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AN APPLICATION OF MARKOV'S EXTENDED INTERVAL ARITHMETIC TO INTERVAL-VALUED SEQUENCE SPACES: A SPECIAL EXAMPLE

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Abstract. In the classical sense, it is known that it is impossible to construct a vector space over the entire set of real numbers with the help of simple interval arithmetic. In this article, it has shown that a vector space can be constructed in the classical sense by helping Markov's extended interval arithmetic on the interval valued Cesaro sequence spaces of non-absolute type. As a result of the positive answers, this idea was extended by us with some theorems. Consequently, a new perspective was gained to the construction of new types of sequence spaces by using different algebraic operations on interval-valued sequence spaces.

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1. Introduction and Preliminaries

As it is known, interval numbers offer great advantages in managing uncertainty and errors in scientific applications, providing ease of calculation and enabling more reliable analysis. This allows the development of models that better fit real-world problems and offer a wider range of applications. In recent years, sequence spaces of interval numbers have also been constructed, and many studies have been carried out. One of the main focuses of research on sequence spaces is to create new sequence spaces and investigate their topological and algebraic properties. For this purpose, the domain of an infinite matrix is generally taken into account, and new sequence spaces are created. There are many articles about this in the relevant literature. Another way to create a new

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sequence space is to use sequences of intervals whose terms are interval numbers instead of sequences with real or complex terms for example you can see [17]. The sequence spaces obtained this way are significantly broader than classical sequence spaces and exhibit distinctive topological and algebraic structures compared to traditional ones. In this context, non-absolute type sequence spaces represent a significant class that has garnered substantial attention in sequence space theory. Convergent sequence spaces of non-absolute type; they were introduced to study the convergence properties of sequences that are not absolutely convergent. These spaces have been the subject of numerous research articles, and many different aspects of their properties have been studied in detail.

One of these non-absolute type sequence spaces is denoted as

$$X_p = \{(x_k) : \left(\sum_k |k^{-1}\sum_{i=1}^{k} x_i|^p\right)^{\frac{1}{p}} < \infty, 1 \leq p < \infty\}, \tag{1}$$

commonly referred to as the non-absolute type Ces'aro sequence space, [16].

Overall, the study of Cesàro sequence spaces of non-absolute type is a fascinating and important area of research in the theory of sequence spaces. The properties of these spaces have important applications in many different areas of mathematics, and their study has led to many interesting and important results. By researching the relevant literature, one can gain a deep understanding of the rich and complex structure of these important sequence spaces.

-*The question of whether the explanations we provided above about X_p would also apply to $X_p(I)$, which is a natural extension of X_p, inspired us to prepare this article, where*

$$X_p(I) \quad = \quad \{([x_k^-, x_k^+]) : \left(\sum_k \max\{|k^{-1}\sum_{i=1}^{k} x_i^-|^p, |k^{-1}\sum_{i=1}^{k} x_i^+|^p\}\right)^{\frac{1}{p}} < \infty, 1 \leq p\}.$$

It is not possible to construct a rich algebraic structure on a set of interval numbers in the classical sense. However, when classical interval arithmetic is used on the set of interval numbers, this is not as simple as it seems, and many rich mathematical concepts are lost on the mathematical structures.

In this context, interval arithmetic was first suggested by P. S. Dwyer [6] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by R. E. Moore [13], [15] in 1959 and 1962. Furthemore, Moore and others [7], [8], [6], [14] have developed applications to differential equations.

After these developments, in [5] Chiao introduced sequence of interval numbers and after them the bounded and convergent sequences spaces of interval numbers have defined by Şengönül and Eryılmaz in [17].

A set consisting of a closed interval of real numbers $\mathbf{u}$ such that $u^- \leq x \leq u^+$, $(u^-, u^+ \in \mathbb{R})$ is called an interval number. A real interval can also be considered as a set. Let us denote the set of all interval numbers with notation $\mathbb{R}(I)$. That is $\mathbb{R}(I) = \{\mathbf{u} : \mathbf{u} \text{ is a interval number}\}$.

It is known that, one of the most problematic areas of the set of interval numbers is related to their algebraic structure. While conventional operations (using the endpoints of the interval) are often used in research, for to eliminate the emerging problematic areas, here we will use new type algebraic operations (*extended interval arithmetics*) of Markov. The new type algebraic operations (*extend interval arithmetics*) belonging to Markov [12] can be extended to sequences of interval number as follows.

Let's define transformation f from $\mathbb{N}$ to $\mathbb{R}(I)$ by $k \to f(k) = \mathbf{u},\ \mathbf{u} = (\mathbf{u}_k)$. Then, $(\mathbf{u}_k)$ is called sequence of interval numbers. The $\mathbf{u}_k$ is called n^{th} term of the interval sequence $(\mathbf{u}_k)$.

Please let's not lose sight of the fact that bolded notations correspond to intervals.

For any $\mathbf{u}_k = [u_k^-, u_k^+]$, $(k \in \mathbb{N})$ we define $\omega(\mathbf{u}_k) = u_k^+ - u_k^-$ and $\chi(\mathbf{u}_k) = \{u_k^-/u_k^+ \text{ if } \mu(\mathbf{u}_k) \geq 0,\ u_k^+/u_k^- \text{ if } \mu(\mathbf{u}_k) < 0\}$ where $\mu(\mathbf{u}_k) = (u_k^- + u_k^+)/2$. In addition to them, we define the functional $\varphi, \psi : \mathbb{R}(I) \times \mathbb{R}(I) \to \{+, -\}$ defined by

$$\varphi(\mathbf{u}_k, \mathbf{v}_k) = \begin{cases} +, & \omega(\mathbf{u}_k) \geq \omega(\mathbf{v}_k) \\ -, & otherwise \end{cases},\ \psi(\mathbf{u}_k, \mathbf{v}_k) = \begin{cases} +, & \chi(\mathbf{u}_k) \geq \chi(\mathbf{v}_k) \\ -, & otherwise \end{cases}.$$

The notations $\mathbf{Z}$, $\mathbf{Z}^*$ and $\mathbb{R}(I) \setminus \mathbf{Z}$ are defined by $\mathbf{Z} = \{\mathbf{u}_k \in \mathbb{R}(I) : 0 \in \mathbf{u}_k,\ k \in \mathbb{N}\}$, $\mathbf{Z}^* = \{\mathbf{u}_k \in \mathbb{R}(I) : u_k^- < 0 < u_k^+, k \in \mathbb{N}\}$ and $\mathbb{R}(I) \setminus \mathbf{Z} = \{\mathbf{u}_k \in \mathbb{R}(I) : u_k^- > 0 \text{ or } 0 > u_k^+,\ k \in \mathbb{N}\}$, respectively.

