+1}$$

where $b_{j,i}$ denotes the coefficients of the augmented change of basis matrix B_α^{*-1}.

If we denote by $\varsigma_{\alpha+1} \in A^1(E_{\alpha+1}^{\alpha+1})$ the class of any hyperplane in $E_{\alpha+1}^{\alpha+1}$ then we have the following intersection products

$$\begin{cases} e_{\alpha+1}^{\alpha+1} \cdot e_{\alpha+1}^{\alpha+1} = -j_{\alpha+1*}(\varsigma_{\alpha+1}), & (7) \\ e_j^{\alpha+1} \cdot e_{\alpha+1}^{\alpha+1} = j_{\alpha+1*}(\varsigma_{\alpha+1}) \quad \text{if } P_{\alpha+1} \to P_j, & (8) \\ e_j^{\alpha+1} \cdot e_{\alpha+1}^{\alpha+1} = 0 \quad \text{otherwise}, & (9) \end{cases}$$

where equation (7) follows from equation (4) and equation (8) is a direct consequence of [5, Corollary 6.7.1], that is $\pi_{\alpha+1}^*(e_j^\alpha) = e_j^{\alpha+1} + e_{\alpha+1}^{\alpha+1}$, and equations (3) and (7). So the following intersection product is 0

$$(e_j^{\alpha+1} + p_{j\alpha+1} e_{\alpha+1}^{\alpha+1}) \cdot e_{\alpha+1}^{\alpha+1} = 0, \tag{10}$$

and we can conclude that

$$e_i^{\alpha+1*} \cdot e_{\alpha+1}^{\alpha+1*} = (e_i^{\alpha+1} + \sum_{j=i+1}^{\alpha} b_{j,i} e_j^{\alpha+1} + (\sum_{j=i}^{\alpha} p_{j\alpha+1} b_{j,i}) e_{\alpha+1}^{\alpha+1}) \cdot e_{\alpha+1}^{\alpha+1} = 0.$$

On the other hand, $h^{\alpha+1*} \cdot e_{\alpha+1}^{\alpha+1} = 0$ is a consequence of the moving lemma (see [5, 11.4 Moving Lemma]). If we make the pull back through $\pi_{s,\alpha+1}^*$ for all α, then it follows that

$$\langle \{x_i \cdot x_j\}_{\substack{i,j=0 \\ i \neq j}}^s \rangle \subset ker(\phi).$$

By [5, Example 16.1.11], $A_0(Z_0)$ is a birational invariant, that is $A_0(Z_i) \cong \mathbb{Z}(h^{i*})^n$ for $i = 1, \ldots, s$, so since $(e_{\alpha+1}^{\alpha+1})^n = (-1)^{n-1} j_{\alpha+1*}(\varsigma_{\alpha+1}^n)$ then $(e_{\alpha+1}^{\alpha+1})^n = (-1)^{n-1}(h^{\alpha+1*})^n$, and by making the pull back through $\pi_{s,\alpha+1}^*$ we conclude that $\langle\{(-1)^n(x_i)^n + (x_0)^n\}_{i=1}^s\rangle \subset Ker(\phi)$.

Now we will prove that $Ker(\phi) \subset \mathcal{J}$. Note that $\phi : \mathbb{Z}[x_0, x_1, \ldots, x_s] \to A^\bullet(Z_s)$ is homogenous, so $ker(\phi)$ is an homogenous ideal, and $\mathcal{J}$ is an homogenous ideal too by construction. Let us suppose that $P\,[x] \in Ker(\phi)/\mathcal{J}$ with $deg(P) = \eta$. Then $2 \leq \eta \leq n$, since $\left\{x_i^{n+1}\right\}_{i=0}^s \in \mathcal{J}$, and $P\,[x]$ must be of the form $P\,[x] = \sum_{i=0}^s a_i x_i^\eta \mod (\mathcal{J})$, since $\{x_i \cdot x_j\}_{\substack{i,j=0\\ i\neq j}}^s \in \mathcal{J}$. Now if $\eta < n$, then $x_i^{n-\eta} P\,[x]$ will be also in $Ker(\phi)$, so $\phi(x_i^{n-\eta} P\,[x]) = a_i(e_i^{s*})^n = 0$, and since $(e_i^{s*})^n \neq 0$ then $a_i = 0$ for $i = 0, 1, \ldots, s$. If $\eta = n$, since $\{(-1)^n(x_i)^n + (x_0)^n\}_{i=1}^s \in Ker(\phi)$ then it follows that $a_0 + (-1)^{n+1}\sum_{i=1}^s a_i = 0$, so $P\,[x] = 0 \mod (\mathcal{J})$. □

