J. Appl. & Pure Math. Vol. **6**(2024), No. 3 - 4, pp. 201 - 210
https://doi.org/10.23091/japm.2024.201

A NUMERICAL INVESTIGATION ON THE STRUCTURE OF THE ZEROS OF q-EULER-FIBONACCI POLYNOMIALS†

CHEON SEOUNG RYOO

Abstract. In this paper, we construcr the q-Bernoulli-Fibonacci numbers and polynomials. Finally, we investigate the distribution of the zeros of the q-Bernoulli-Fibonacci polynomials by using computer.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.
Key words and phrases : Bernoulli numbers and polynomials, Fibonacci numbers, Bernoulli-Fibonacci polynomials, q-Bernoulli-Fibonacci polynomials.

1. Introduction

In this paper, we define the q-Bernoulli-Fibonacci numbers and polynomials and investigate the distribution of zeros of the q-Bernoulli-Fibonacci polynomials by using computer. Throughout this paper, we always make use of the following notations: $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of complex numbers, respectively.

The authors [1, 2, 3, 4, 5, 6] introduced generating functions for Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$B as follow

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}, \quad \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt}.$$

Now, we give some definitions that we will use throughout the article. The F-factorial is defined as

$$F_n! = F_n \cdot F_{n-1} \cdot F_{n-2} \cdots F_1, \quad F_0! = 1.$$

Received April 9, 2024. Revised July 22, 2024. Accepted July 25, 2024.
†This work was supported by 2023 Hannam University Research Fund.

where F_n is n-th Fibonacci numbers. The Fibonomial coefficients are defined as $(0 \leq k \leq n)$ as

$$\binom{n}{k}_F = \frac{F_n!}{F_{n-k}!F_k!}$$

with $\binom{n}{0}_F = \binom{n}{n}_F = 1$ and $\binom{n}{k}_F = 0$ for $n < k$(see [7]).

The binomial theorem for the F-analogues (or-Golden binomial theorem) are given by

$$(x+y)_F^n = \sum_{k=0}^{n} (-1)^{\binom{n}{2}} \binom{n}{k}_F x^{n-k} y^k$$

The F-exponential functions $e_F(x)$ and $E_F(x)$ are defined as

$$e_F(x) = \sum_{n=0}^{\infty} \frac{x^n}{F_n!}, \quad E_F(x) = \sum_{n=0}^{\infty} (-1)^{\binom{n}{2}} \frac{x^n}{F_n!}.$$

The quantum q- Fibonacci number is defined as

$$[F_n]_q = \frac{1-q^{F_n}}{1-q}.$$

for F_n is n-th Fibonacci numbers with $q \neq 1$.

Now, we give some definitions that we will use throughout the article. The q-F-factorial is defined as

$$[F_n]_q! = [F_n]_q \cdot [F_{n-1}]_q \cdot [F_{n-2}]_q \cdots [F_1]_q, \quad [F_0]_q! = 1.$$

where F_n is n-th Fibonacci numbers. The q-Fibonomial coefficients are defined as $(0 \leq r \leq m)$ as

$$\begin{bmatrix} m \\ r \end{bmatrix}_{q,F} = \frac{[F_m]_q!}{[F_{m-r}]_q![F_r]_q!},$$

where m and r are non-negative integers.

The q-F-exponential functions $e_{q,F}(x)$ is defined as

$$e_{q,F}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[F_n]_q!}.$$

We define generating functions for q-Bernoulli-Fibonacci numbers $B_{n,q,F}$ and q-Bernoulli-Fibonacci polynomials $B_{n,q,F}(x)$ as follow

$$\sum_{n=0}^{\infty} B_{n,q,F} \frac{t^n}{[n]_q!} = \frac{t}{e_{q,F}(t)-1},$$

$$\sum_{n=0}^{\infty} B_{n,q,F}(x) \frac{t^n}{[n]_q!} = \frac{t}{e_{q,F}(t)-1} e_{q,F}(xt).$$

Theorem 1.1. *For $n \geq 1$, we have*

$$B_{n,q,F}(x) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{q,F} B_{l,q,F} x^{n-l}.$$