Define a sing functional $\sigma : \mathbb{R}(I) \setminus \mathbf{Z}^* \to \{-, +\}$ by means of

$$\sigma(\mathbf{u}_k) = \begin{cases} +, & 0 \leq u_k^- \\ -, & u_k^+ \leq 0,\ \mathbf{u}_k \neq [0, 0]. \end{cases}$$

Then the addition and multiplication sequences of the interval numbers $(\mathbf{u}_k)$ and $(\mathbf{v}_k)$ is defined by

$$\mathbf{u}_k + \mathbf{v}_k = [u_k^{-\alpha} + v_k^{\alpha}, u_k^{\alpha} + v_k^{-\alpha}],\ \alpha = \varphi(\mathbf{u}_k, \mathbf{v}_k) \tag{2}$$

$$\mathbf{u}_k\mathbf{v}_k = \begin{cases} [u_k^{\sigma(v_k)\epsilon} v_k^{-\sigma(u_k)\epsilon}, u_k^{-\sigma(v_k)\epsilon} v_k^{\sigma(u_k)\epsilon}], & \epsilon = \psi(\mathbf{u}_k, \mathbf{v}_k), \mathbf{u}_k, \mathbf{v}_k \in \mathbb{R}(I) \setminus \mathbf{Z}, \\ [u_k^{-\delta} v_k^{-\delta}, u_k^{-\delta} v_k^{\delta}], & \delta = \sigma(\mathbf{u}_k), \mathbf{u}_k \in \mathbb{R}(I) \setminus \mathbf{Z}, \mathbf{v}_k \in \mathbf{Z}, \\ [u_k^{-\delta} v_k^{-\delta}, u_k^{\delta} v_k^{-\delta}], & \delta = \sigma(\mathbf{v}_k), \mathbf{v}_k \in \mathbb{R}(I) \setminus \mathbf{Z}, \mathbf{u}_k \in \mathbf{Z}, \\ [\min\{u_k^- v_k^+, u_k^+ v_k^-\}, \max\{u_k^- v_k^-, u_k^+ v_k^+\}], & \mathbf{u}_k, \mathbf{v}_k \in \mathbf{Z} \end{cases} \tag{3}$$

Together with the operations in (2) and (3), defined as above, a extended algebraic structure can be constructed on the set of sequences of intervals. This algebraic construct can satisfy most of conditions vector space in classic manner. Most importantly, it satisfies the property of $\mathbf{u} - \mathbf{u} = \theta$, which is not present in standard interval arithmetic. Further information about this algebraic structure can be found in Markov's article [12]. Therefore, we will leave it here without providing more information.

The set of all interval numbers $\mathbb{R}(I)$ is a metric space [13] defined by

$$d(\mathbf{u}, \mathbf{v}) = \max\{|u^- - v^-|, |u^+ - v^+|\}. \tag{4}$$

Moreover it is known that $\mathbb{R}(I)$ is a complete metric space.

Definition 1.1. [5] A sequence $\mathbf{u} = (\mathbf{u}_k)$ of interval numbers is said to be convergent to the interval number $\mathbf{u}_0$ if for each $\varepsilon > 0$ there exists a positive integer n_0 such that $\tilde{d}(\mathbf{u}_k, \mathbf{u}_0) < \varepsilon$ for all $k \geq n_0$, and we denote it by writing $\lim_k \mathbf{u}_k = \mathbf{u}_0$.

The space of all null, $c_0(I)$, convergent, $c(I)$, bounded, $\ell_\infty(I)$, sequences of interval number are subsets of $w(I)$ and for all $(\mathbf{u}_k), (\mathbf{v}_k) \in c_0(I)$ $(or\ c(I),\ \ell_\infty(I))$ are complete metric spaces defined by $\tilde{d}(\mathbf{u}_k, \mathbf{v}_k) = \sup_k\{\max(|u_k^- - v_k^-|, |u_k^+ - v_k^+|)\}$ [17].

Let A be an infinite matrix of real or complex numbers. We define $c_A = \{\mathbf{u} : A\mathbf{u} \in c,\ \text{with}\ u_k^- = u_k^+\}$ where c is convergent sequences space of real or complex numbers. This type sequence space has been studies by many authors; see, for example, [1].

In general, let V be a given sequence space. We define $U = \{x : Ax \in V\}$. In follows, we assume that the mapping of A from U to V is one-one and onto. In particular, when A is a Cesaro matrix C of order one which is a lower triangular matrix defined by

$$c_{nk} = \begin{cases} \frac{1}{n+1} & ,0 \leq k \leq n, \\ 0 & ,k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$ and $V = \ell_p$ (where ℓ_p is classicial manner), for $1 \leq p < \infty$ then the set U is called Cesàro sequence space of a non-absolute type [19] and denote it by X_p. In other words, $x \in X_p$ for $1 \leq p < \infty$ if and only if

$$\left(\sum_k |k^{-1} \sum_{i=1}^{k} x_i|^p\right)^{\frac{1}{p}} < \infty. \tag{5}$$

Now we consider the absolute version. Let A be an infinite matrix and V a sequence space in classical manner. We consider $U = \{x; A|x| \in V\}$. In particular,when A is a Cesaro matrix C and $V = \ell_p$ for $1 < p < \infty$ we call to U Cesaro sequences space of an absolute type and denote it by ces_p. In other words, $x \in ces_p$ for $1 < p < \infty$ and only if

$$\left(\sum_k k^{-1} \sum_{i=1}^{k} |x_i|^p\right)^{\frac{1}{p}} < \infty. \tag{6}$$

The space ces_p was first defined by Shiue [20], its α-dual given by Jagers [4] for $1 < p < \infty$, and by Ng and Lee [15] for $p = \infty$.