Remark 3.4. Note that

$$\langle x_0, x_1, \ldots, x_s\rangle Ker(\phi) = \langle \{x_i x_j x_k\}_{\substack{i,j,k=0\\ i\neq j\\ j\neq k}}^s, \left\{x_i^{n+1}\right\}_{i=0}^s\rangle,$$

so we have that $Ker(\phi)/\langle x_0, x_1, \ldots, x_s\rangle Ker(\phi)$ is a free $\mathbb{Z}-$module of finite rank $\binom{n+1}{2} + n$. Any set of generators of the ideal $Ker(\phi)$ is a set of generators of $Ker(\phi)/\langle x_0, x_1, \ldots, x_s\rangle Ker(\phi)$ as $\mathbb{Z}-$module, so

$$\{\{x_i \cdot x_j\}_{\substack{i,j=0\\ i\neq j}}^s, \{(-1)^n(x_i)^n + (x_0)^n\}_{i=1}^s\}$$

is a minimal set of generators for $Ker(\phi)$.

Corollary 3.5. *Given two sequences of point blow-ups* $(Z_0, \ldots, Z_s, \pi)$ *and* $(Z_0', \ldots, Z_{s'}', \pi')$, *if* $s = s'$ *then* $A^\bullet(Z_s) \cong A^\bullet(Z_{s'}')$.

Proof. It follows directly from equation (6) in Theorem 3.3. □

We can use $\{\widetilde{h}^s, \{e_i^s\}_{i=1}^s\}$ as generators of the Chow ring $A^\bullet(Z_s)$ as $\mathbb{Z}-$algebra instead.

Theorem 3.6. *A presentation of* $A^\bullet(Z_s)$ *using* $\{\widetilde{h}^s, \{e_i^s\}_{i=1}^s\}$ *as generators is the following one*:

$$A^\bullet(Z_s) \cong \frac{\mathbb{Z}\,[y_0, y_1, \ldots, y_s]}{\mathcal{A}}, \tag{11}$$

where

$$\begin{aligned}\mathcal{A} = ((\{y_0 \cdot y_i\}_{i=1}^s, &\{(y_i + \sum_{k=i+1}^s b_{k,i} y_k) \cdot (y_j + \sum_{l=j+1}^s b_{l,j} y_l)\}_{\substack{i,j=1\\ i\neq j}}^s,\\ &\{(y_i)^n + ((-1)^n + \#\{j\}_{j\to i})(y_0)^n\}_{i=1}^s))\end{aligned} \tag{12}$$

by sending y_0 *to* $\widetilde{h}^s$ *and* y_i *to* e_i^s *for* $i = 1, \ldots, s$.

Proof. In this case there exists a surjective morphism

$$\phi^{'} : \mathbb{Z}\left[y_0, y_1, \ldots, y_s\right] \to A^{\bullet}(Z_s)$$

with $\phi^{'}(y_0) = h^{s*}$ and $\phi^{'}(y_i) = e_i^s$ for $i = 1, \ldots, s$. Moreover we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z}\left[x_0, \ldots, x_s\right] & & \\ \rho \uparrow & \searrow^{\phi} & \\ \mathbb{Z}\left[y_0, \ldots, y_s\right] & \xrightarrow{\phi^{'}} & A^{\bullet}(Z_s) \end{array}$$

where $\rho : \mathbb{Z}\left[y_0, \ldots, y_s\right] \to \mathbb{Z}\left[x_0, \ldots, x_s\right]$ is the isomorphism induced by the augmentated change of basis matrix B_s^*, that is $\rho(y_0) = x_0$ and $\rho(y_i) = x_i - \sum_{j=i+1}^{s} p_{ij} x_j$. Now, by considering the following images through ρ:

$$\begin{aligned}
(13) \qquad & \rho((y_i)^n + ((-1)^n + \#\left\{j\right\}_{j \to i})(y_0)^n) \\
&= (x_i - \sum_{k=i+1}^{s} p_{ik} x_k)^n + ((-1)^n + \#\left\{j\right\}_{j \to i})(x_0)^n \\
&= (x_i)^n + (-1)^n \sum_{k=i+1}^{s} p_{ik}(x_k)^n + ((-1)^n + \#\left\{j\right\}_{j \to i})(x_0)^n \\
&\quad + \sum_{\substack{n_i + n_{i+1} + \ldots + n_s = n \\ n_i, \ldots, n_s \neq n}} (-1)^{n - n_i} \binom{n}{n_i, n_{i+1} \ldots, n_s} \prod_{\beta = i}^{s} (p_{i\beta} x_\beta)^{n_\beta} \\
&= (-1)^n((-1)^n (x_i)^n + (x_0)^n) + \sum_{k=i+1}^{s} p_{ik}((-1)^n (x_k)^n + (x_0)^n) \\
&\quad + \sum_{\substack{n_i + n_{i+1} + \ldots + n_s = n \\ n_i, \ldots, n_s \neq n}} (-1)^{n - n_i} \binom{n}{n_i, n_{i+1} \ldots, n_s} \prod_{\beta = i}^{s} (p_{i\beta} x_\beta)^{n_\beta},
\end{aligned}$$

$$(14) \qquad \rho(y_0 \cdot y_i) = x_0 \cdot (x_i - \sum_{k=i+1}^{s} p_{ik} x_k) = x_0 \cdot x_i - \sum_{k=i+1}^{s} p_{i,k} x_0 \cdot x_k,$$

$$(15) \qquad \rho((y_i + \sum_{k=i+1}^{s} b_{k,i} y_k) \cdot (y_j + \sum_{l=j+1}^{s} b_{l,j} y_l)) = x_i \cdot x_j,$$

we can conclude that $\mathcal{A} \subset Ker(\phi^{'})$. The inclusion $Ker(\phi^{'}) \subset \mathcal{A}$ is straightforward by Remark 3.4. □

The next examples illustrate some interesting consequences of Corollary 3.5. In particular, the first one shows how the presentation of the Chow ring of the sky of a sequence of point blow-ups in terms of the total transforms of the exceptional components fails to detect the proximity configuration of the sequence.

Example 3.7. Let us consider all possible proximity configurations for a sequence of point blow-ups of length 4 verifying that at least $P_{i+1} \to P_i$, that is

(1) P_1, $P_2 \to P_1$, $P_3 \to P_2$ and $P_4 \to P_3$,
(2) P_1, $P_2 \to P_1$, $P_3 \to \{P_1, P_2\}$ and $P_4 \to P_3$,
(3) P_1, $P_2 \to P_1$, $P_3 \to \{P_1, P_2\}$ and $P_4 \to \{P_2, P_3\}$,
(4) P_1, $P_2 \to P_1$, $P_3 \to \{P_1, P_2\}$ and $P_4 \to \{P_1, P_3\}$,
(5) P_1, $P_2 \to P_1$, $P_3 \to \{P_1, P_2\}$ and $P_4 \to \{P_1, P_2, P_3\}$,
(6) P_1, $P_2 \to P_1$, $P_3 \to P_2$ and $P_4 \to \{P_2, P_3\}$.

We can compute a presentation of the Chow ring of the skies of these 6 proximity configurations using both the total transforms of the exceptional components and the strict ones as generators. Firstly, we give the presentations in terms of the strict transforms:

(1) $A^\bullet(Z_4) \cong \mathbb{Z}[h, e_1, e_2, e_3, e_4]/\mathcal{A}_1$ where

$$\begin{aligned}\mathcal{A}_1 = (&\{h \cdot e_i\}_{i=1}^4, (e_1 + e_2) \cdot (e_2 + e_3), e_1 \cdot e_3, (e_2 + e_3) \cdot (e_3 + e_4),\\ &e_1 \cdot e_4, e_2 \cdot e_4, (e_3 + e_4) \cdot e_4, (e_1)^n, (e_2)^n, (e_3)^n,\\ &(-1)(e_4)^n + (h)^n) \text{ if } n \text{ is odd,}\end{aligned}$$

$$\begin{aligned}\mathcal{A}_1 = (&\{h \cdot e_i\}_{i=1}^4, (e_1 + e_2) \cdot (e_2 + e_3), e_1 \cdot e_3, (e_2 + e_3) \cdot (e_3 + e_4),\\ &e_1 \cdot e_4, e_2 \cdot e_4, (e_3 + e_4) \cdot e_4, (e_1)^n + 2(h)^n, (e_2)^n + 2(h)^n,\\ &(e_3)^n + 2(h)^n, (e_4)^n + (h)^n) \text{ if } n \text{ is even.}\end{aligned}$$