For the first few q-Bernoulli-Fibonacci numbers we have,

$$
\begin{aligned}
B_{0,q,F} &= 1\\
B_{1,q,F} &= -1,\\
B_{2,q,F} &= \frac{q}{1-q}\\
B_{3,q,F} &= -\frac{q^3}{(1-q)(1+q)(1+q+q^2)} + \frac{q^5}{(1-q)(1+q)(1+q+q^2)},\\
B_{4,p,F} &= \frac{1}{(1-q)^2} - \frac{q^2}{(1-q)^2} - \frac{q^3}{(1-q)^2} + \frac{q^5}{(1-q)^2} + \frac{1}{(1-q)^2(1+q)^2}\\
&- \frac{q^2}{(1-q)^2(1+q)^2} - \frac{q^3}{(1-q)^2(1+q)^2} + \frac{q^5}{(1-q)^2(1+q)^2}\\
&- \frac{3}{(1-q)^2(1+q)} + \frac{3q^2}{(1-q)^2(1+q)} + \frac{3q^3}{(1-q)^2(1+q)}\\
&- \frac{3q^5}{(1-q)^2(1+q)} + \frac{2}{(1-q)^2(1+q)(1+q+q^2)}\\
&- \frac{2q^2}{(1-q)^2(1+q)(1+q+q^2)}\\
&- \frac{2q^3}{(1-q)^2(1+q)(1+q+q^2)}\\
&+ \frac{2q^5}{(1-q)^2(1+q)(1+q+q^2)}\\
&- \frac{1}{(1-q)^2(1+q)(1+q+q^2)(1+q+q^2+q^3+q^4)}\\
&+ \frac{q^2}{(1-q)^2(1+q)(1+q+q^2)(1+q+q^2+q^3+q^4)}\\
&+ \frac{q^3}{(1-q)^2(1+q)(1+q+q^2)(1+q+q^2+q^3+q^4)}\\
&- \frac{q^5}{(1-q)^2(1+q)(1+q+q^2)(1+q+q^2+q^3+q^4)}.
\end{aligned}
$$

2. Zeros of the q-Bernoulli-Fibonacci polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the q- Bernoulli-Fibonacci polynomials $B_{n,q,F}(x)$. The Bernoulli-Fibonacci polynomials $B_{n,q,F}(x)$. can be determined explicitly.

A few of them are

$$B_{0,q,F}(x) = 1,$$

$$B_{1,q,F}(x) = -1 + x,$$

$$B_{2,q,F}(x) = \frac{q}{1+q} - x + x^2,$$

$$\begin{aligned}
B_{3,q,F}(x) &= -\frac{q^3}{(1-q)(1+q)(1+q+q^2))} + \frac{q^5}{(1-q)(1+q)(1+q+q^2)} \\
&+ \frac{qx}{(1-q)(1+q)} - \frac{q^3x}{(1-q)(1+q)} - \frac{x^2}{(1-q)} + \frac{q^2x^2}{(1-q)} + x^3,
\end{aligned}$$

$$\begin{aligned}
B_{4,q,F}(x) &= \frac{1}{(1-q)^2} - \frac{q^2}{(1-q)^2} - \frac{q^3}{(1-q)^2} + \frac{q^5}{(1-q)^2} + \frac{1}{(1-q)^2(1+q)^2)} \\
&- \frac{q^2}{(1-q)^2(1+q)^2)} - \frac{q^3}{(1-q)^2(1+q)^2} + \frac{q^5}{(1-q)^2(1+q)^2} \\
&- \frac{3}{(1-q)^2(1+q)} + \frac{3q^2}{(1-q)^2(1+q)} + \frac{3q^3}{(1-q)^2(1+q)} \\
&- \frac{3q^5}{(1-q)^2(1+q)} + \frac{2}{(1-q)^2(1+q)(1+q+q^2)} \\
&- \frac{2q^2}{(1-q)^2(1+q)(1+q+q^2)} - \frac{2q^3}{(1-q)^2(1+q)(1+q+q^2)} \\
&+ \frac{2q^5}{(1-q)^2(1+q)(1+q+q^2)} \\
&- \frac{1}{(1-q)^2(1+q)(1+q+q^2)(1+q+q^2+q^3+q^4)} \\
&+ \frac{q^2}{(1-q)^2(1+q)(1+q+q^2)(1+q+q^2+q^3+q^4)} \\
&+ \frac{q^3}{(1-q)^2(1+q)(1+q+q^2)(1+q+q^2+q^3+q^4)} \\
&- \frac{q^5}{(1-q)^2(1+q)(1+q+q^2)(1+q+q^2+q^3+q^4)} \\
&- \frac{q^3x}{(1-q)^2(1+q)(1+q+q^2)} + \frac{q^5x}{(1-q)^2(1+q)(1+q+q^2)} \\
&+ \frac{q^6x}{(1-q)^2(1+q)(1+q+q^2)} - \frac{q^8x}{(1-q)^2(1+q)(1+q+q^2)} \\
&+ \frac{qx^2}{(1-q)^2(1+q)} - \frac{q^3x^2}{(1-q)^2(1+q)} - \frac{q^4x^2}{(1-q)^2(1+q)} \\
&+ \frac{q^6x^2}{(1-q)^2(1+q)} - \frac{x^3}{1-q} + \frac{q^3x^3}{1-q} + x^4.
\end{aligned}$$