Let us define the sets $ces_p(I)$ and $\ell_p(I)$ of interval numbers as follows:

$$ces_p(I)$$

$$= \{(\mathbf{x}_k) = ([x_k^-, x_k^+]) : \left(\sum_k \max\{k^{-1} \sum_{i=1}^{k} |x_i^-|^p, k^{-1} \sum_{i=1}^{k} |x_i^+|^p\}\right)^{\frac{1}{p}} < \infty, p \geq 1\},$$

$$\ell_p(I) = \{(\mathbf{x}_k) = ([x_k^-, x_k^+]) : \left(\sum_k \max\{|x_k^-|^p, |x_k^+|^p\}\right)^{\frac{1}{p}} < \infty, p \geq 1\},$$

$$\ell_\infty(I) = \{(\mathbf{x}_k) = ([x_k^-, x_k^+]) : \sup_k\{\max\{|x_k^-|, |x_k^+|\}\} < \infty\},$$

$$c(I) = \{(\mathbf{x}_k) = ([x_k^-, x_k^+]) : \lim_k [x_k^-, x_k^+] = [x_0^-, x_0^+], [x_0^-, x_0^+] \in I\},$$

[17]. The set $ces_p(I)$ is called Cesaro sequence spaces of interval numbers and the set $\ell_p(I)$ is called absolute convergent series space of p^{th} order of interval numbers.

2. Main Results: Construction Of The Space $X_p(I)$.

Now we will construct $X_p(I)$, which is the main subject of this study. First, let's provide the necessary preliminary information. In the classical manner non-absolute type Cesàro sequence space, as mentioned above, has defined by Yee in [19] as :

$$\begin{aligned} X_p &= \{(x_k) : \left(\sum_k |k^{-1}\sum_{i=1}^k x_i|^p\right)^{\frac{1}{p}} < \infty, 1 \leq p < \infty\} \\ &= \{(x_k) : \left(k^{-1}\sum_{i=1}^k x_k\right) \in \ell_p, 1 \leq p < \infty\}. \end{aligned} \tag{7}$$

If we replace sequences of real numbers with sequences of interval numbers in (7), we obtain $X_p(I)$. In other words, non-absolute type Cesàro sequence space of intervals is defined as follows:

$$\begin{aligned} X_p(I) &= \{([x_k^-, x_k^+]) : \left(\sum_k \max\{|k^{-1}\sum_{i=1}^k x_i^-|^p, |k^{-1}\sum_{i=1}^k x_i^+|^p\}\right)^{\frac{1}{p}} < \infty, 1 \leq p\} \\ &= \{(\mathbf{x}_k) = ([x_k^-, x_k^+]) : \left(k^{-1}\sum_{i=1}^k \mathbf{x}_i\right) \in \ell_p(I), 1 \leq p < \infty\}. \end{aligned} \tag{8}$$

The set $X_p(I)$ is not empty. The $X_p(I)$ is not linear space in conventional manner. It is clear that, if $x_k^- = x_k^+$ for all $k \in \mathbb{N}$ then the space $X_p(I)$ is equal to famous the space X_p. The set $X_p(I)$ has a algebraic structure with extended interval arithmetic which these algebraic operations allow it to perform addition and subtraction operations just like in real numbers on $X_p(I)$.

The norm function on the classical sequence spaces can be extended to the sequence spaces of the interval numbers. Let suppose that $\lambda(I)$ is a subset of $w(I) = \{\mathbf{u} : \mathbf{u} \text{ is sequence of interval numbers}\}$.

Let $\lambda(I)$ be the set of interval sequences $\mathbf{u} = ([u_k^-, u_k^+])_{k\in\mathbb{N}}$. A functional ρ from $\lambda(I)$ into the non-negative extended real number system is called a semi-norm for intervals if

(1) $g(\theta) = 0$, where $\theta = [0, 0]$
(2) $g(\alpha\mathbf{u}) = |\alpha|\rho(\mathbf{u})$
(3) $g(\mathbf{u} + \mathbf{v}) \leq g(\mathbf{u}) + g(\mathbf{v})$, where $\mathbf{u} = [u^-, u^+]$ and $\mathbf{v} = [v^-, v^+]$.

If instead of 1. g satisfies the condition that $g(\mathbf{u}) = 0$ if and only if $\mathbf{u} = \theta$, then g is called a norm. Also, it know that the norm $g(x)$ of x is the distance from x to 0 in the sequences space real numbers (see, [10]). In many sources $g(\mathbf{u})$ is written as $||\mathbf{u}||_{\lambda(I)}$.

If a normed space $\lambda(I)$ contains a sequence (e_n) with the property that for every $\mathbf{u} \in \lambda(I)$there is a unique sequence of scalars (ν_n) such that $||\mathbf{u} - (\nu_1\mathbf{u}_1 + \nu_2\mathbf{u}_2 + \cdots + \nu_n\mathbf{u}_n)||_{\lambda(I)} \to \theta,\ (n \to \infty)$ then (ν_n) is called a Schauder basis (or basis) for $\lambda(I)$.

When considering [12] (Markov's "*extended interval arithmetic*") and convergence on set $\ell_p(I)$, the following lemma can be easily proven.