(2) $A^\bullet(Z_4) \cong \mathbb{Z}[h, e_1, e_2, e_3, e_4]/\mathcal{A}_2$ where

$$\begin{aligned}\mathcal{A}_2 = (&\{h \cdot e_i\}_{i=1}^4, (e_1 + e_2) \cdot (e_2 + e_3), (e_1 + e_3) \cdot (e_3 + e_4),\\ &(e_2 + e_3) \cdot (e_3 + e_4), e_1 \cdot e_4, e_2 \cdot e_4, (e_3 + e_4) \cdot e_4, (e_1)^n + (h)^n,\\ &(e_2)^n, (e_3)^n, (-1)(e_4)^n + (h)^n) \text{ if } n \text{ is odd,}\end{aligned}$$

$$\begin{aligned}\mathcal{A}_2 = (&\{h \cdot e_i\}_{i=1}^4, (e_1 + e_2) \cdot (e_2 + e_3), (e_1 + e_3) \cdot (e_3 + e_4),\\ &(e_2 + e_3) \cdot (e_3 + e_4), e_1 \cdot e_4, e_2 \cdot e_4, (e_3 + e_4) \cdot e_4, (e_1)^n + 3(h)^n,\\ &(e_2)^n + 2(h)^n, (e_3)^n + 2(h)^n, (e_4)^n + (h)^n) \text{ if } n \text{ is even.}\end{aligned}$$

(3) $A^\bullet(Z_4) \cong \mathbb{Z}[h, e_1, e_2, e_3, e_4]/\mathcal{A}_3$ where

$$\begin{aligned}\mathcal{A}_3 = (&\{h \cdot e_i\}_{i=1}^4, (e_1 + e_2) \cdot (e_2 + e_3 + 2e_4), (e_1 + e_3) \cdot (e_3 + e_4),\\ &(e_2 + e_3) \cdot (e_3 + e_4), e_1 \cdot e_4, (e_2 + e_4) \cdot e_4, (e_3 + e_4) \cdot e_4,\\ &(e_1)^n + (h)^n, (e_2)^n + (h)^n, (e_3)^n, (-1)(e_4)^n + (h)^n)\\ &\text{if } n \text{ is odd,}\end{aligned}$$

$$\mathcal{A}_3 = (\{h \cdot e_i\}_{i=1}^4, (e_1 + e_2) \cdot (e_2 + e_3 + 2e_4), (e_1 + e_3) \cdot (e_3 + e_4),$$

$(e_2+e_3)\cdot(e_3+e_4), e_1\cdot e_4, (e_2+e_4)\cdot e_4, (e_3+e_4)\cdot e_4,$
$(e_1)^n+3(h)^n, (e_2)^n+3(h)^n, (e_3)^n+2(h)^n, (e_4)^n+(h)^n)$
if n is even.

(4) $A^\bullet(Z_4)\cong\mathbb{Z}\left[h,e_1,e_2,e_3,e_4\right]/\mathcal{A}_4$ where

$$\begin{aligned}\mathcal{A}_4=(&\{h\cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3+e_4), (e_1+e_3)\cdot(e_3+e_4),\\ &(e_2+e_3)\cdot(e_3+e_4), (e_1+e_4)\cdot e_4, e_2\cdot e_4, (e_3+e_4)\cdot e_4,\\ &(e_1)^n+2(h)^n, (e_2)^n, (e_3)^n, (-1)(e_4)^n+(h)^n) \text{ if } n \text{ is odd,}\end{aligned}$$

$$\begin{aligned}\mathcal{A}_4=(&\{h\cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3+e_4), (e_1+e_3)\cdot(e_3+e_4),\\ &(e_2+e_3)\cdot(e_3+e_4), (e_1+e_4)\cdot e_4, e_2\cdot e_4, (e_3+e_4)\cdot e_4,\\ &(e_1)^n+4(h)^n, (e_2)^n+2(h)^n, (e_3)^n+2(h)^n, (e_4)^n+(h)^n)\\ &\text{if } n \text{ is even.}\end{aligned}$$

