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THEORY OF HYPERSURFACES OF A FINSLER SPACE WITH THE GENERALIZED SQUARE METRIC

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Abstract. The emergence of generalized square metrics in Finsler geometry can be attributed to various classification concerning (α, β)-metrics. They have excellent geometric properties in Finsler geometry. Within the scope of this research paper, we have conducted an investigation into the generalized square metric denoted as $F(x,y) = \frac{[\alpha(x,y)+\beta(x,y)]^{n+1}}{[\alpha(x,y)]^n}$, focusing specifically on its application to the Finslerian hypersurface. Furthermore, the classification and existence of first, second, and third kind of hyperplanes of the Finsler manifold has been established.

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1. Introduction

Let M be an n-dimensional differential manifold. Also, let TM be a tangent bundle of M, which is a disjoint union of tangent spaces at all points of $p \in M$. Define a Finsler metric on the differentiable manifold M. This Finsler metric is known as Finsler fundamental fuction on the manifold M. Let us first define what is exactly mean by Finsler fundamental function.

Definition 1.1. (Finsler metric)
We say a function $F : TM \to R$ is a Finsler metric or Finsler fundamental function ([8]) on the manifold M if F satisfy the following conditions:

(1) F is C^∞ away from zero vectors of the tangent spaces.

(2) Positive homogeneity of function F:
$F(x, \lambda y) = \lambda F(x,y)$, $\forall\ \lambda > 0$; for $x \in M$ and all $y \in T_xM$.

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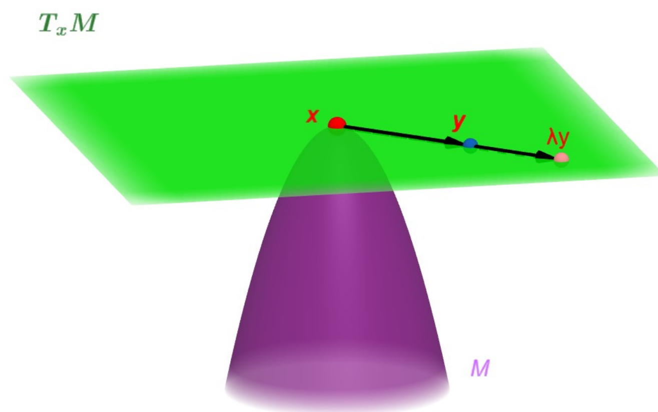

FIGURE 1. This figure shows the tangent vector $\lambda y \in T_xM$, $\lambda > 0$, is λ times the tangent vector $y \in T_xM$.

(3) Strict convexity of the function F:
$F : TM \to R$ is strictly convex over the tangent bundle TM.

It is important to note that strictly convex condition is equivalent to the hessian matrix $[g_{ij}]$, where $i, j \in \{1, 2, 3, ...dim(M)\}$, defined by $\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y) = g_{ij}(x, y)$ is positive definite for any $(x, y) \in TM$. It is the convexity condition on the Finsler metric F that guaranties for the arc length minimization, given by the following formula of the admissible curves $\gamma : [a, b] \to M$ belonging to the set $C^\infty[a, b]$

$$s[\gamma(t)] = \int_a^b F(\gamma(t), \dot{\gamma}(t))dt \tag{1.1}$$

can be achieved.

In other words, the convexity condition on the Finsler metric is a geometric requirement that makes sure that the length of a curve in Figure 2 can be defined and that the arc length functional $s[\gamma(t)]$ is well-behaved, particularly for the purpose of minimizing or finding geodesics with respect to the given Finsler metric F. Without convexity, the concept of length and the corresponding optimization problems will not make sense or may not have unique solutions. One can see above facts relating convexity condition in any text book of calculus of variations.

Definition 1.2. (Finsler Manifold)
A differentiable manifold denoted as M, when equipped with a Finsler metric

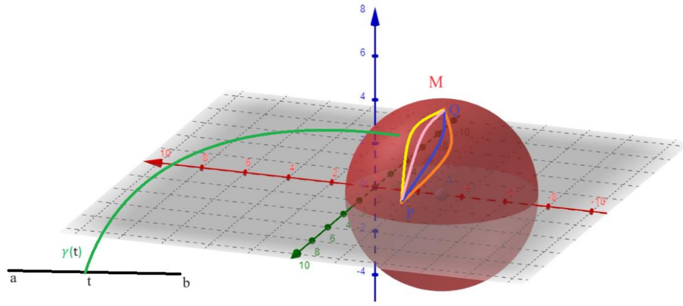

FIGURE 2. This figure shows the curves $\gamma : [a, b] \to M$ over the manifold M such that $\gamma(a) = P$ and $\gamma(b) = Q$.

$F(x, y)$, is referred to as a Finsler manifold or Finsler space. This is typically denoted as (M, F).

Definition 1.3. ((α, β)-metric)
Consider a Finsler space denoted as $(M, F(x, y))$, where $F(x, y)$ represents the Finsler fundamental function. This space is said to possess an (α, β)-metric ([10]) if the fundamental function $F(x, y)$ can be expressed as

$$F(x, y) = F\left(\alpha(x, y), \beta(x, y)\right).$$

That is, F is a composite function of x and y.

