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NUMERICAL STUDY OF THE SERIES SOLUTION METHOD TO ANALYSIS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this article, the Series Solution Method (SSM) is employed to solve the linear or non-linear Volterra integro-differential equations. Numerous examples have been presented to explain the numerical results, which is the comparison between the exact solution and the numerical solution, and it is found through the tables. The amount of error between the exact solution and the numerical solution is very small and almost nonexistent, and it is also illustrated through the graph how the exact solution completely applies to the numerical solution. This proves the accuracy of the method, which is the Series Solution Method (SSM) for solving the linear or non-linear Volterra integro-differential equations using Mathematica. Furthermore, this approach yields numerical results with remarkable accuracy, speed, and ease of use.

AMS Mathematics Subject Classification : 65R20, 65R99, 45G99.
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1. Introduction

These types of equations, such as ordinary and partial differential equations and Integro-differential equations, are usually the result of mathematical modelling in real-life problems. The use of integro-differential equations in Mathematical problems in physical sight is widespread, and they are used in various areas, such as biological models, fluid dynamics, and economics [1-20, 34]. Moreover, it is challenging to solve integro-differential equations analytically, so it is necessary to find an effective approximate solution to determine nonlinear problems. In almost every field of science, the Series Solution Method (SSM) is a

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popular analytical technique that plays a significant role in dealing with nonlinear phenomena [1–5].

With this method, you can obtain a highly effective conclusion for analytical approximations associated with nonlinear problems. But sometimes, when complicated nonlinearity occurs, this method may be difficult to use. There are advanced techniques in the literature that are more efficient and easier to compute results with. This work's main objective is to introduce a recent analytical technique, such as the series solution method (SSM) [8–12]. This is a method that is both simple and effective, and it is expected to become more significant in the field of nonlinear science. The SSM method was successfully applied to numerous problems. In recent times, many mathematicians have been using these techniques.

The main purpose of this paper is the introduction of the Series Solution Method (SSM) for the integro-differential equations. In tabular and graphical format, the proposed method is compared to the exact solution. Numerical and graphical results are derived through the use of Wolfram Mathematica software.

In real life, the fundamental sciences that describe physical, chemical, engineering, and medical phenomena are known as integral or integro-differential equations. Furthermore, they provide significant contributions to achieving analytical and numerical solutions to these phenomena in different areas of our lives [20–21]. We discover that nonlinear integro-differential equation solutions are more challenging to solve than linear integro-differential equation solutions. The references contain numerous analytical and numerical methods that can be used to solve linear and nonlinear integro-differential equations [22–25]. The numerical solution of the Volterra integro-differential equation of the 2nd type [27–28] is the topic of our discussion. In [29], MATLAB and Mathematica are used to illustrate the series solution approach for the second class of the Fredholm integro-differential (FIDE) problem. The Series Solution Method (SSM) approach was used to resolve the Fredholm integro-differential problems of the second kind [30].

In this article, we have applied the series solution method to a variety of examples using the Wolfram Mathematica algorithm. The process involves determining the approximate solution, comparing it to the exact solution, and calculating the error between them. The main purpose of this work is to use the Series Solution Method (SSM) to solve the second-order Volterra integro-differential equations with Wolfram Mathematica. The paper is arranged as follows: In Section 2, the Volterra integro-differential equation; in Section 3, the Series Solution Method (SSM); in Section 4, numerical examples are also considered to show the ability of the proposed method; and at the end, a conclusion is drawn in Section 5.

2. Volterra integro-differential equations

Early in the 1900s, Volterra made the first use of this type of equation. Through his research, Volterra founded the field of integro-differential equations while working on a population growth model with a focus on hereditary factors. Scientists and researchers researched the topic of integro-differential equations through their work in science applications such as heat transfer, diffusion processes in general, neutron diffusion, and biological species coexisting with growing and decreasing rates of generation. More information regarding the sources of these equations can be derived in publications on advanced integral equations as well as physics, biology, and engineering applications [23, 32]. The linear or non-linear Volterra integro-differential equations are characterized by at least one variable limit of integration.