Lemma 2.1. *The sequence* $(E_n) = (\delta_{ni}) = \begin{cases} 1, & n = i \\ 0, & n \neq i \end{cases}$ *is a Schauder basis for the* $\ell_p(I)$ *in conventional manner.*

Proof. Let us suppose that $\mathbf{u} = (\mathbf{u}_k) = ([u_k^-, u_k^+]) \in \ell_p(I)$ and define the sequence $(\mathbf{v}_n) = ([v_n^-, v_n^+])$ as follows:

$$\begin{aligned}
\mathbf{v}_n &= [v_n^-, v_n^+] = [u_k^-, u_k^+] - \sum_{i=1}^{n}[u_i^-, u_i^+]E_i \\
&= ([u_1^-, u_1^+], [u_2^-, u_2^+], \cdots, [u_n^-, u_n^+], \cdots) - ([u_1^-, u_1^+]E_1 + [u_2^-, u_2^+]E_2 + \cdots + [u_n^-, u_n^+]E_n) \\
&= ([u_1^-, u_1^+] - [u_1^-, u_1^+]E_1, [u_2^-, u_2^+] - [u_2^-, u_2^+]E_2, \cdots, [u_n^-, u_n^+] - [u_n^-, u_n^+]E_n, [u_{n+1}^-, u_{n+1}^+], \cdots) \\
&= ([u_1^-, u_1^+] - [u_1^-, u_1^+], [u_2^-, u_2^+] - [u_2^-, u_2^+], \cdots, [u_n^-, u_n^+] - [u_n^-, u_n^+], \cdots, [u_{n+1}^-, u_{n+1}^+], \cdots)
\end{aligned} \tag{9}$$

If we consider *extended interval arithmetics of Markov's* in the (9) then we have

$$([v_n^-, v_n^+]) = (\theta, \theta, \cdots, \theta, [u_{n+1}^-, u_{n+1}^+], \cdots).$$

Since $v_n \to \theta$, $(n \to \infty)$ we can write

$$||\mathbf{v}_n||_{\ell_p(I)} = \sum_{k \geq n+1} \left(\max\{|v_k^-|^p, |v_k^+|^p\}\right) \to 0.$$

This shows that $\mathbf{v} = \sum_k \mathbf{u}_k E_k$. Now, let us suppose another representation as $\mathbf{v} = \sum_k \mathbf{t}_k E_k$. Under condition $\omega(t_k) \leq \omega(u_k)$ (for all $k \in \mathbb{N}$), we can write

$$\begin{aligned}
||\sum_{k=1}^{n}([t_k^-, t_k^+] - [u_k^-, u_k^+])E_k||_{\ell_p(I)} &= ||\sum_{k=1}^{n}([t_k^-, t_k^+] + [-u_k^+, -u_k^-])E_k||_{\ell_p(I)} \\
&= ||\sum_{k=1}^{n}[t_k^- - u_k^-, t_k^+ - u_k^+]E_k||_{\ell_p(I)} \to 0, (n \to \infty).
\end{aligned}$$

This shows that for all $k \in \mathbb{N}$, $t_k^- = u_k^-$ and $t_k^+ = u_k^+$, which implies $\mathbf{u} = \mathbf{t}$. This completes the proof. □

Let

$$\rho(\mathbf{u}, \mathbf{v}) = \left(\sum_k \max\{|k^{-1}\sum_{i=1}^{k} u_i^- - v_i^-|^p, |k^{-1}\sum_{i=1}^{k} u_i^+ - v_i^+|^p\}\right)^{\frac{1}{p}}.$$

The function ρ is metric function on $X_p(I)$ and we can easily prove that the couple $(X_p(I), \rho)$ is a complete metric space. Let

$$\begin{aligned}
\rho(\mathbf{u}, \theta) &= \left(\sum_k \max\{|k^{-1}\sum_{i=1}^{k} u_i^- - 0|^p, |k^{-1}\sum_{i=1}^{k} u_i^+ - 0|^p\}\right)^{\frac{1}{p}} \\
&= \left(\sum_k \max\{|k^{-1}\sum_{i=1}^{k} u_i^-|^p, |k^{-1}\sum_{i=1}^{k} u_i^+|^p\}\right)^{\frac{1}{p}}
\end{aligned}$$

where $\theta = [0, 0]$. Then $\rho(\mathbf{u}, \theta)$ is a norm on the $X_p(I)$ and denoted by $||\mathbf{u}||_{X_p(I)}$. Therefore, we can give following proposition.

Proposition 2.2. *The $X_p(I)$ is complete normed non-absolute type Cesàro space of interval numbers with the norm*

$$||\boldsymbol{u}||_{X_p(I)}^p = \sum_k \max\{|k^{-1}\sum_{i=1}^{k} u_i^-|^p, |k^{-1}\sum_{i=1}^{k} u_i^+|^p\} \tag{10}$$

and extended interval arithmetics of Markov's.

Proof. It is easy to prove that the norm conditions are satisfied. Let's prove that $X_p(I)$ is complete according to the relevant norm. Let $(\mathbf{u}^{(n)})_{n\in\mathbb{N}}$ be any Cauchy sequence in the space $X_p(I)$. We write $\mathbf{u}^{(m)} = (\mathbf{u}_i^{(m)})_{i\in\mathbb{N}} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)}, ...)$. From definition of Cauchy sequence we have

$$\begin{aligned}&||\mathbf{u}^{(m)} - \mathbf{u}^{(n)}||_{X_p(I)}\\&= \left(\sum_k \max\{|k^{-1}\sum_{i=1}^k u_i^{(m)-} - u_i^{(n)-}|^p, |k^{-1}\sum_{i=1}^k u_i^{(m)+} - u_i^{(n)+}|^p\}\right)^{\frac{1}{p}}\\&< \epsilon\end{aligned}$$

for all $n, m \geq N(\epsilon)$ and given $\epsilon > 0$. Then for every fixed i, the sequence $(\mathbf{u}_i^{(m)})_{m\in\mathbb{N}}$ converges. In this case, let suppose that $\lim_m(\mathbf{u}_i^{(m)}) = \mathbf{u}_i$. Then for all $m > N(\epsilon)$ and for all j we can write

$$\sum_{k=1}^j \max\{|k^{-1}\sum_{i=1}^k u_i^{(m)-} - u_i^-|^p, |k^{-1}\sum_{i=1}^k u_i^{(m)+} - u_i^+|^p\} \leq \epsilon^p.$$