The class of (α, β)-metrics was originally introduced by the renowned geometer M. Matsumoto [8]. Let us define generalized square (α, β)-metric as follows:

Example 1.4. The metric defined by$F(x, y) = \frac{[\alpha(x,y)+\beta(x,y)]^{n+1}}{[\alpha(x,y)]^n}$ is called generalized square metric. We say the space $\left(M, F(x, y) = \frac{[\alpha(x,y)+\beta(x,y)]^{n+1}}{[\alpha(x,y)]^n}\right)$ constructed with generalized square metric the generalized square space.

This was the "generalized square metric" that draws our attention to work with hypersurface of a Finsler space. Let us first define the meaning of a hypersurface:

Definition 1.5. ([7]) A hypersurface is an embedded submanifold of codimension 1. That is, a submanifold of dimension less that 1 of a given manifold is called hypersuraface of the underlying manifold.

A submanifold of dimension $n - 1$ is generally denoted by the symbol M^{n-1}. In this paper, we will also denote a hupersurafce of a manifold M by the symbol M^{n-1}.

Example 1.6. Let $M = R^2$ be a manifold. Then a straight line in Figure 3 denoted by $M^{n-1} = M^{2-1} = M^1$ and defined by linear equation in two variables x and y, i.e., $ax + by + c = 0$ is a hypersuarface of the underlying manifold $M = R^2$, because dim(M)=dim(R^2)=2 while the dim(M^{n-1})=$n-1 = 2-1 = 1$.

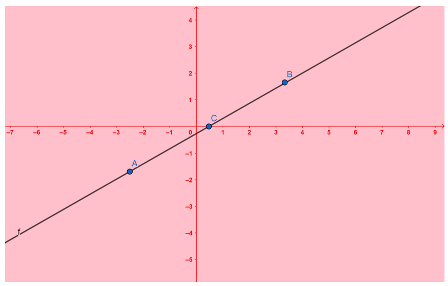

FIGURE 3. The line represented by eqution ax+by+c=0 is a hypersurface of the 2-dimensional manifold R^2.

Example 1.7. Let $M = R^3$ be a manifold. Then the unit sphere in Figure 4 denoted by $M^{n-1} = M^{3-1} = M^2$ and defined by $x^2 + y^2 + z^2 = 1$ is a hypersuarface of the underlying manifold $M = R^3$, because dim(M)=dim(R^3)=3 while the dim(M^{n-1})=$n - 1 = 3 - 1 = 2$.

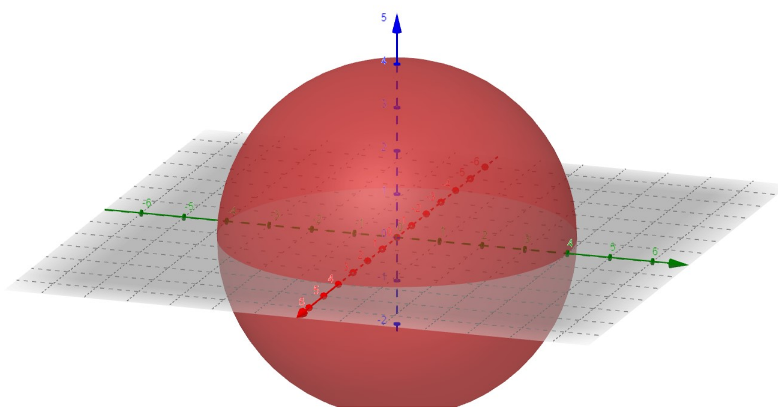

FIGURE 4. The sphere represented by equation $x^2 + y^2 + z^2 = 16$ is a hypersurface of the 3-dimensional manifold R^3.

Matsumoto [9], a prominent Finslerian, was the first person who studies the hypersurfaces and charecterised the special hypersurfaces M^{n-1} of a Finsler manifold. He, specifically, characterised the properties of hypersurface M^{n-1} of Randers space [13]. After this, the number of Finslerians dramatically increased to show their interest in Finsler hypersurface M^{n-1}. Many authors around the world ([6], [5], [3], [14], [15], [18], [2], [11], [1], [4], [19], [20], [17]) did study the properties of special hypersurface M^{n-1} and derived the conditions under which a Finsler hypersurface M^{n-1} of a Finsler manifold $(M, F(x, y))$ becomes a hyperplane of first kind, second kind but not of the third kind. Aim of the present paper is to investigate the hypersurface M^{n-1} of Finsler space using generalized square metric $F(x, y) = \frac{[\alpha(x,y)+\beta(x,y)]^{n+1}}{[\alpha(x,y)]^n}$.