The linear Volterra integro-differential equation (LVIDE) of the second kind as follows:

$$\phi^n(z) = g(z) + \lambda \int_0^z K(z,t)\phi(t)dt$$

and the non-linear Volterra integro-differential equation (NVIDE) of the second kind as follows:

$$\phi^n(z) = g(z) + \lambda \int_0^z K(z,t)G(\phi(t))dt \tag{1}$$

where $\phi^n(z)$ indicate the nth derivative of $\phi(z)$ such as $\phi^n = \frac{d^n\phi}{dz^n}$, initial conditions $\phi^p(0) = d_p; 0 \leq p \leq (n-1)$ such as $\phi(0), \phi^{'}(0), \phi^{''}(0), \ldots, \phi^{n-1}(0)$, the function $g(z)$ are given real valued functions, $K(z,t)$ is the kernel of integral equation, λ is suitable constant and d_p are constants that define the initial conditions.

The function $G(\phi(z))$ is a non-linear function of $\phi(z)$ such as $\phi^2(z), sin(\phi(z))$, and $e^{\phi(z)}$. Because the equation in (1) combines the differential operator and the integral operator, it is necessary to define initial conditions for the determination of the particular solution $\phi(z)$ of the nonlinear Volterra integro-differential equation.

Several techniques, such as the Adomian Decomposition Method (ADM) [31, 32, 34, 35], the Variation Iteration Method (VIM) [32, 33], and the combined Laplace transform-Adomian Decomposition Method [31], have been applied to solving these problems. The advantage of these methods is their capability of combining the two efficient methods for obtaining exact solutions to nonlinear equations. In order to demonstrate the utility of the suggested method, some examples of the system of Volterra integro-differential equations [26] are given, which are solved using the established method. The obtained results are compared with the exact solutions. In all cases, the present algorithm performed excellently.

3. The Series Solution Method (SSM)

This is a traditional method that is based on the Taylor series and has been used in differential and integral equations as well. However, the method is mainly used for solving Volterra integral equations. In what follows, we present a brief idea about the method, whose details can be found in many references, such as [29–32].

Given that $\phi(z)$ is an analytic function and represented by the Taylor Series given by

$$\phi(z) = \sum_{p=0}^{\infty} b_p z^p, \tag{2}$$

In this section, we'll prove a very effective method that finds analytic functions from the Taylor series, and then we will solve the Volterra integro-differential equations (1).

Where b_p are the recursively determined constants. The initial few coefficients can be determined by using the prescribed initial conditions, where we may use

$$\begin{aligned} b_0 &= \phi(0), \\ b_1 &= \phi^{'}(0), \\ b_2 &= \frac{1}{2!}\phi^{''}(0), \\ &\vdots \end{aligned} \tag{3}$$

Substituting (3) into both sides of (1), and assuming that the kernel $K(z,t)$ is separable as $K(z,t) = g_1(z).g_2(t)$, we obtain

$$\sum_{p=0}^{\infty} (b_p z^p)^{(n)} = T(g(z)) + \lambda g_1(z) \int_0^z g_2(t) G\left(\sum_{p=0}^{\infty} b_p t^p\right) dt. \tag{4}$$

Generally, we have

$$(b_0 + b_1 z + b_2 z^2 + \dots)^{(n)} = T(g(z)) + \lambda g_1(z) \int_0^z g_2(t) G(b_0 + b_1 t + b_2 t^2 + \dots) dt. \tag{5}$$

The Taylor series for $g(z)$ is $T(g(z))$. Then we solved equation $g(z)$. But first we solved the integral in equation (5), so we integrated the unknown function $\phi(z)$. Terms of the form $t^p, p \geq 0$ will be integrated. We are finding a series of solutions.

Further, if $g(z)$ includes elementary functions like logarithmic, exponential, trigonometric, etc., then $g(z)$ also expands through the Taylor series. After integrating and expanding the Taylor series in Eq. $g(z)$, So, we compared both sides of the equation. Therefore, we collect the like powers of $z's$ coefficients. So, we solved the coefficients, and then we found a recurrence relation in $b_i, i \geq 0$. So, we solved this relationship.

After we find the coefficients $b_i, i \geq 0$ these coefficients are put in Eq. (5). So, we obtained an exact solution if there exists one. But if there is no exact solution, then we obtain a series that can be used for a numerical solution.

4. Problems

In this section, we give several test examples to confirm our analysis. The main purpose here is to solve the four linear Volterra integro-differential equations and three non-linear Volterra integro-differential equations (VIDEs) by applying the Series Solution Method (SSM) given above. All the computations were performed using software, both MATLAB and Mathematica.