Thus letting $j \to \infty$ then we have $||\mathbf{u}^{(m)} - \mathbf{u}||_{X_p(I)} < \epsilon$ for all m. Therefore, for $1 \leq p < \infty$ the $X_p(I)$ is complete. □

Theorem 2.3. *Let $1 \leq p < \infty$ and λ be defined on $X_p(I)$ by $\xi(\boldsymbol{u}) = (\xi_n(\boldsymbol{u}))$, $\xi_n(\boldsymbol{u}) = n^{-1}\sum_{i=1}^n \boldsymbol{u}_i$*

Then ξ is an one-to-one bounded linear transformation from $X_p(I)$ onto the sequence space $\ell_p(I)$ with operator norm 1 and extended interval arithmetics of Markov's.

Proof. The proof is easy so we omit it. □

Theorem 2.4. *$ces_p(I) \subset X_p(I)$ for $1 \leq p < \infty$ and $ces_p(I) \neq X_p(I)$.*

Proof. The inclusion $ces_p(I) \subset X_p(I)$ for $1 \leq p < \infty$ is clear from definition of $ces_p(I)$ and $X_p(I)$ for $1 \leq p < \infty$. Now let us consider second part of theorem. Now let us consider the sequence of intervals as follows:

$$(\mathbf{u}_k) = ([u_k^-, u_k^+]) = (\underbrace{[-1,0]}_{\text{1. place}}, [0,2], [-2,0], ..., \overbrace{[0,2]}^{\text{k is even}}, \underbrace{[-2,0]}_{\text{k is odd}}, ...).$$

Then

$$\begin{aligned}\left(\sum_k |k^{-1}\sum_{i=1}^k [u_i^-, u_i^+]|^p\right)^{\frac{1}{p}} &= \Big(|[-1,0]|^p + |\frac{1}{2}([-1,0]+[0,2])|^p\\&\quad +|\frac{1}{3}([-1,0]+[0,2]+[-2,0])|^p + ...\Big)^{\frac{1}{p}}\\&= \Big(\max\{|-1|,|0|\})^p + (\frac{1}{2}\max\{|0|,|1|\})^p\\&\quad +(\frac{1}{3}\max\{|-1|,|0|\})^p + \cdots\Big)^{\frac{1}{p}} = \left(\sum_k (\frac{1}{k})^p\right)^{\frac{1}{p}}.\end{aligned}$$

Since the last series convergent for $p > 1$, we see that the sequence $(\mathbf{u}_k) = ([u_k^-, u_k^+])$ in $X_p(I)$ but not in the space $ces_p(I)$ by using similar way. That is $ces_p(I) \neq X_p(I)$. □

Theorem 2.5. *The sequence space $X_p(I)$ is linearly isomorphic to the space $\ell_p(I)$, that is $X_p(I) \cong \ell_p(I)$.*

Proof. We should show the existence of a linear bijection between the spaces $X_p(I)$ and $\ell_p(I)$ Consider the transformation C define, defined as follows:

$$C : X_p(I) \longmapsto \ell_p(I),\ \mathbf{u} \longmapsto C\mathbf{u} = \mathbf{v}, \quad \mathbf{v} = (\mathbf{v}_i), \quad \mathbf{v}_i = \frac{1}{i}\sum_{j=1}^{i} \mathbf{u}_j = [\frac{1}{i}\sum_{j=1}^{i} u_j^-, \frac{1}{i}\sum_{j=1}^{i} u_j^+],$$

$(i \in \mathbb{N})$. Under operations which are given in (2) and (3) the linearity of C is clear. Further, it is trivial that $\mathbf{u} = \theta = [0, 0]$ whenever $C\mathbf{u} = \theta$ and hence C is injective. Let $\mathbf{v} \in \ell_p(I)$ and define the sequence $\mathbf{u} = (\mathbf{u}_k)$ by $\mathbf{u}_k = k\mathbf{v}_k - (k-1)\mathbf{v}_{k-1} = [\sum_{j=k-1}^{k} j(-1)^{k-j} v_j^-, \sum_{j=k-1}^{k} j(-1)^{k-j} v_j^+]$, $(k \in \mathbb{N})$. Then

$$\begin{aligned}
\|\mathbf{u}\|_{X_p(I)}^p &= \sum_k \max\{|k^{-1}\sum_{j=1}^{k} u_j^-|^p, |k^{-1}\sum_{j=1}^{k} u_j^+|^p\} \\
&= \sum_k \max\{|k^{-1}\sum_{i=1}^{k}(\sum_{j=k-1}^{k} j(-1)^{k-j} v_j^-)|^p, |k^{-1}\sum_{i=1}^{k}(\sum_{j=k-1}^{k} j(-1)^{k-j} v_j^+)|^p\} \\
&= \sum_k \max\{|v_j^-|^p, |v_j^+|^p\} = \|\mathbf{v}\|_{\ell_p(I)}^p
\end{aligned}$$

which says us that $\mathbf{u} \in X_p(I)$, consequently C is surjective. Hence, C is linear bijection map between the spaces $X_p(I)$ and $\ell_p(I)$. This completes proof. □

Now let $(\mathbf{u}_k)$ and $\mathbf{u}_k = [-1, 0]$ if k is odd; $\mathbf{u}_k = [0, 1]$ if k is even. Then $(\mathbf{u}_k)$ is in $X_p(I)$ but $|\mathbf{u}_k| = \max\{|u_k^-|, |u_k^+|\} = 1 = [1, 1]$ is not in $X_p(I)$ for $1 < p < \infty$. That is $X_p(I)$ is non-absolute type. Since any space X solid then X is also absolute type. Thus, since $X_p(I)$ is non-absolute type it cannot be solid. This thought suggests that the α-, β-, and γ-duals of $X_p(I)$ may differ, which may be addressed in subsequent sections.