2. Preliminaries

We consider the Finsler space (M, F), where F is the generalized square metric, that is given by

$$F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}. \tag{2.1}$$

Calculate all the partial derivatives of equation (2.1) up to second order, we get

$$F_\alpha = \frac{(\alpha - n\beta)(\alpha+\beta)^n}{\alpha^{n+1}}, \tag{2.2}$$

$$F_\beta = \frac{(n+1)(\alpha+\beta)^n}{\alpha^n}, \tag{2.3}$$

$$F_{\alpha\alpha} = \frac{n(n+1)\beta^2(\alpha+\beta)^{n-1}}{\alpha^{n+2}}, \tag{2.4}$$

$$F_{\beta\beta} = \frac{n(n+1)(\alpha+\beta)^{n-1}}{\alpha^n}, \tag{2.5}$$

$$F_{\alpha\beta} = -\frac{n(n+1)\beta(\alpha+\beta)^{n-1}}{\alpha^{n+1}}. \tag{2.6}$$

We already know that, in a general Finsler manifold (M, F), the normalized element of support $l_i = \frac{\partial F}{\partial y_i}$ and the angular metric tensor h_{ij} [13] are evaluated by the following formula:

$$l_i = \frac{F_\alpha y_i}{\alpha} + F_\beta b_i, \tag{2.7}$$

$$h_{ij} = pa_{ij} + q_0 b_i b_j + q_1(b_i y_j + b_j y_i) + q_2 y_i y_j \tag{2.8}$$

and the coefficients are defined and calculated as follows:

$$y_i = a_{ij} y^j,$$

$$p = \frac{FF_\alpha}{\alpha} = \frac{(\alpha - n\beta)(\alpha+\beta)^{2n+1}}{\alpha^{2n+2}}, \tag{2.9}$$

$$q_0 = FF_{\beta\beta} = \frac{n(n+1)(\alpha+\beta)^{2n}}{\alpha^{2n}}, \tag{2.10}$$

$$q_1 = \frac{FF_{\alpha\beta}}{\alpha} = -\frac{n(n+1)\beta(\alpha+\beta)^{2n}}{\alpha^{2n+2}}, \tag{2.11}$$

$$\begin{aligned} q_2 &= \frac{F(F_{\alpha\alpha} - \frac{F_\alpha}{\alpha})}{\alpha^2}, \\ &= \frac{(\alpha+\beta)^{2n}\left\{n\beta(n+2\beta+n\beta) - \alpha(\alpha+\beta)\right\}}{\alpha^{2n+4}}. \end{aligned} \tag{2.12}$$

We also know that, in a general Finsler manifold (M, F), the fundamental metric tensor $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is evaluated by [13] the following formula:

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_1(b_i y_j + b_j y_i) + p_2 y_i y_j, \tag{2.13}$$

whereas its coefficients p, p_0, p_1 and p_2 are defined and calculated as follows:

$$\begin{aligned} p &= \frac{FF_\alpha}{\alpha}, \\ &= \frac{(\alpha - n\beta)(\alpha+\beta)^{2n+1}}{\alpha^{2n+2}}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} p_0 &= q_0 + F_\beta^2, \\ &= \frac{(n+1)(2n+1)(\alpha+\beta)^{2n}}{\alpha^{2n}}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} p_1 &= q_1 + \frac{pF_\beta}{F}, \\ &= \frac{(n+1)(\alpha+\beta)^{2n}(\alpha - 2n\beta)}{\alpha^{2n+2}}, \end{aligned} \tag{2.16}$$

$$\begin{aligned} p_2 &= q_2 + \frac{p^2}{F^2}, \\ &= \frac{\beta(\alpha+\beta)^{2n}\left\{2n^2\beta + 2n\beta - n\alpha - \alpha\right\}}{\alpha^2(n+2)}. \end{aligned} \tag{2.17}$$

We know that, in a Finsler manifold (M, F), reciprocal metric tensor of a fundamental metric tensor $g_{ij} = \frac{1}{2}\frac{\partial\partial F^2}{\partial y^i \partial y^j}$ is denoted by g^{ij} and is evaluated by the formula [13]

$$g^{ij} = \frac{a^{ij}}{p} - S_0 b^i b^j - S_1(b^i y^j + b^j y^i) - S_2 y^i y^j, \tag{2.18}$$

whereas its coefficients b^i, S_0, S_1 and S_2 are evaluated by the following formulae:

$$b^i = a^{ij} b_j,$$

$$S_0 = \frac{pp_0 + (p_0 p_2 - p_1^2)\alpha^2}{p\zeta}, \tag{2.19}$$

$$S_1 = \frac{pp_1 + (p_0p_2 - p_1^2)\beta}{p\zeta}, \tag{2.20}$$

$$S_2 = \frac{pp_2 + (p_0p_2 - p_1^2)b^2}{p\zeta}, \tag{2.21}$$

$$\zeta = p(p + p_0b^2 + p_1\beta) + (p_0p_2 - p_1^2)(\alpha^2b^2 - \beta^2), \tag{2.22}$$

where $b^2 = a_{ij}b^ib^j$.