4.1. Non-linear Volterra integro-differential equations.

4.1.1. Problem. Let us consider the non-linear Volterra integro-differential equation (NVIDE):

$$\phi^{'}(z) = ze^{z} + \frac{1}{3}z^{3}e^{2z} - \int_{0}^{z} e^{2(z-t)}[\phi^{'}(t)]^{2}dt,$$

With initial condition $\phi(0) = 0$ and exact solution is $\phi(z) = e^{z}(z-1)+1$.

Using SSM: The initial few coefficients,

$$b_0 = 0, b_1 = 0, b_2 = \frac{1}{2}, b_3 = \frac{1}{3}, b_4 = \frac{1}{8}, b_5 = \frac{1}{30}, \ldots \tag{6}$$

Substituting (6) into (2), gives the following series solution

$$\phi(z) = \frac{1}{2}z^{2} + \frac{1}{3}z^{3} + \frac{1}{8}z^{4} + \frac{1}{30}z^{5} + \ldots.$$

TABLE 1. Comparison between Exact and SSM with errors.

z	$Exact \quad Sol^{n}.$	SSM	$Error = \|Exact - SSM\|$
0	0	0	0
0.1	0.00534617373	0.00534616667	$7.06000000e^{-9}$
0.2	0.02287779347	0.02287733333	$4.60140000e^{-7}$
0.3	0.05509883470	0.05509350000	$5.33470000e^{-6}$
0.4	0.10490518142	0.10487466667	$3.05147500e^{-5}$
0.5	0.17563936465	0.17552083333	$1.18531320e^{-4}$
0.6	0.27115247984	0.27079200000	$3.60479840e^{-4}$
0.7	0.39587418776	0.39494816667	$9.26021090e^{-4}$
0.8	0.55489181430	0.55278933333	$2.10248097e^{-3}$
0.9	0.75403968888	0.74969550000	$4.34418888e^{-3}$
1.0	1	0.99166666667	$8.33333333e^{-3}$

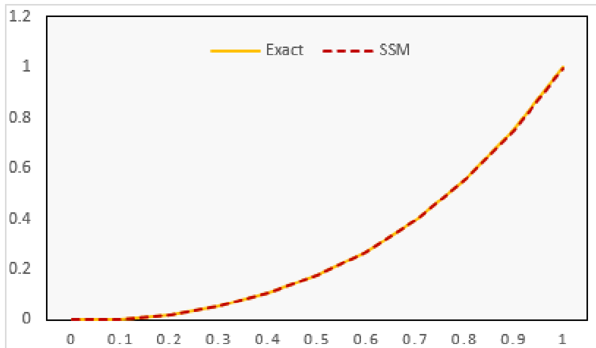

FIGURE 1. Comparison between Exact and SSM.

4.1.2. Problem. Let us consider the non-linear Volterra integro-differential equation (NVIDE):

$$\phi^{''}(z) = sinh(z) + \frac{1}{2}z + \frac{1}{2}cosh(z)sinh(z) - \int_0^z [\phi^{'}(t)]^2 dt,$$

With initial conditions $\phi(0) = 0, \quad \phi^{'}(0) = 1$ and exact solution is $\phi(z) = sinh(z)$.

Using SSM: The first few coefficients,

$$b_0 = 0, b_1 = 1, b_2 = 0, b_3 = \frac{1}{3!}, b_4 = 0, b_5 = \frac{1}{5!}, b_6 = 0, b_7 = \frac{1}{7!}, \ldots \qquad (7)$$

Substituting (7) into (2), gives the following series solution

$$\phi(z) = z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \frac{1}{7!}z^7 + \ldots.$$

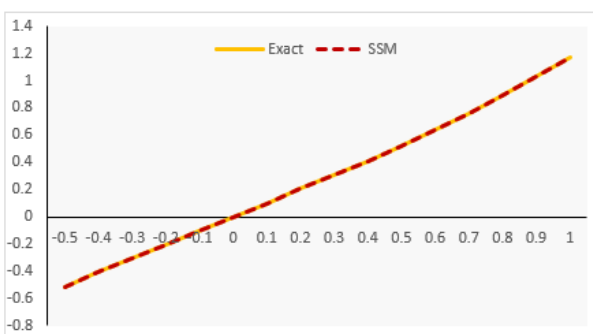

FIGURE 2. Comparison between Exact and SSM.