(5) $A^\bullet(Z_4)\cong\mathbb{Z}\left[h,e_1,e_2,e_3,e_4\right]/\mathcal{A}_5$ where

$$\begin{aligned}\mathcal{A}_5=(&\{h\cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3+2e_4), (e_1+e_3)\cdot(e_3+e_4),\\ &(e_2+e_3)\cdot(e_3+e_4), (e_1+e_4)\cdot e_4, (e_2+e_4)\cdot e_4, (e_3+e_4)\cdot e_4,\\ &(e_1)^n+2(h)^n, (e_2)^n+(h)^n, (e_3)^n, (-1)(e_4)^n+(h)^n)\\ &\text{if } n \text{ is odd,}\end{aligned}$$

$$\begin{aligned}\mathcal{A}_5=(&\{h\cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3+2e_4), (e_1+e_3)\cdot(e_3+e_4),\\ &(e_2+e_3)\cdot(e_3+e_4), (e_1+e_4)\cdot e_4, (e_2+e_4)\cdot e_4, (e_3+e_4)\cdot e_4,\\ &(e_1)^n+4(h)^n, (e_2)^n+3(h)^n, (e_3)^n+2(h)^n, (e_4)^n+(h)^n)\\ &\text{if } n \text{ is even.}\end{aligned}$$

(6) $A^\bullet(Z_4)\cong\mathbb{Z}\left[h,e_1,e_2,e_3,e_4\right]/\mathcal{A}_6$ where

$$\begin{aligned}\mathcal{A}_6=(&\{h\cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3+2e_4), e_1\cdot e_3,\\ &(e_2+e_3)\cdot(e_3+e_4), e_1\cdot e_4, (e_2+e_4)\cdot e_4, (e_3+e_4)\cdot e_4, (e_1)^n,\\ &(e_2)^n+(h)^n, (e_3)^n, (-1)(e_4)^n+(h)^n) \text{ if } n \text{ is odd,}\end{aligned}$$

$$\begin{aligned}\mathcal{A}_6=(&\{h\cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3+2e_4), e_1\cdot e_3,\\ &(e_2+e_3)\cdot(e_3+e_4), e_1\cdot e_4, (e_2+e_4)\cdot e_4, (e_3+e_4)\cdot e_4,\\ &(e_1)^n+2(h)^n, (e_2)^n+3(h)^n, (e_3)^n+2(h)^n, (e_4)^n+(h)^n)\\ &\text{if } n \text{ is even.}\end{aligned}$$

However, the presentations of the Chow ring of the skies of these 6 different proximity configurations coincide when considering the total transforms as generators:

$$A^\bullet(Z_4)\cong\mathbb{Z}\left[h^*,e_1^*,e_2^*,e_3^*,e_4^*\right]/\mathcal{A}, \tag{16}$$

where $\mathcal{A}=(\{h^*\cdot e_i^*\}_{i=1}^4, \{e_i^*\cdot e_j^*\}_{\substack{i,j=1\\ i\neq j}}^4, \{(-1)^n(e_i^*)^n+(h^*)^n\}_{i=1}^4)$.

Now, if we restrict ourselves to the study of sequences of point blow-ups with a fixed proximity configuration, the following example exhibits that even although the skies of two sequences may not be isomorphic, there will exist an isomorphism between their Chow rings.

Example 3.8. Let us consider all sequences of point blow-ups of length 5 with $Z_0 \cong \mathbb{P}^2$ and the following proximity configuration: P_1, $P_2 \to P_1$, $P_3 \to P_1$, $P_4 \to P_1$ and $P_5 \to P_1$. Then a presentation of the Chow ring of any of the skies of these sequences using the strict transforms of the exceptional components as generators is

$$A^\bullet(Z_5) \cong \mathbb{Z}\left[h, e_1, e_2, e_3, e_4, e_5\right]/\mathcal{B}, \tag{17}$$

where

$$\mathcal{B} = \left(\{h \cdot e_i\}_{i=1}^5, \{(e_1 + e_j) \cdot e_j\}_{j=2}^5, \{e_j \cdot e_k\}_{\substack{j,k=2\\ j\neq k}}^5, (e_1)^2 + 5(h)^2, \left\{(e_j)^2 + (h)^2\right\}_{j=2}^5\right).$$

Nonetheless, since $E_1^1 \cong \mathbb{P}^1$, it is clear that if we choose two sequences of point blow-ups as above such that the centers $\{P_2, P_3, P_4, P_5\}$ and $\{P_2', P_3', P_4', P_5'\}$ have different cross ratios, then the skies of the associated sequences will not be isomorphic but their Chow rings will do.