Let us define the hv-torsion tensor $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ as follows [16]:

$$C_{ijk} = \frac{p_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k}{2p} \tag{2.23}$$

and its coefficients γ_1 and m_i are evaluated by the formulae

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_1q_0, m_i = b_i - \frac{y_i\beta}{\alpha^2}. \tag{2.24}$$

Now we put

$$2E_{ij} = b_{ij} + b_{ji}, \tag{2.25}$$

$$2F_{ij} = b_{ij} - b_{ji}, \tag{2.26}$$

where $b_{ij} = \nabla_j b_i$. Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ be Cartan connection of (M, F). The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \Gamma^i_{jk}$ of the special Finsler manifold (M, F) is given by [16]

$$\begin{aligned} D^i_{jk} =& B^iE_{jk} + F^i_kB_j + F^i_jB_k + B^i_jb_{0k} + B^i_kb_{0j} - b_{0m}g^{im}B_{jk} - C^i_{jm}A^m_k \\ & - C^i_{km}A^m_j + C_{jkm}A^m_sg^{is} + \lambda^s(C^i_{jm}Csk^m + C^i_{km}C^m_{sj} - C^m_{jk}C^i_{ms}), \end{aligned} \tag{2.27}$$

where

$$B_k = p_0b_k + p_1y_k, \tag{2.28}$$

$$B^i = g^{ij}B_j, \tag{2.29}$$

$$B_{ij} = \frac{p_1(a_{ij} - \frac{y_iy_j}{\alpha^2}) + \frac{\partial p_0}{\partial \beta}m_im_j}{2}, \tag{2.30}$$

$$B^k_i = g^{kj}B_{ji}, \tag{2.31}$$

$$A^m_k = B^m_kE_{00} + B^mE_{k0} + B_kF^m_0 + B_0F^m_k, \tag{2.32}$$

$$\lambda^m = B^mE_{00} + 2B_0F^m_0, \tag{2.33}$$

$$F^k_i = g^{kj}F_{ji}, \tag{2.34}$$

$$B_0 = B_iY^i. \tag{2.35}$$

Here as well as henceforward '0' denotes tensorial contraction with y^i besides p_0,q_0 and S_0.

3. Induced Cartan Connection

Let (M,F) be a Finsler manifold, where $F(\alpha,\beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$ is generalized square metric. Also, let M^{n-1} be a hypersurface of the Finsler manifold (M,F) whose hypothetical picture is depicted in the Figure 4.

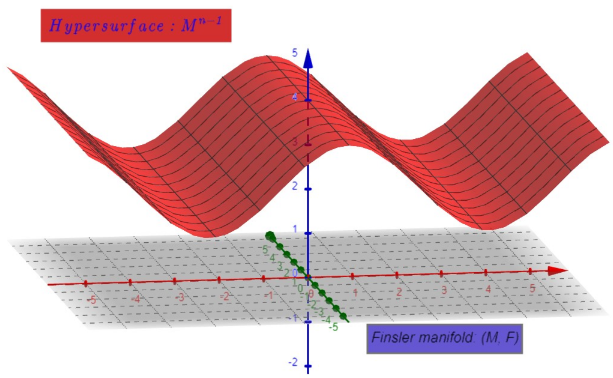

FIGURE 5. A hypothetical picture of hypersurface M^{n-1} of the 3-dimensional Finsler manifold (M,F).

We will describe this hypersurface M^{n-1} by following parametric equations:

$$x^i = x^i(u^\alpha), (i = 1,2,3,........n; \alpha = 1,2,3,........n-1), \tag{3.1}$$

where u^α is a parameter that represents coordinates on the hypersurface M^{n-1}. Now differentiating the equation (3.1) of the hypersurface with respect to parameters u^α, we get $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$. Here each $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ for (α=1,2,3,........n-1) represents components of tangent vectors and these tangent vectors B^i_α represent a tangent space at a point p of the hypersurface M^{n-1} . Let the matrix corresponding to first derivative $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ be $[B^i_\alpha] = [\frac{\partial x^i}{\partial u^\alpha}]$, and it has maximal rank , namely, (n-1). To introduce a Finsler structure in the hypersurface M^{n-1} , the supporting element y^i at a point u^α of M^{n-1} is assumed to be tangential to M^{n-1}, so that we may write

$$y^i = B^i_\alpha(u)v^\alpha. \tag{3.2}$$

Therefore v^α is the element of support of hypersurface M^{n-1} at the point u^α. The metric tensor $g_{\alpha\beta}$ and hv-torsion tensor $C_{\alpha\beta\gamma}$ of hypersurface M^{n-1} are defined by

$$g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta, C_{\alpha\beta\gamma} = C_{ijk}B^i_\alpha B^j_\beta B^k_\gamma. \tag{3.3}$$

Now the unit normal vector $N^i(u,v)$ at an arbitrary point u^α of the hypersurface M^{n-1} is defined as follows:

Definition 3.1. A vector $N^i(u,v)$ at a point u^α of the hypersurface M^{n-1} is said to be unit normal vector if

$$g_{ij}(x(u,v),y(u,v))B^i_\alpha N^j = 0, g_{ij}(x(u,v),y(u,v))N^iN^j = 1. \tag{3.4}$$