TABLE 2. Comparison between Exact and SSM with errors.

z	$Exact \quad Sol^n.$	SSM	$Error = \|Exact - SSM\|$
-0.5	-0.52109530550	-0.52109530010	$5.40000e^{-9}$
-0.4	-0.41075232580	-0.41075232510	$0.70000e^{-9}$
-0.3	-0.30452029340	-0.30452029340	0
-0.2	-0.20133600250	-0.20133600250	0
-0.1	-0.10016675000	-0.10016675000	0
0	0	0	0
0.1	0.10016675002	0.10016675002	0
0.2	0.20133600254	0.20133600254	0
0.3	0.30452029345	0.30452029339	$0.60000e^{-10}$
0.4	0.41075232580	0.41075232508	$7.20000e^{-10}$
0.5	0.52109530549	0.52109530010	$5.39000e^{-9}$
0.6	0.63665358215	0.63665355429	$2.78600e^{-8}$
0.7	0.75858370184	0.75858359014	$1.11700e^{-7}$
0.8	0.88810598219	0.88810561016	$3.72030e^{-7}$
0.9	1.02651672571	1.02651565018	$1.07553e^{-6}$
1.0	1.17520119364	1.17519841270	$2.78094e^{-6}$

4.1.3. Problem. Let us consider the non-linear Volterra integro-differential equation (NVIDE):

$$\phi^{'}(z) = \frac{-1}{2} + \int_0^z [\phi^{'}(t)]^2 dt,$$

With initial condition $\phi(0) = 0$ and exact solution is $\phi(z) = -ln(\frac{1}{2}z + 1)$.

Using SSM: The first few coefficients,

$$b_0 = 0, b_1 = \frac{-1}{2}, b_2 = 0, b_3 = \frac{1}{12}, b_4 = 0, b_5 = 0, b_6 = 0, b_7 = \frac{1}{1008}, \dots \quad (8)$$

Substituting (8) into (2), gives the following series solution

$$\phi(z) = \frac{1}{2}z + \frac{1}{12}z^3 + \frac{1}{1008}z^7 + \dots.$$

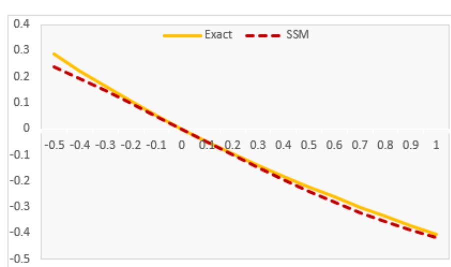

FIGURE 3. Comparison between Exact and SSM.

TABLE 3. Comparison between Exact and SSM with errors.

z	$Exact \quad Sol^n.$	SSM	$Error = \|Exact - SSM\|$
-0.5	0.28768207245	0.23957558284	$4.810648961e^{-2}$
-0.4	0.22314355131	0.19466504127	$2.847851004e^{-2}$
-0.3	0.16251892950	0.14774978304	$1.476914646e^{-2}$
-0.2	0.10536051566	0.09933332063	$6.027195030e^{-3}$
-0.1	0.05129329439	0.04991666657	$1.376627820e^{-3}$
0	0	0	0
0.1	-0.04879016420	-0.04991666660	$1.12650240e^{-3}$
0.2	-0.09531017980	-0.09933332060	$4.02314080e^{-3}$
0.3	-0.13976194240	-0.14774978300	$7.9878406e^{-3}$
0.4	-0.18232155680	-0.19466504130	$1.23434845e^{-2}$
0.5	-0.22314355130	-0.23957558280	$1.64320315e^{-2}$
0.6	-0.26236426450	-0.28197222860	$1.96079641e^{-2}$
0.7	-0.30010459250	-0.32133496600	$2.12303735e^{-2}$
0.8	-0.33647223660	-0.35712528260	$2.06530460e^{-2}$
0.9	-0.37156355640	-0.38877549910	$1.72119427e^{-2}$
1.0	-0.40546510810	-0.41567460320	$1.02094951e^{-2}$

4.2. Linear Volterra integro-differential equations.

4.2.1. Problem. Let us consider the linear Volterra integro-differential equation (LVIDE):

$$\phi^{'}(z) = 1 - 2zsinz + \int_0^z \phi^{'}(t)dt,$$

With initial condition $\phi(0) = 0$ and exact solution is $\phi(z) = zcosz$.