We investigate the zeros of the q-Bernoulli-Fibonacci polynomials $B_{n,q,F}(x) = 0$. by using a computer. We plot the zeros of the q-Bernoulli-Fibonacci polynomials $B_{n,q,F}(x) = 0$ for $x \in \mathbb{C}$(Figure 1).

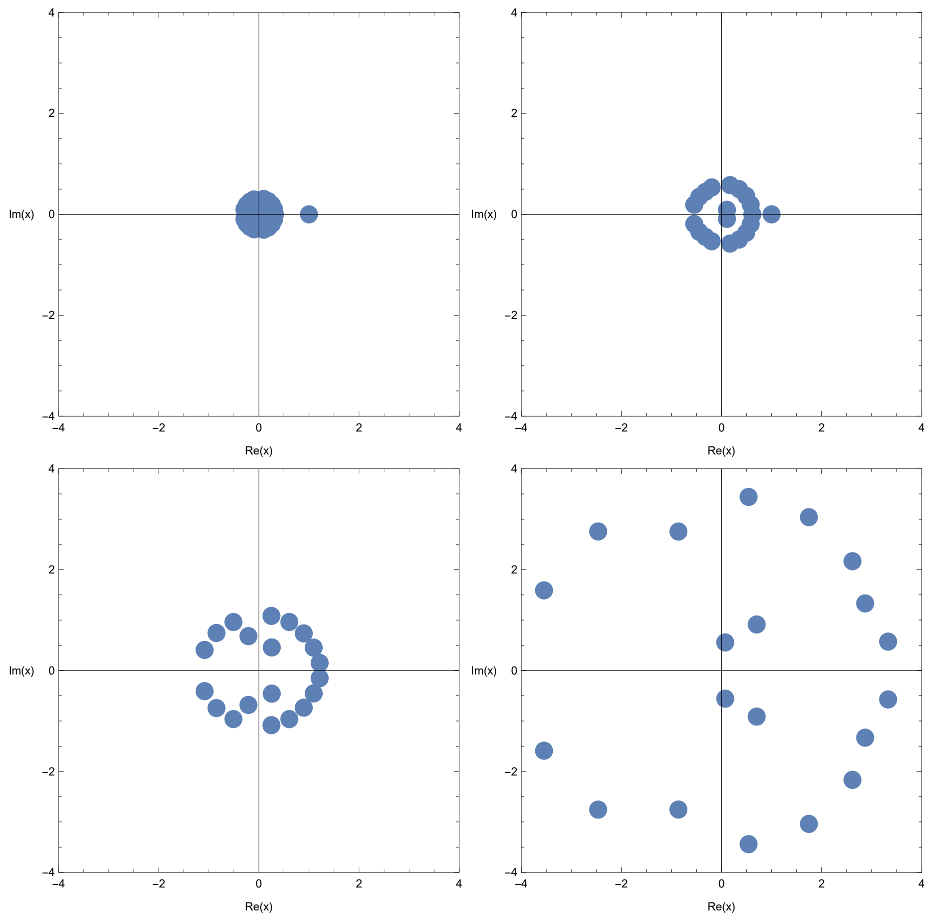

FIGURE 1. Zeros of $B_{n,q,F}(x) = 0$

In Figure 1(top-left), we choose $n = 20, q = \frac{3}{10}$. In Figure 1(top-right), we choose $n = 20, q = \frac{5}{10}$. In Figure 1(bottom-left), we choose $n = 20, q = \frac{7}{10}$. In Figure 1(bottom-right), we choose $n = 20, q = \frac{9}{10}$.