Definition 3.2. We say the tensor h_{ij} an angular metric tensor, if h_{ij} satisfies the following conditions:

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta, h_{ij}B^i_\alpha N^j = 0, h_{ij}N^iN^j = 1. \tag{3.5}$$

The induced Cartan connection $\mathcal{I}C\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ on hypersurface M^{n-1} induced from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ is given by [9] $\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma, G^\alpha_\beta = B^\alpha_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta)$,

$$C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk}B^j_\beta B^k_\gamma, \tag{3.6}$$

where second fundamental v-tensor $M_{\beta\gamma}$ is defined by $M^\alpha_\beta = g^{\alpha\gamma}M_{\beta\gamma}$ and normal curvature vector H_β is defined by $H_\beta = N_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta)$, where $B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial U^\gamma}$, $B^i_{0\beta} = B^i_{\alpha\beta}v^\alpha$. The quantities $M_{\beta\gamma}$ and H_β appeared in above equations are called the second fundamental v-tensor and normal curvature vector respectively [9]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [9]

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M_\beta H_\gamma, \tag{3.7}$$

where

$$M_\beta = C^i_{jk}B^j_\beta N_i N^k = C_{ijk}B^i_\beta N^j N^k. \tag{3.8}$$

The relative h-covariant derivative and v-covariant derivative of projection factor B^i_α are respectively given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta}N^i, \tag{3.9}$$

$$B^i_{\alpha|\beta} = M_{\alpha\beta}N^i. \tag{3.10}$$

The equation (3.8) shows that $H_{\beta\gamma}$ is not always symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta. \tag{3.11}$$

Thus the above equation simplifies to

$$H_{0\gamma} = H_\gamma, H_{\gamma 0} = H_\gamma + M_\gamma H_0. \tag{3.12}$$

Lemma 3.3 ([9]). *The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if normal curvature vector H_β vanishes.*

Lemma 3.4 ([9]). *A hypersurface $M^{n-1}$is a hyperplane of first kind if and only if $H_\alpha = 0$.*

Lemma 3.5 ([9]). *A hypersurface M^{n-1} is a hyperplane of second kind with respect to Cartan connection $C\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

Lemma 3.6 ([9]). *A hypersurface M^{n-1} is a hyperplane of third kind with respect to Cartan connection $C\Gamma$ if and only if $H_\alpha = 0$, $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$.*

4. Hypersurface M^{n-1} of the special Finsler space

In this paper we are specifically confined to Finslerian hypersurfaces M^{n-1}. Let us proof the following propositions in context of Finslerian hypersurfaces.

Proposition 4.1. *Let (M, F) be a Finsler manifold, where $F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric and M^{n-1} be its hypersurface. Then fundamental function of the hypersurface M^{n-1} induced from the Finsler manifold (M, F) is a Riemannian metric.*

Proof. Let level equation of the hypersurface M^{n-1} be given by $b(x) = c$, where c is a real number. Take the gradient of the level equation representing hypersurface M^{n-1}, we get $b_i(x) = \partial_i b$. Again consider the parametric equation of the same hypersurface M^{n-1} as $x^i = x^i(u^\alpha)$. Differentiating the equation of hypersurface $b(x(u)) = c$ with respect to parameter u^α, we get

$$\frac{\partial b(x(u))}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = 0,$$
$$b_i(x)B^i_\alpha = 0,$$

where $b_i(x) = \frac{\partial b(x(u))}{\partial x^i}$ and $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$.

This implies that $b_i(x)$ are normal vector field (covariant component) of hypersurface M^{n-1}. Thus at any point of the hypersurface M^{n-1} we now have

$$b_i B^i_\alpha = 0 \tag{4.1}$$

$$b_i y^i = 0, i.e., \beta = 0. \tag{4.2}$$

Now, we will see how generalized square metric $F = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, induces a metric on the hypersurface M^{n-1}. In this case we will denote induced metric by $\bar{F}$. First consider the generalized square metric

$$\begin{aligned} F &= \frac{(\alpha+\beta)^{n+1}}{\alpha^n} \\ &= \frac{(\sqrt{a_{ij}y^iy^j} + b_iy^i)^{n+1}}{(\sqrt{a_{ij}y^iy^j})^n} \\ &= \frac{\left(\sqrt{a_{ij}B^i_\alpha(u)B^j_\beta(u)v^\alpha v^\beta} + b_iy^i\right)^{n+1}}{\left(\sqrt{a_{ij}B^i_\alpha(u)B^j_\beta(u)v^\alpha v^\beta}\right)^n}, \end{aligned}$$

which is the general induced metric on the corresponding hypersurface M^{n-1}. Using equation (4.2), general induced metric of the hypersurface becomes

$$F(u, v) = \sqrt{a_{\alpha\beta}v^\alpha v^\beta}, \tag{4.3}$$

where $a_{\alpha\beta} = a_{ij}B^i_\alpha(u)B^j_\beta(u)$.
Thus function represented by equation (4.3) is fundamental function or the metric of the hypersurface M^{n-1} induced from the ambient Finsler manifold (M, F). The fundamental function of the hypersurface M^{n-1} represented by equation (4.3) do not have β component as $\beta = b_i y^i = 0$ over the hypersurface M^{n-1} therefore fundamental function of the hypersurface M^{n-1} induced from the Finsler manifold (M, F) is a Riemannian metric. □