Using SSM: The first few coefficients,

$$b_0 = 0, b_1 = 1, b_2 = 0, b_3 = \frac{-1}{2}, b_4 = 0, b_5 = \frac{1}{24}, b_6 = 0, b_7 = \frac{-1}{720}, \dots \quad (9)$$

Substituting (9) into (2), gives the following series solution

$$\begin{aligned}\phi(z) &= z - \frac{1}{2}z^3 + \frac{1}{24}z^5 - \frac{1}{720}z^7 + \dots, \\ &= z - \frac{1}{2!}z^3 + \frac{1}{4!}z^5 - \frac{1}{6!}z^7 + \dots.\end{aligned}$$

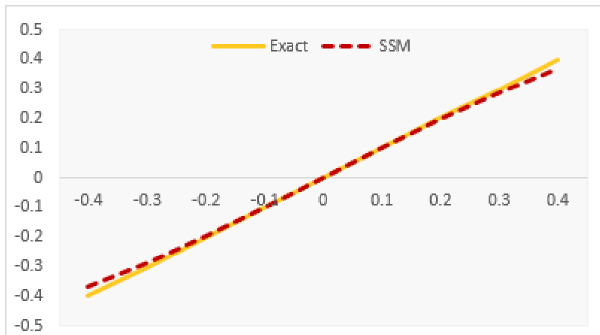

FIGURE 4. Comparison between Exact and SSM.

TABLE 4. Comparison between Exact and SSM with errors.

z	$Exact \quad Sol^n.$	SSM	$Error = \|Exact - SSM\|$
-0.5	-0.49998096150	-0.4387912326	$6.118972890e^{-2}$
-0.4	-0.39999025230	-0.3684243911	$3.156586120e^{-2}$
-0.3	-0.29999588770	-0.2866009463	$1.339494140e^{-2}$
-0.2	-0.19999878150	-0.1960133156	$3.985465900e^{-3}$
-0.1	-0.09999984770	-0.0995004165	$4.994312000e^{-4}$
0	0	0	0
0.1	0.09999984769	0.09950041653	$4.994311600e^{-4}$
0.2	0.19999878152	0.19601331556	$3.985465960e^{-3}$
0.3	0.29999588768	0.28660094625	$1.339494143e^{-2}$
0.4	0.39999025228	0.36842439111	$3.156586117e^{-2}$
0.5	0.49998096153	0.43879123264	$6.118972889e^{-2}$

4.2.2. Problem. Let us consider the linear Volterra integro-differential equation (LVIDE):

$$\phi^{'}(z) = -1 + \frac{1}{2}z^2 - ze^z - \int_0^z t\phi(t)dt,$$

With initial condition $\phi(0) = 0$ and exact solution is $\phi(z) = 1 - e^z$.

Using SSM: The first few coefficients,

$$b_0 = 0, b_1 = -1, b_2 = \frac{-1}{2}, b_3 = \frac{-1}{6}, b_4 = \frac{-5}{24}, b_5 = \frac{-7}{120}, b_6 = \frac{-9}{720}, \dots \tag{10}$$

Substituting (10) into (2), gives the following series solution

$$\begin{aligned}\phi(z) &= -z - \frac{1}{2}z^2 - \frac{1}{6}z^3 - \frac{5}{24}z^4 - \frac{7}{120}z^5 - \frac{9}{720}z^6 + \dots, \\ &= -z - \frac{1}{2!}z^2 - \frac{1}{3!}z^3 - \frac{5}{4!}z^4 - \frac{7}{5!}z^5 - \frac{9}{6!}z^6 + \dots.\end{aligned}$$

TABLE 5. Comparison between Exact and SSM with errors.

z	$Exact \quad Sol^n.$	SSM	$Error = \|Exact - SSM\|$
-0.5	0.39346934029	0.38444010417	$9.02923612e^{-3}$
-0.4	0.32967995396	0.32587946667	$3.80048726e^{-3}$
-0.3	0.25918177932	0.25794513750	$1.23664182e^{-3}$
-0.2	0.18126924692	0.18101786667	$2.51380250e^{-4}$
-0.1	0.09516258196	0.09514640417	$1.61777900e^{-5}$
0	0	0	0
0.1	-0.10517091810	-0.10518809580	$1.71777000e^{-5}$
0.2	-0.22140275820	-0.22168613330	$2.83375010e^{-4}$
0.3	-0.34985880760	-0.35133836250	$1.47955490e^{-3}$
0.4	-0.49182469760	-0.49664853330	$4.82383570e^{-3}$
0.5	-0.64872127070	-0.66087239580	$1.21511251e^{-2}$
0.6	-0.82211880040	-0.84811920000	$2.60003996e^{-2}$
0.7	-1.01375270750	-1.06346219580	$4.97094883e^{-2}$
0.8	-1.22554092850	-1.31305813330	$8.75172048e^{-2}$
0.9	-1.45960311120	-1.60427576250	$1.446726513e^{-1}$
1.0	-1.71828182850	-1.94583333330	$2.275515048e^{-1}$