Because of the isomorphism C , defined in Theorem 2.5, is onto the inverse image of the basis of those space $\ell_p(I)$ the basis of the new space $X_p(I)$ repsectively. Therefore, we have the following:

Proposition 2.6. *Define the sequence $\boldsymbol{b}^{(k)} = \{\boldsymbol{b}_n^{(k)}\}_{n\in\mathbb{N}}$ of the elements of the space $X_p(I)$ by*

$$(\boldsymbol{b}_n^{(k)}) = \begin{cases} k(-1)^{k-n}, & (k-1 \leq n \leq k) \\ \theta, & others \end{cases}$$

for every fixed $k \in \mathbb{N}$. Then the sequence $(\boldsymbol{b}_n^{(k)})$ is a basis for $X_p(I)$ in classical manner and any $\boldsymbol{u} \in X_p(I)$ has a unique representation of the form: $\boldsymbol{u} = \sum_k \boldsymbol{v}_k b^{(k)}$ where $\boldsymbol{v}_k = (C\boldsymbol{u})_k$ for all $k \in \mathbb{N}$.

In fact, it is interesting that a sequence of real numbers can serve as a basis for the sequence spaces of interval numbers. I don't currently know if there exists a basis for the sequence spaces of interval numbers that consists of a sequence of interval numbers.

3. Some Matrix Transformations on $\ell_\infty(I)$, $\ell_p(I)$ and $X_p(I)$

In this section, we will represent an infinite matrix with each term being a real number using matrix A, and an infinite matrix with each term being an interval number using matrix $\mathbf{A} = ([a^-_{nk}, a^+_{nk}])$.

Let $\mathbf{A} = (\mathbf{a}_{nk})$ be an infinite matrix of the interval numbers and $\lambda(I), \mu(I) \subset w(I)$. For $\mathbf{u} = (\mathbf{u}_k) \in \lambda(I))$ if $\mathbf{Au} = \mathbf{v} \in \mu(I)$ then the matrix $\mathbf{A}$ is called matrix transform from $\lambda(I)$ to $\mu(I)$. The meaning of this is that the matrix $\mathbf{A}$ transforms every element $\lambda(I)$ in the form of $(\mathbf{u}_k)$ to $\mu(I)$. The product of the $\mathbf{A}$ and $\mathbf{u}$ gives to us

$$\left[\sum_k [a^-_{1k}, a^+_{1k}][u^-_k, u^+_k] \ \sum_k [a^-_{2k}, a^+_{2k}][u^-_k, u^+_k] \ \cdots \ \sum_k [a^-_{nk}, a^+_{nk}][u^-_k, u^+_k] \ \cdots \right]^T_{1\times\infty}$$

as a column matrix. Clearly, every $n \in \mathbb{N}$ the series $\sum_k [a^-_{nk}, a^+_{nk}][u^-_k, u^+_k]$ must be convergent.

Let $\mathbf{u} \in \lambda(I))$ and $\omega(\mathbf{u}_k) \geq \omega(\mathbf{u}_{k+1})$. Given $\mathbf{A} = (\mathbf{a}_{nk}) = a^-_{nk} = a^+_{nk} > 0$ of real numbers and (a_{nk}) be a decreasing sequence of real numbers, that is for all $k, n \in \mathbb{N}$, $a_{nk} \geq a_{n+1,k+1}$. Under *extend interval arithmetics operations* of Markov's and due to the hypothesis above, the following equation is valid.

$$\sum_k a_{nk}[u^-_k, u^+_k] = [a_{n1}u^-_1 + \sum_{k\geq 2} a_{nk}u^+_k, a_{n1}u^+_1 + \sum_{k\geq 2} a_{nk}u^-_k] \tag{11}$$

Some authors, such as Başar [3], Başar and Çolak [4], Kuttner [9], Lorentz and Zeller [11] worked on the dual summability methods. Now, we give a extension of dual summability methods for interval type sequence spaces:

Let us suppose that the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ map the sequences of intervals $\mathbf{u} = (\mathbf{u}_k)$ and $\mathbf{v} = (\mathbf{v}_k)$ which are connected by the relation $\mathbf{v}_n = [v^-_n, v^+_n] = \frac{1}{n}\sum_{k=1}^n \mathbf{u}_k = \frac{1}{n}\sum_{k=1}^n [u^-_k, u^+_k] = [\frac{1}{n}\sum_{k=1}^n] u^-_k, \frac{1}{n}\sum_{k=1}^n u^+_k]$ to the sequences of interval numbers $\mathbf{z} = (\mathbf{z}_n)$ and $\mathbf{t} = (\mathbf{t}_n)$, respectively, i.e.,

$$\mathbf{z}_n = (A\mathbf{x})_n = \sum_k a_{nk}\mathbf{x}_k, \ (n \in \mathbb{N}) \tag{12}$$

and

$$\mathbf{t}_n = (B\mathbf{y})_n = \sum_k b_{nk}\mathbf{y}_k, \ (n \in \mathbb{N}). \tag{13}$$

It is clear here that the method B is applied to the C-transform of the sequence of intervals $\mathbf{x} = (\mathbf{x}_k)$ while the method A is directly applied to the entries of the sequence of intervals $\mathbf{x} = (\mathbf{x}_k)$. So, the methods A and B are essentially different.