Proposition 4.2. *Let (M, F) be a Finsler manifold, where $F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric and M^{n-1} be its associated hypersurface. Then the covariant and contravariant components of normal vector field on the hypersurface M^{n-1} are given by*

(1) $b_i = \sqrt{\frac{b^2}{1+n(n+1)}}N_i$,
(2) $b^i = \sqrt{b^2\{1+n(n+1)\}}N^i + \frac{b^2}{\alpha}y^i$.

Proof. It is given that M^{n-1} is a hypersurface of the manifold (M, F), where $F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric. Moreover, we know from equation (4.2) that $\beta = 0$ over the hypersurface M^{n-1}. Let us calculate the value of p, p_0, p_1 and p_2. For that, substitute the value of $\beta = 0$ into equations (2.14), (2.15), (2.16), and (2.17), we get

$$p = 1, p_0 = (n+1)(2n+1), p_1 = \frac{n+1}{\alpha}, p_2 = 0. \tag{4.4}$$

Now put the values of p, p_0, p_1, p_2 into equations (2.19), (2.20), (2.21) and (2.22), we get

$$S_0 = \frac{n(n+1)}{1+n(n+1)b^2} \tag{4.5}$$

$$S_1 = \frac{n+1}{\alpha\{1+n(n+1)b^2\}} \tag{4.6}$$

$$S_2 = -\frac{(n+1)^2b^2}{\alpha^2\{1+n(n+1)b^2\}} \tag{4.7}$$

$$\zeta = 1 + n(n+1)b^2. \tag{4.8}$$

Substituting the values of p, S_0, S_1, S_2 from the equations (4.4), (4.5), (4.6) and (4.7) into equation (2.18), we have

$$g^{ij} = \frac{a^{ij}}{1} - \frac{n(n+1)}{1+n(n+1)b^2} \times b^i b^j - \frac{n+1}{\alpha\{1+n(n+1)b^2\}} \times (b^i y^j + b^j y^i) + \frac{(n+1)^2b^2}{\alpha^2\{1+n(n+1)b^2\}} \times y^i y^j. \tag{4.9}$$

Multiplying equation 4.9 by $b_i b_j$ and using $\beta = b_i y^i = 0$, over the hypersurface M^{n-1}, it becomes $g^{ij}b_i b_j = \frac{b^2}{1+n(n+1)}$. Now from the above equation and using the equation (3.4), we get

$$b_i = \sqrt{\frac{b^2}{1+n(n+1)}}N_i, \tag{4.10}$$

which is the covariant component of the normal vector field on the hypersurface M^{n-1}. Now from (4.9) and (4.10) we get

$$\begin{aligned} b^i &= a^{ij}b_j \\ &= \sqrt{b^2\left\{1+n(n+1)\right\}}N^i + \frac{b^2}{\alpha}y^i, \end{aligned} \tag{4.11}$$

which is the contravariant component of the normal vector field on the hypersurface M^{n-1}. □

Proposition 4.3. *Let (M, F) be a Finsler manifold, where $F(\alpha,\beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric and M^{n-1} be its associated hypersurface. Then second fundamental v-tensor of hypersurface M^{n-1} is given by*

$M_{\alpha\beta} = \frac{(n+1)}{2\alpha}\left(\sqrt{\frac{b^2}{1+n(n+1)}}\right)h_{\alpha\beta}$

and second fundamental h-tensor $H_{\alpha\beta}$ is symmetric, i.e., $H_{\alpha\beta} = H_{\beta\alpha}$.

Proof. It is given that M^{n-1} is a hypersurface of the manifold (M, F), where $F(\alpha,\beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric. Moreover, we know from equation (4.2) that $\beta = 0$ over the hypersurface M^{n-1}. Put the value of $\beta = 0$ into equations (2.14), (2.15), (2.16), and (2.17), we get

$$p = 1, p_0 = (n+1)(2n+1), p_1 = \frac{n+1}{\alpha}, p_2 = 0.$$

Now, put the values of p, p_0, p_1 and p_2 obtained above into equation 2.13, we get fundamental metric tensor of the hypersurace M^{n-1}

$$g_{ij} = a_{ij} + (n+1)(2n+1)b_ib_j + \frac{(n+1)}{\alpha}(b_iy_j + b_jy_i). \tag{4.12}$$