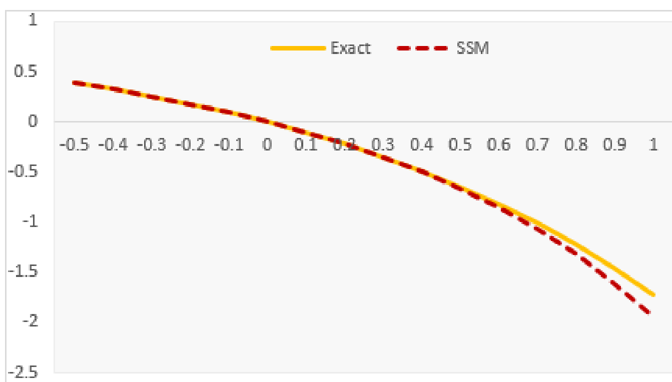

FIGURE 5. Comparison between Exact and SSM.

4.2.3. Problem. Let us consider the linear Volterra integro-differential equation (LVIDE):

$$\phi^{''}(z) = -8 - \frac{1}{3}(z^3 - z^4) + \int_0^z (z-t)\phi(t)dt, \tag{11}$$

With initial condition $\phi(0) = 0, \quad \phi^{'}(0) = 2$ and exact solution is $\phi(z) = 2z - 4z^2$.

Using SSM: Putting Eq. (2) into both sides of the Eq. (11)

$$p(p-1)\sum_{p=2}^{\infty} b_p z^{p-1} = -8 - \frac{1}{3}(z^3 - z^4) + \int_0^z (z-t)\sum_{p=0}^{\infty} b_p z^p dt.$$

Using the initial conditions $b_0 = 0$ and $b_1 = 2$.

Evaluating and the first few coefficients,

$$b_0 = 0, b_1 = 2, b_2 = -4, b_3 = 0, b_4 = 0, b_5 = 0, b_6 = 0, \dots \quad (12)$$

Substituting (12) into (2), gives the following series solution

$$\phi(z) = b_0 z^0 + b_1 z^1 + b_2 z^2 + b_3 z^3 + b_4 z^4 + \dots,$$
$$= 2z - 4z^2.$$

Therefore, it gives an exact solution.

4.2.4. Problem. Let us consider the linear Volterra integro-differential equation (LVIDE):

$$\phi^{''}(z) = \frac{1}{2}z^2 - zcoshz - \int_0^z t\phi(t)dt,$$

With initial condition $\phi(0) = 1, \quad \phi^{'}(0) = -1$ and exact solution is $\phi(z) = 1 - sinhz$.

Using SSM: The initial few coefficients,

$$b_0 = 1, b_1 = -1, b_2 = 0, b_3 = \frac{-1}{6}, b_4 = 0, b_5 = \frac{-1}{120}, b_6 = 0, b_7 = \frac{-1}{5040}, \dots \quad (13)$$

Substituting (13) into (2), gives the following series solution

$$\phi(z) = 1 - z - \frac{1}{6}z^3 - \frac{1}{120}z^5 - \frac{1}{5040}z^7 - \dots,$$
$$= 1 - z - \frac{1}{3!}z^3 - \frac{1}{5!}z^5 - \frac{1}{7!}z^7 - \dots.$$

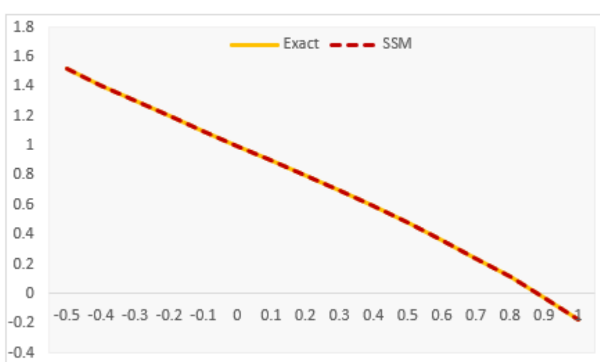

FIGURE 6. Comparison between Exact and SSM.