In Example 3.7 we can foresee that the proximity relations of a sequence of point blow-ups are encoded in some way in the presentation of the Chow ring of the sky when using the strict transforms of the exceptional components as generators. Now we formalize it.

Definition 3.9. We define for a given a sequence of blow-ups $(Z_0, \ldots, Z_s, \pi)$ that an irreducible component E_i is final if there not exists j such that P_j is proximate to P_i.

Remark 3.10. In [1] the definition of final divisor is different, but in the case of sequences of point blow-ups both definitions are equivalent.

Also, in [1], final divisors are characterized in terms of the intersection of the irreducible components of the exceptional divisor. We can now use Theorem 3.6 to refine the characterization.

Corollary 3.11. *E_i is final if and only if its class in $A^1(Z_s)$, that is e_i^s, satisfies the following two conditions*

$$\begin{cases} (e_i^s)^n = (-1)^r (e_i^s)^{n-r} (e_j^s)^r & (18)\\ (e_j^s)^{n-1} e_i^s = (h^{s*})^n & (19)\end{cases}$$

for every j such that $e_i^s \cdot e_j^s \neq 0$.

Proof. If E_i is final then there not exists k such that P_k is proximate to P_i. By equation (10) $(e^i_j + e^i_i) \cdot e^i_i = 0$ if P_i is proximate to P_j and $e^i_i \cdot e^i_j = 0$ otherwise. Since E_i is final then it follows that

$$\begin{cases} (e^s_j + e^s_i) \cdot e^s_i = 0 & \text{if } P_i \to P_j \quad (20) \\ e^s_i \cdot e^s_j = 0 & \text{otherwise} \quad (21) \end{cases}$$

From equation (20) we can deduce that $(e^s_i)^n = (-1)^r (e^s_i)^{n-r} (e^s_j)^r$. Moreover $(h^{s*})^n = (-1)^{n+1} (e^{s*}_i)^n$, so $(h^{s*})^n = (-1)^{2n} e^s_i (e^s_j)^{n-1} = e^s_i (e^s_j)^{n-1}$.

Now we will prove that if E_i is not final, then some of the above conditions fails. Among all the index $\{\beta\}$ satisfying $P_\beta \to P_i$ there must exist an index j such that $P_j \to P_i$ but that there not exists k with $P_k \to P_i$ and $P_k \to P_j$. Since E^j_j is final for the sequence $(Z_0, \ldots, Z_j, \pi_{j,0})$, then $(e^j_j) \cdot (e^j_i)^{n-1} = (h^{j*})^n$ and $(e^j_i)^{n-1-\beta} (e^j_j)^{1+\beta} = (-1)^\beta (e^j_i)^{n-1} e^j_j$. Moreover, since there not exists P_k with P_k proximate to both P_i and P_j, then we can conclude that $(e^s_j) \cdot (e^s_i)^{n-1} = (h^{s*})^n$ and $(e^s_i)^{n-1-\beta} (e^s_j)^{1+\beta} = (-1)^\beta (e^s_i)^{n-1} e^s_j$. If n is even, although $(e^s_j)^{n-1} e^s_i = (h^{s*})^n$ since $n-2$ is even too, $(e^s_i)^n \neq (-1)^{n-1} (e^s_i)(e^s_j)^{n-1}$ since by Theorem 3.6 $(e^s_i)^n = -(1 + \#\{\beta\})(h^{s*})^n$ with $\#\{\beta\} \geq 1$ so condition (18) fails.

If n is odd, $(e^s_j)^{n-1} e^s_i = -(h^{j*})^n$, since $n-2$ is odd too, so condition (19) fails. □

Some comments about the Chow ring of blow-ups at more general centers.