Let us calculate the value of q_0, q_1 and q_2. For that, substitute the value of $\beta = 0$ into equations (2.10), (2.11), and (2.12), we get

$$q_0 = n(n+1), q_1 = 0, q_2 = -\frac{1}{\alpha^2}.$$

Substituting the values of p, q_0, q_1 and q_2 in equation 2.8, we get angular metric tensor of the hypersurface M^{n-1}

$$h_{ij} = a_{ij} + n(n+1)b_ib_j - \frac{1}{\alpha^2}y_iy_j. \tag{4.13}$$

Differentiating equation 2.15 with respect to β, we have

$$\frac{\partial p_0}{\partial \beta} = \frac{2n(n+1)(2n+1)(\alpha+\beta)^{2n-1}}{\alpha^{2n}}.$$

We know from equation (4.2) that $\beta = 0$ over the hypersurface M^{n-1} so put the value of $\beta = 0$ into above equation and equation 2.24, we get

$$\begin{aligned}\frac{\partial p_0}{\partial \beta} &= \frac{2n(n+1)(2n+1)}{\alpha}\\ \gamma_1 &= \frac{n(n^2-1)}{\alpha}\\ m_i &= b_i.\end{aligned}$$

Using the values of p, p_1, γ_1 and m_i in equation 2.23, hv-torsion tensor on the hypersurface M^{n-1}, becomes

$$C_{ijk} = \frac{(n+1)\left[(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) + n(n^2-1)b_ib_jb_k\right]}{2\alpha}. \tag{4.14}$$

Substituting the value of C_{ijk} from equation 4.14 in equation 3.8, we get

$$M_{\alpha\beta} = \frac{(n+1)}{2\alpha}\left(\sqrt{\frac{b^2}{1+n(n+1)}}\right)h_{\alpha\beta}. \tag{4.15}$$

Again, substituting the value of C_{ijk} from equation 4.14 into equation 3.8, we get

$$M_\alpha = 0. \tag{4.16}$$

Substituting the value of M_α from the equation 4.16 in equation 3.12, we get

$$H_{\alpha\beta} = H_{\beta\gamma},$$

which shows that $H_{\alpha\beta}$ is symmetric. □

Theorem 4.4. *Let (M, F) be a Finsler manifold, where $F(\alpha,\beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric and M^{n-1} be its associated hypersurface. Then the hypersurface M^{n-1} will be hyperplane of first kind if and only if $2b_{ij} = b_ic_j + b_jc_i$. Moreover we show that second fundamental tensor $H_{\alpha\beta}$ of M^{n-1} is proportional to it's angular metric tensor $h_{\alpha\beta}$. That is, $H_{\alpha\beta} = \frac{c_0 b}{\sqrt{1+n(n+1)}}h_{\alpha\beta}$.*

Proof. Let us differentiate equation 4.1 with respect to β, we get

$$b_{i|\beta}B^i_\alpha + b_iB^i_{\alpha|\beta} = 0. \tag{4.17}$$

Put the value of $B^i_{\alpha|\beta}$ from equation 3.10 and $b_{i|\beta} = b_{i|j}B^j_\beta + b_{i|j}N^jH_\beta$ into equation 4.17, we get

$$b_{i|j}B^j_\beta B^i_\alpha + b_{i|j}N^jH_\beta B^i_\alpha + b_iH_{\alpha\beta}N^i = 0. \tag{4.18}$$

We know that $b_{i|j} = -b_hC^h_{ij}$. Put the value of b_h from equation 4.10 into above expression produces $b_{i|j} = 0$. Using $b_{i|j}B^i_\alpha N^j = 0$ and equation 4.10 in the equation 4.18 and then using the fact that $N_iN^i = 1$, we get

$$b_{i|j}B^j_\beta B^i_\alpha + \sqrt{\frac{b^2}{1+n(n+1)}}H_{\alpha\beta} = 0. \tag{4.19}$$

It is obvious that $b_{i|j}$ is symmetric. Now contracting 4.19 with v^β first and then with v^α respectively and using the equations 3.2, 3.12 and 4.16, we get

$$b_{i|j}B^i_\alpha y^j + \sqrt{\frac{b^2}{1+n(n+1)}}H_\alpha = 0 \tag{4.20}$$

$$b_{i|j}y^iy^j + \sqrt{\frac{b^2}{1+n(n+1)}}H_0 = 0. \tag{4.21}$$

We know from the Lemma 3.3 and Lemma 3.4, a hypersurface M^{n-1} is a hyperplane of first kind if and only if normal curvature vanishes, i.e., $H_0 = 0$. Using the value $H_0 = 0$ in equation 4.21 we find that hypersurface M^{n-1} is a hyperplane of first kind if and only if $b_{i|j}y^iy^j = 0$. This $b_{i|j}$ is the covariant derivative of with respect to Cartan connection $C\Gamma$ of Finsler space F, it may depend on y^i. Moreover $\nabla_j b_i = b_{ij}$ is the covariant derivative of b_i with respect to Riemannian connection Γ^i_{jk} constructed from $a_{ij}(x)$, therefore b_{ij} dose not depend on y^i. We shall consider the difference $b_{i|j} - b_{ij}$ of above covariant derivatives in further discussion. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \Gamma^i_{jk}$ is given by equation 2.27. Since b_i is a gradient vector, from equations 2.25 and 2.26 we have