Table 6. Comparison between Exact and SSM with errors.

z	$Exact \quad Sol^n.$	SSM	$Error = \|Exact - SSM\|$
-0.5	1.5210953055	1.52109530010	$5.4e^{-9}$
-0.4	1.4107523258	1.41075232508	$7.2000006e^{-10}$
-0.3	1.3045202934	1.30452029339	$1.0000001e^{-11}$
-0.2	1.2013360025	1.20133600254	$4.0000003e^{-11}$
-0.1	1.1001667500	1.10016675002	$2.0000002e^{-11}$
0	1	1	0
0.1	0.89983324998	0.89983324998	0
0.2	0.79866399746	0.79866399746	0
0.3	0.69547970655	0.69547970661	$6.0000005e^{-11}$
0.4	0.58924767420	0.58924767492	$7.1999995e^{-10}$
0.5	0.47890469451	0.47890469990	$5.3900000e^{-9}$
0.6	0.36334641785	0.36334644571	$2.7860000e^{-8}$
0.7	0.24141629816	0.24141640986	$1.1170000e^{-7}$
0.8	0.11189401781	0.11189438984	$3.7200000e^{-7}$
0.9	-0.02651672570	-0.02651565020	$1.0755000e^{-6}$
1.0	-0.17520119360	-0.17519841270	$2.7809000e^{-6}$

5. Conclusion

The Series Solution Method is a powerful technique that is capable of handling higher-order linear or non-linear Volterra Integro-differential equations. This technique has been successfully employed for higher-order Volterra Integro differential equations. When the approximation results are found by applying the Series Solution Method (SSM) and compared to the exact solutions with existing results, Also, it is observed that the convergences are quite close to the exact solution. We've noticed that this technique is simple to implement and only achieves the desired accuracy in a few terms compared to other methods. It has been observed that increasing the number of terms in numerical computation leads to clear convergence of the method. The method's ability to run for almost large intervals is demonstrated by some experiments. The application of the method for linear or non-linear Volterra integro-differential equations has been successfully evaluated. This demonstrates the method's accuracy, which is the Series Solution Method (SSM) for solving the linear or non-linear Volterra integro-differential equations using Mathematica. Thus, the Series Solution Method can be the best alternative method for solving linear or non-linear Volterra Integro-differential equation. And this technique is distinguished by its simplicity, quickness, and high accuracy in producing numerical or approximate results.

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References

1. G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic, Boston, 1994.
2. W. Chen, Z. Lu, *An Algorithm for Adomian Decomposition Method*, Applied Mathematics and Computation **159** (2004), 221-235.
3. D.J. Evans, K.R. Raslan, *The Adomian Decomposition Method for Solving Delay Differential Equation*, International Journal of Computer Mathematics **82** (2005), 49-54.
4. A.U. Keskin, *Boundary Value Problems for Engineers with MATLAB Solutions, chapter Adomian Decomposition Method (ADM)*, Springer Cham, (2019) 311-359.
5. E.U. Haq, Q.M.U. Hassan, J. Ahmad, K. Ehsan, *Fuzzy solution of system of fuzzy fractional problems using a reliable method*, Alexandria Engineering Journal **61** (2022), 3051-3058.
6. N.H. Sweilam, *Fourth order integro-differential equations using variational iteration method*, Computers & Mathematics with Applications **54** (2007), 1086-1091.
7. N. Khan, Q.M.U. Hassan, E.U. Haq, M.Y. Khan, K. Ayub, J. Ayub, *Analytical Technique With Lagrange Multiplier For Solving Specific Nonlinear Differential Equations*, Journal of Science and Arts **21** (2021), 5-14.
8. Kumar, Pramod, *Numerical Solutions of Linear Fredholm Integro-Differential Equations by Non-Standard Finite Difference Method*, Applications and Applied Mathematics **10** (2015), 1019-1025.
9. N. Ahmad, B. Singh, *Numerical solution of Integral Equation by Using New Modified Adomian Decomposition Method and Newton Raphson Method*, International Journal of Innovative Technology and Exploring Engineering (IJITEE) **10** (2021), 5-8.
10. Ahmad N., Singh B., *Study of Numerical solution of nonlinear Integral Equations by Using Adomian Decomposition Method and HE's Polynomial*, Journal of Mathematical Control Science and Applications **6** (2), (2020) 93-95.
11. N. Anjum, J.H. He, *Laplace transform: Making the variational iteration method easier*, Applied Mathematics Letters **92** (2019), 134-138.
12. N. Ahmad, B. Singh, *Numerical solution of Integral Equation by Using Galerkin Method with Hermite, Chebyshev and Orthogonal Polynomials*, Journal of Science and Arts **50** (2020), 35-42.
13. S. Abbasbandy and S. Elyas, *Series Solution of the System of Integro-Differential Equations*, A Journal of Physical Sciences **64a** (2009), 811-818.
14. A. Rani, M. Saeed, Q.M. Ul-Hassan, M. Ashraf, M.Y. Khan, K. Ayub, *Solving system of differential equations of fractional order by Homotopy analysis method*, Journal of Science and Arts **17** (2017), 457-468.
15. A.A. Soliman, *Exact solutions of KdV–Burgers' equation by Exp-function method*, Chaos, Solitons & Fractals **41** (2009), 1034-1039.
16. M. Senthilvelan, *On the extended applications of Homogenous Balance Method*, Applied Mathematics and Computation **123** (2001), 381-388.
17. W. Majid Abdul, *A Reliable Modification of Adomian Decomposition Method*, Applied Mathematics and Computation **102** (1999), 77-86.