Let us assume that the matrix product BC exists which is a much weaker assumption than the conditions on the matrix B belonging to any matrix class, in general. The methods A and B in (12), (13) are called *dual summability methods* if $\mathbf{z}_n$ reduces to $\mathbf{t}_n$ (or $\mathbf{t}_n$ reduces to $\mathbf{z}_n$) under the application of formal summation by parts. This leads us to the fact that BC exists and is equal to A and $(BC)\mathbf{x} = B(C\mathbf{x})$ formally holds, if one side exists. This statement is equivalent to the following relation between the entries of the matrices $A = (a_{nk})$ and $B = (b_{nk})$:

$$a_{nk} := \sum_{j=k}^{\infty} \frac{1}{j+1} b_{nj} \quad \text{or} \quad b_{nk} := (k+1)(a_{nk} - a_{n,k+1}) = (k+1)\Delta a_{nk} \tag{14}$$

for all $n, k \in \mathbb{N}$.

Let's assume that the following are given

$$\sup_n \sum_{k\geq 2} |a_{nk}| < \infty \quad (15)$$

$$\sup_n |a_{n1}| < \infty \quad (16)$$

$$\sup_K \sum_n |\sum_{k\in K, k\geq 2} a_{nk}|^p < \infty \quad (17)$$

$$\sum_n |a_{n1}|^p < \infty \quad (18)$$

$$\sum_n |n^{-1}\sum_{i=1}^{n} a_{i1}|^p < \infty \quad (19)$$

$$\sup_n \sum_k |a_{nk}|^q < \infty, \ q = p(p-1)^{-1} \quad (20)$$

$$\sup_n |a_{n1}|^q < \infty \quad (21)$$

$$\sup_K \sum_n |\sum_{k\in K, k\geq 2} (k+1)(a_{nk} - a_{nk+1})|^p < \infty \quad (22)$$

$$\sum_n |2(a_{n1} - a_{n2})|^p < \infty \quad (23)$$

$$\sup_K \sum_n |\sum_{k\in K, k\geq 2} n^{-1}\sum_{i=1}^{n} a_{ik}|^p < \infty \quad (24)$$

3.1. Matrix Transformations on $\ell_\infty(I)$.

Theorem 3.1. *$A \in (\ell_\infty(I) : \ell_\infty(I))$ if and only if (15) and (16) holds.*

Proof. Let us suppose that $\mathbf{u} = ([u_k^-, u_k^+]) \in \ell_\infty(I)$ with $\omega(\mathbf{u}_k) \geq \omega(\mathbf{u}_{k+1})$ and given $\mathbf{A}$ in the form $(\mathbf{a}_{nk}) = a_{nk}^- = a_{nk}^+ > 0$ and $a_{nk} \geq a_{nk+1}$. Then

$$\begin{aligned} ||\mathbf{Au}||_{\ell_\infty(I)} &= ||\sum_k a_{nk}[u_k^-, u_k^+]||_{\ell_\infty(I)} \\ &= \sup_n (\max\{|a_{n1}u_1^- + \sum_{k\geq 2} a_{nk}u_k^+|, |a_{n1}u_1^+ + \sum_{k\geq 2} a_{nk}u_k^-|\}) \end{aligned} \quad (25)$$

Since $([u_k^-, u_k^+]) \in \ell_\infty(I)$, we have $\sup_k\{|u_k^-|, |u_k^+|\} < \infty$. Thus equality (25) can rearrangement as follows:

$$||\mathbf{Au}||_{\ell_\infty(I)} \leq \sup_n |a_{n1}|(\max\{|u_1^-|, |u_1^+|\}) + (\sup_n \sum_{k\geq 2} |a_{nk}|)(\max_{k\geq 2}\{|u_k^-|, |u_k^+|\}) \quad (26)$$

Let we write $M_1 = \max\{|u_1^-|, |u_1^+|\}$, $M_2 = \max\{|u_k^-|, |u_k^+|\}$ and $M = \max\{M_1, M_2\}$. Then the last side of the equality (30) is $||\mathbf{Au}||_{\ell_\infty(I)} \leq M\left(\sup_n |a_{n1}| + \sup_n \sum_{k\geq 2} |a_{nk}|\right)$ which it yields that $\sup_n |a_{n1}| < \infty$ and $\sup_n \sum_{k\geq 2} |a_{nk}| < \infty$ for all $k \in \mathbb{N}$.

Conversely, let suppose that $\mathbf{u} = ([u_k^-, u_k^+]) \in \ell_\infty(I)$ and the conditions $\sup_n |a_{n1}| < \infty$ and $\sup_n \sum_{k\geq 2} |a_{nk}|$ (for all $k \in \mathbb{N}$) are holds. Then it is easily see that $\mathbf{Au} \in \ell_\infty(I)$. This completes proof. □

Theorem 3.2. *$A \in (\ell_\infty(I) : \ell_p(I))$ if and only if (17) and (18) holds.*

Proof. Let us suppose that $A \in (\ell_\infty(I), \ell_p(I))$. Then for all $\mathbf{u} \in \ell_\infty(I)$ the sequence $(A(\mathbf{u})_n)$ exists and in $\ell_p(I)$. If we consider (11) we have

$$\begin{aligned} ||A\mathbf{u}||_{\ell_p(I)} &= \left(\sum_n |(A\mathbf{u})|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_n (\max\{|a_{n1}u_1^- + \sum_{k\geq 2} a_{nk}u_k^+|, |a_{n1}u_1^+ + \sum_{k\geq 2} a_{nk}u_k^-|\})^p\right)^{\frac{1}{p}} \end{aligned}$$

$$\le \left(\sum_n (\max\{|a_{n1}u_1^-| + |\sum_{k\ge 2} a_{nk}u_k^+|, |a_{n1}u_1^+| + |\sum_{k\ge 2} a_{nk}u_k^-|\})^p \right)^{\frac{1}{p}}$$

$$= \left(\sum_n (\max\{|a_{n1}u_1^-|, |a_{n1}u_1^+|\} + \max\{|\sum_{k\ge 2} a_{nk}u_k^+|, |\sum_{k\ge 2} a_{nk}u_k^-|\})^p \right)^{\frac{1}{p}}. \tag{27}$$