Stacks of zeros of the q-Bernoulli-Fibonacci polynomials $B_{n,q,F}(x) = 0$ for $1 \leq n \leq 20$ from a 3-D structure are presented(Figure 3).

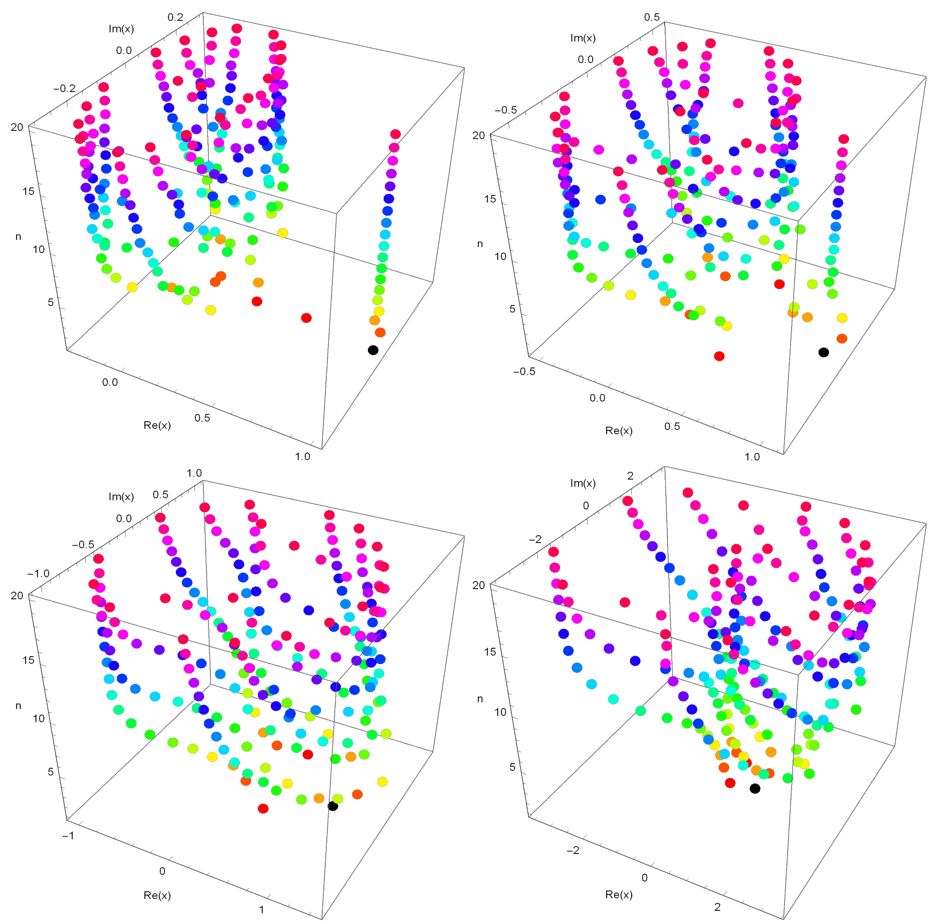

FIGURE 2. Zeros of $B_{n,q,F}(x) = 0$

In Figure 2(top-left), we choose $q = \frac{3}{10}$. In Figure 2(top-right), we choose $q = \frac{5}{10}$. In Figure 2(bottom-left), we choose $q = \frac{7}{10}$. In Figure 2(bottom-right), we choose $q = \frac{9}{10}$.

Stacks of zeros of the q-Bernoulli-Fibonacci polynomials $B_{n,q,F}(x) = 0$ for $1 \leq n \leq 20, q = \frac{99}{100}$ from a 3-D structure are presented(Figure 3).