18. A.W. Majid, S.M. El-Sayed, *A New Modification of the Adomian Decomposition Method for Linear and Nonlinear Operators*, Applied Mathematics and Computation **122** (2001), 393-405.
19. W. Majid, Abdul, *Linear and Nonlinear Integral Equations*, Higher Education Press, Beijing, 2011.
20. L. Peter, *Analytical and Numerical Methods for Volterra Equations*, Studies in Applied Mathematics, SIAM, Philadelphia, 1985.
21. M.G. Amani, A.M. Dalal, Z.B. Badreeh, *Numerical Solution of Volterra Integral Equation of Second Kind Using Implicit Trapezoidal*, Journal of Advances in Mathematics **8** (2014), 1540-1553.
22. A.M. Dalal, *The Adomian Decomposition Method of Fredholm Integral Equation of the Second Kind Using Maple*, Journal of Advances in Mathematics **9** (2014), 1868-1875.
23. M. Malaikah Hunida , *The Adomian Decomposition Method for Solving Volterra-Fredholm Integral Equation Using Maple*, Applied Mathematics **11** (2020), 779-787.
24. A. Ameera, M Dalal, A. Hashim, *Adomian Decomposition Method for Solving Boussinesq Equations Using Maple*, Applied Mathematics **14** (2023), 121-129.
25. A.M. Dalal, *Application of Adomian Decomposition Method for Solving of Fredholm Integral Equation of the Second Kind*, European Journal of Science and Engineering **9** (2014), 1-9.
26. A.M. Dalal, *Adomian Decomposition Method for Solving of Fredholm Integral Equation of the Second Kind Using MATLAB*, International Journal of GEOMATE **11** (2016), 2830-2833.
27. A.M. Dalal, M.M. H., *Numerical Solution of System of Three Nonlinear Volterra Integral Equation Using Implicit Trapezoidal*, Journal of Mathematics Research **10** (2018), 44-58.
28. A.M. Wazwaz, *A First Course in Integral Equations*, World Scientific, 1997.
29. N. Ahmad, B. Singh, *Numerical Solution of Volterra Nonlinear Integral Equation by using Laplace Adomian Decomposition Method*, International Journal of Applied Mathematics **35** (2022), 39-48.
30. A. Asiya, N. Ahmad, *Numerical Accuracy of Fredholm linear Integro-differential equations by using Adomian Decomposition Method, Modified Adomian Decomposition Method and Variational Iteration Method*, Journal of Science and Arts **23** (2023), 625-638.
31. A. Asiya, N. Ahmad, *Numerical Accuracy of Fredholm Integro-differential equations by using Adomian Decomposition Method and Modified Adomian Decomposition Method*, Bull. Cal. Math. Soc. **115** (2023), 567-578.
32. D.I. Lanlege, F.M. Edibo and S.O. Momoh, *Solution of Fredholm Integro-Differential Equation by Variational Iteration Method*, FUDMA Journal of Sciences (FJS) **7** (2023), 1-8.
33. A. Asiya, N. Ahmad and D. Farah, *Application of The Direct Computation Method for Solving a General Fredholm Integro-Differential Equations*, Global and Stochastic Analysis **11** (2024), 65-74.
34. A. Asiya, N. Ahmad, *Numerical Solution for nonlinear Volterra-Fredholm Integro-Differential Equations using Adomian and Modified Adomian Decomposition Method*, Transylvanian Review **31** (2023), 16321-16327.

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