The last expression is holds, for all $\mathbf{u} \in \ell_\infty(I)$. Let K be finite subset of $\mathbb{N}$. For $n \in K$ we may chose a sequence $\mathbf{u}$ as follows:

$$\mathbf{u} = \mathbf{u}_k^n = \begin{cases} [0,1], & n = k \\ \theta = [0,0], & others \end{cases}$$

then the expression (27) is equal to

$$\left(\sum_n (\max\{0, |a_{n1}|\} + \max\{|\sum_{k\ge 2} a_{nk}u_k^+|, 0\})^p \right)^{\frac{1}{p}}$$

$$= \left(\sum_n (|a_{n1}| + |\sum_{k\ge 2} a_{nk}u_k^+|)^p \right)^{\frac{1}{p}}$$

$$\le \left(\sum_n |a_{n1}|^p \right)^{\frac{1}{p}} + \left(\sum_n |\sum_{k\ge 2} a_{nk}u_k^+|^p \right)^{\frac{1}{p}}.$$

And from here we see that the conditions (17) and (18) are necessary. Conversely, if the conditions (17) and (18) are holds then easily we see that $A \in (\ell_\infty(I), \ell_p(I))$ for all $\mathbf{u} \in \ell_\infty(I)$. □

Let's assume that there exists a relation

$$e_{nk} = n^{-1} \sum_{i=1}^{n} d_{ik} \tag{28}$$

between matrices D and E. And $\lambda(I)$ given a sequence space of intervals. Then

Theorem 3.3. *$D \in (\lambda(I) : X_p(I))$ if and only if $E \in (\lambda(I) : \ell_p(I))$ holds.*

Proof. The proof of this theorem can be easily demonstrated using a similar approach as utilized by Şengönül [18]. Therefore, we did not provide the proof. □

Proposition 3.4. *Let the equality (28) holds. Then the result of the Theorem 3.3 we have $A \in (\ell_\infty(I) : X_p(I))$ if and only if (24) and (19) holds.*

3.2. Matrix Transformations on $\ell_p(I)$.

Theorem 3.5. *$A \in (\ell_p(I) : \ell_\infty(I))$ if and only if (20) and (21) holds.*

Proof. Let us suppose that $A \in (\ell_p(I) : \ell_\infty(I))$ hold. Then for all $\mathbf{u} = ([u_k^-, u_k^+]) \in \ell_p(I)$ with $\omega(\mathbf{u}_k) \geq \omega(\mathbf{u}_{k+1})$ and given $\mathbf{A}$ in the form $(\mathbf{a}_{nk}) = a_{nk}^- = a_{nk}^+ > 0$ and $a_{nk} \geq a_{nk+1}$ we have

$$||\mathbf{Au}||_{\ell_\infty(I)} = ||\sum_k a_{nk}[u_k^-, u_k^+]||_{\ell_\infty(I)}$$
$$= \sup_n(\max\{|a_{n1}u_1^- + \sum_{k\geq 2} a_{nk}u_k^+|, |a_{n1}u_1^+ + \sum_{k\geq 2} a_{nk}u_k^-|\}) \tag{29}$$

Since $([u_k^-, u_k^+]) \in \ell_p(I)$, we have $\sup_k\{|u_k^-|, |u_k^+|\} < \infty$. Thus equality (29) can rearrangement as follows:

$$||\mathbf{Au}||_{\ell_\infty(I)} \leq \sup_n |a_{n1}|(\max\{|u_1^-|, |u_1^+|\}) + (\sup_n \sum_{k\geq 2} |a_{nk}|)(\max_{k\geq 2}\{|u_k^-|, |u_k^+|\}) \tag{30}$$

Under assumptions $M_1 = \max\{|u_1^-|, |u_1^+|\}$, $M_2 = \max\{|u_k^-|, |u_k^+|\}$ and $M = \max\{M_1, M_2\}$; we see that last side of the equality (30) is $||\mathbf{Au}||_{\ell_\infty(I)} \leq M\big(\sup_n |a_{n1}| + \sup_n \sum_{k\geq 2} |a_{nk}|\big)$ which it yields that $\sup_n |a_{n1}|^q < \infty$ and $\sup_n \sum_{k\geq 2} |a_{nk}|^q < \infty$.

Now let suppose that $\mathbf{u} = ([u_k^-, u_k^+]) \in \ell_p(I)$ and the conditions $\sup_n |a_{n1}|^q < \infty$ and $\sup_n \sum_{k\geq 2} |a_{nk}|^q$ are holds. Then it is easily see that $\mathbf{Au} \in \ell_\infty(I)$. This completes proof. □

4. Results and Discussion

Sequences of interval numbers, each component of which is an interval, are not a vector space according to classical interval arithmetic. Due to this deficiency, the possibility of obtaining rich algebraic structures on interval number sequences is lost. In other words, it is not possible to elevate the spaces of interval number sequences to richer mathematical structures according to these algebraic operations. In this study, an attempt was made to obtain the linear space structure of interval number sequence spaces by using the algebraic operations suggested by Markov for intervals. Significant results were obtained. By using these operations, the door has been opened to investigate the properties of interspaced number sequence spaces such as symmetry, monotonicity, normality or rotundness. For example, it has been shown that a sequence of real numbers can form the basis for $X_p(I)$. Additionally, in this study, the infinite matrix A is considered as an infinite matrix of real numbers. If the elements of the matrix A are also intervals, it remains an open problem whether matrix transformations can be performed. But my expectation is that there may be nothing more than a few additional conditions.

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