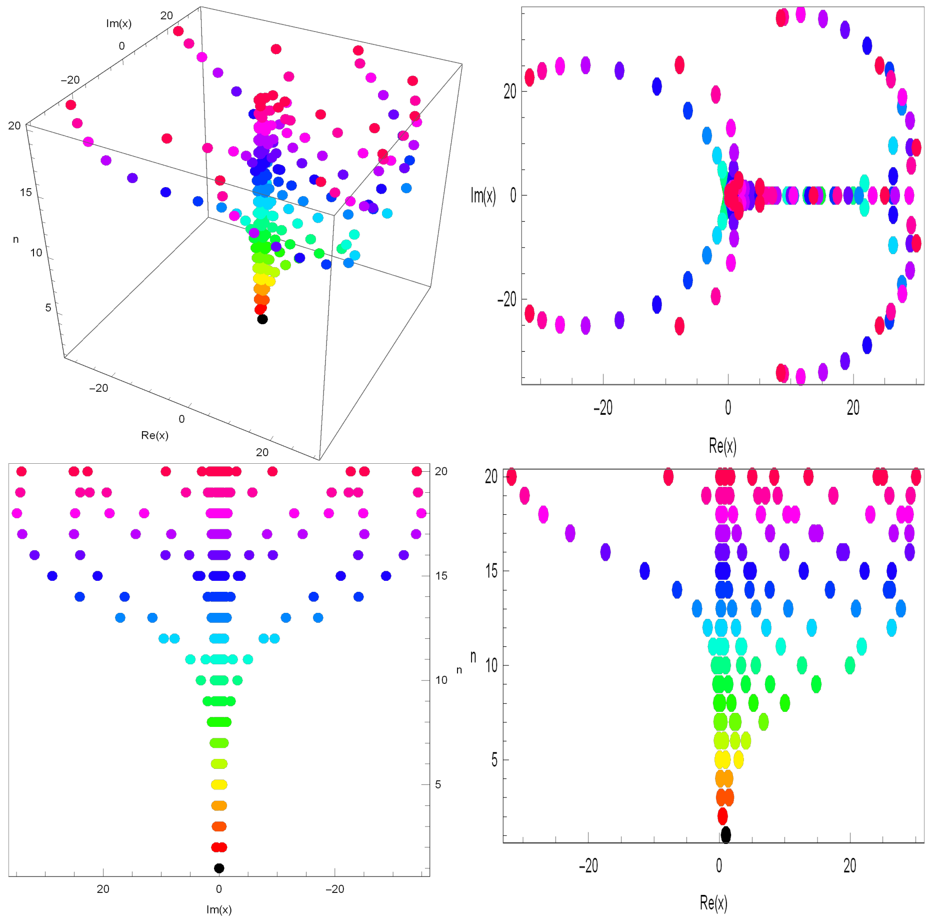

FIGURE 3. Zeros of $B_{n,q,F}(x) = 0$

In Figure 3(top-left), we draw stacks of zeros of the q-Bernoulli-Fibonacci polynomials in the three dimensions. In Figure 3(top-right), we draw x and y axes but no z axis in the three dimensions. In Figure 3(bottom-left), we draw y and z axes but no x axis in the three dimensions. In Figure 3(bottom-right), we draw x and z axes but no y axis in the three dimensions.

The plot of real zeros of q-Bernoulli-Fibonacci polynomials $B_{n,q,F}(x) = 0$ for $1 \leq n \leq 20$ structure are presented(Figure4).