The main result in the literature about the structure of the Chow ring of a blow-up at a center of arbitrary dimension is the following one

Theorem 3.12 ([6, Appendix Theorem 1])**.** *Suppose the map of bivariant rings*

$$i^*_{\alpha+1} : A^\bullet(Z_\alpha) \to A^\bullet(C_{\alpha+1})$$

is surjective, then $A^\bullet(Z_{\alpha+1})$ is isomorphic to

$$A^\bullet(Z_\alpha)\,[T]\,/(P(T), (T \cdot Ker(i^*_{\alpha+1}))),$$

where $P(T) \in A^\bullet(Z_\alpha)\,[T]$ is any polynomial whose constant term is $[C_{\alpha+1}]$ and whose restriction to $A^\bullet(C_{\alpha+1})$ is the Chern polynomial of the normal bundle $\mathcal{N}_{C_{\alpha+1}/Z_\alpha}$ i.e.

$$i^*_{\alpha+1}(P(T)) = t^d + c_1(\mathcal{N}_{C_{\alpha+1}/Z_\alpha}) T^{d-1} + \cdots + c_{d-1}(\mathcal{N}_{C_{\alpha+1}/Z_\alpha}) T + c_d(\mathcal{N}_{C_{\alpha+1}/Z_\alpha}),$$

(where $d = codim(C_{\alpha+1}, Z_\alpha)$). This isomorphism is induced by

$$\pi^*_{\alpha+1} : A^\bullet(Z_\alpha) \to A^\bullet(Z_{\alpha+1})$$

and by sending $-T$ to the class of the exceptional divisor.

However, the surjectivity hypothesis of the theorem is quite restrictive. For example, let us consider the blow-up of a rational curve $C_1 \subset Z_0 \cong \mathbb{P}^3$ of degree

$\gamma > 1$.

$$\begin{array}{ccc} E_1^1 & \xrightarrow{j_1} & Z_1 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ C_1 & \xrightarrow{i_1} & Z_0 \end{array}$$

Note that in this case the restriction map $i_1^* : A^\bullet(Z_0) \to A^\bullet(C_1)$ is not surjective since $i_1^*(h) = \gamma[P]$, where $[P]$ denotes the class of a point $P \in C_1$, so we can not apply Keel formula of theorem 3.12. $A^\bullet(Z_1)$ is no longer generated by $\{h^{1*}, e_1^{1*}\}$ and we need to add an extra generator $r_1 = j_{1*}[g_1^{-1}(P)]$, which geometrically is the fiber of a point in the blow-up.

One can prove that

$$A^\bullet(Z_1) \cong \mathbb{Z}\left[x_0, x_1, w_1\right]/\mathcal{I}$$

where

$$\begin{aligned} \mathcal{I} = (&(x_0)^2 \cdot x_1, x_0 \cdot x_1 - \gamma w_1, x_0 \cdot w_1, (w_1)^2, \\ &(x_1)^2 + (-4\gamma + 2)w_1 + \gamma(x_0)^2, (x_0)^3 + x_1 \cdot w_1) \end{aligned}$$

by sending x_0, x_1 and w_1 to h^{1*}, e_1^1 and r_1^1, respectively.

Acknowledgements. I would like to thank the referee for his/her detailed reading of this paper and thoughtful suggestions.

References

[1] D. Camazón and S. Encinas, *Sequences of point blow-ups from a combinatorial point of view*, preprint.

[2] A. Campillo, G. Gonzalez-Springberg, and M. Lejeune-Jalabert, *Clusters of infinitely near points*, Math. Ann. **306** (1996), no. 1, 169–194. https://doi.org/10.1007/BF01445246

[3] A. Campillo, G. Gonzalez-Springberg, and F. Monserrat, *Configurations of infinitely near points*, São Paulo J. Math. Sci. **3** (2009), no. 1, 115–160. https://doi.org/10.11606/issn.2316-9028.v3i1p115-160

[4] D. Eisenbud and J. Harris, *3264 and All That—A Second Course in Algebraic Geometry*, Cambridge Univ. Press, Cambridge, 2016. https://doi.org/10.1017/CBO9781139062046

[5] W. Fulton, *Intersection Theory*, second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 2, Springer, Berlin, 1998. https://doi.org/10.1007/978-1-4612-1700-8

[6] S. Keel, *Intersection theory of moduli space of stable n-pointed curves of genus zero*, Trans. Amer. Math. Soc. **330** (1992), no. 2, 545–574. https://doi.org/10.2307/2153922

Daniel Camazón Portela
Department of Algebra, Analysis, Geometry and Topology
University of Valladolid
Valladolid 47011, Spain
Email address: daniel.camazon@uva.es