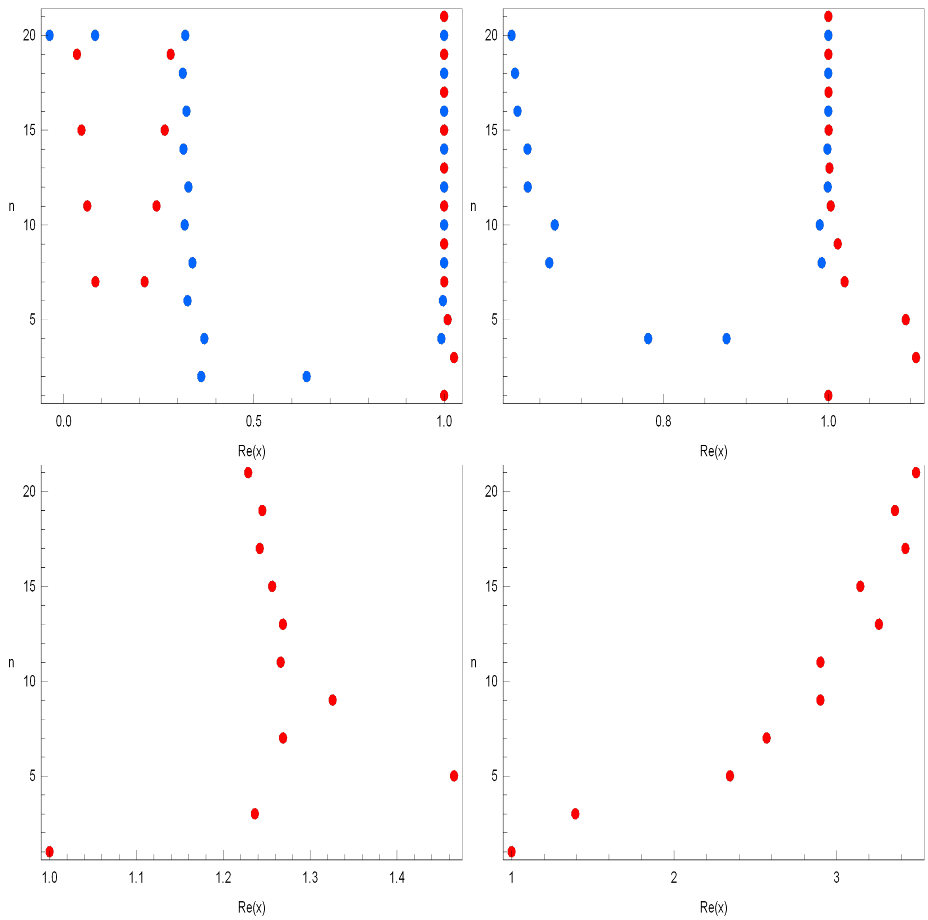

FIGURE 4. Zeros of $B_{n,q,F}(x) = 0$

In Figure 4(top-left), we choose $q = \frac{3}{10}$. In Figure 4(top-right), we choose $q = \frac{5}{10}$. In Figure 4(bottom-left), we choose $q = \frac{7}{10}$. In Figure 4(bottom-right), we choose $q = \frac{9}{10}$.

Next, we calculated an approximate solution satisfying q-Bernoulli-Fibonacci polynomials $B_{n,q,F}(x) = 0$ for $x \in \mathbb{R}, q = \frac{3}{10}$. The results are given in Table 1.

Table 1. Approximate solutions of $B_{n,q,F}(x) = 0$

degree n	x
1	1.0000
2	0.36132, 0.63868
3	1.0261
4	0.36950, 0.99271
5	1.0092
6	0.32541, 0.99685
7	0.083202, 0.21251, 1.0002
8	0.33821, 0.99995
9	1.0001
10	0.31785, 0.99997
11	0.061770, 0.24382, 1.0000
12	0.32779, 0.24382, 1.0000

Conflicts of interest : The author declares no conflict of interest.

Data availability : Not applicable

References

1. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables, Courier Corporation*, USA, 1972.
2. G.E. Andrews, R. Askey, R. Roy, *Special Functions*, Vol. 71, Combridge Press, Cambridge, UK, 1999.
3. N.S. Jung, C.S. Ryoo, *Identities involving q- analogue of modified tangent polynomials*, J. App. Math. & Informatics **39** (2021), 643-654.
4. N.S. Jung, C.S. Ryoo, *Fully Modified (p, q)-poly-tangent polynomials with two variables*, J. App. Math. & Informatics **41** (2023), 753-763.
5. C.S. Ryoo, *Zeros of the Bernoulli-fibonacci Polynomials*, J. App. & Pure Math. Submitted.
6. M. Kaneko, *Poly-Bernoulli numbers*, J. Théor. Nombres Bordeaux **9** (1997), 199-206.
7. M. Ozvatan, *Generalized Golden-Fibonacci calculus and applications*, Master Thesis, The Graduate School of Engineering and Sciences of Izmir Institute of Technology, Izmir, 2018.

Cheon Seoung Ryoo received Ph.D. degree from Kyushu University. His research interests focus on the numerical verification method, scientific computing, functional analysis, and analytic number theory. More recently, he has been working with differential equations, dynamical systems, quantum calculus, and special functions.

Department of Mathematics, Hannam University, Daejeon, 306-791, Korea.
e-mail: ryoocs@hnu.kr