$$E_{ij} = b_{ij}, F_{ij} = 0, F^i_j = 0. \tag{4.22}$$

Using equation 4.22 into equation 2.27, we get

$$\begin{aligned} D^i_{jk} = b_{jk}B^i + b_{0k}B^i_j + b_{0j}B^i_k - b_{0m}g^{im}B_{jk} \\ -A^m_kC^i_{jm} - A^m_jC^i_{km} + A^m_sC_{jkm}g^{is} \\ +\lambda^s(C^m_{sk}C^i_{jm} + C^m_{sj}C^i_{km} - C^i_{ms}C^m_{jk}). \end{aligned} \tag{4.23}$$

Using the equations 4.2, 4.4, 4.5 and 4.6 into equations 2.28 to 2.23, we get

$$B_k = (n+1)(2n+1)b_k + \frac{n+1}{\alpha}y_k, B^i = bb^i + by^i \tag{4.24}$$

$$B_{ij} = \frac{(n+1)\left\{a_{ij}\alpha^2 - y_iy_j + 2n(n+1)b_ib_j\alpha\right\}}{2\alpha^3} \tag{4.25}$$

$$B^i_j = 0 \tag{4.26}$$

$$A^m_k = 0, \lambda^m = B^m b_{00}. \tag{4.27}$$

Using tensor contraction operation with equations 4.25 and 4.26 by y^j, we get $B_{i0} = 0$, $B^i_0 = 0$. Further contracting equation 4.27 by y^k and using the fact that $B^i_0 = 0$, we get $A^m_0 = B^m b_{00}$. Contracting equation 4.23 by y^k and using the facts $B_{i0} = 0$, $B^i_0 = 0$, $A^m_0 = B^m b_{00}$ and $C^m_{s0} = 0$, $C^i_{0m} = 0$, $C^m_{j0} = 0$ obtained by contracting equations 4.25, 4.26, 4.27 and 3.6, we get

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - b_{00} B^m C^i_{jmi} \tag{4.28}$$

$$D^i_{00} = bb^i b_{00} + by^i b_{00}. \tag{4.29}$$

Multiplying equation 4.25 by b_i and then using equations 4.2, 4.21 and 4.23, we get

$$b_i D^i_{j0} = bb_{j0} + bb_j b_{00} - bb_i b^m C^i_{jm} b_{00}. \tag{4.30}$$

Now multiplying equation 4.26 by b_i and then using equation 4.2 we get

$$b_i D^i_{00} = \frac{b^2}{1 + n(n+1)} b_{00}. \tag{4.31}$$

From equations 4.14 and 4.16 it is clear that

$$b^m b_i C^i_{jm} B^j_\alpha = \frac{b^2}{1 + n(n+1)} M_\alpha = 0. \tag{4.32}$$

Contracting the expression $b_{i|j} = b_{ij} - b_r D^r_{ij}$ by y^i and y^j respectively and then using equation 4.31 we get

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = \frac{b^2}{1 + n(n+1)} b_{00}.$$

Put $b_{i|j} = b_{ij} - b_r D^r_{ij}$ in equations 4.17 and 4.18 and then using equations 4.27, 4.1 and 4.29 and the value of $b_{i|j} y^i y^j$ above, equations 4.20 and 4.21 can be written as

$$\sqrt{\frac{b^2}{1 + n(n+1)}} b_{i0} B^i_\alpha + bH_\alpha = 0 \tag{4.33}$$

$$\sqrt{\frac{b^2}{1 + n(n+1)}} b_{00} + bH_0 = 0. \tag{4.34}$$

From the equation 4.31 it is clear that the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where b_{ij} is independent of y^i. Since y^i satisfy equation 4.2, the condition can be written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^j)$ for some $c_j(x)$, so that we have

$$2b_{ij} = b_i c_j + b_j c_i. \tag{4.35}$$

Thus we shown that a Finslerian hypersurface M^{n-1} will be hyperplane of first kind if and only if $2b_{ij} = b_i c_j + b_j c_i$. Now we try to show that second fundamental tensor $H_{\alpha\beta}$ of the hypersurface M^{n-1} is proportional to its angular metric tensor $h_{\alpha\beta}$. For that, contracting equation 4.35 and using the fact that $b_i y^j = 0$, we get $b_{00} = 0$. This implies that the condition $b_{00} = 0$ and $2b_{ij} = b_i c_j + b_j c_i$ are equivalent. Multiplying equation 4.35 by B^i_α and then B^j_β and using equations (4.1) and (4.2), we have $b_{ij} B^i_\alpha B^j_\beta = 0$. Again, multiplying equation 4.35 by B^i_α and y^j and then using Equation (4.1), we have $b_{i0} B^i_\alpha = 0$. Again, consider equation $2b_{ij} = b_i c_j + b_j c_i$. Multiplying by y^j to both sides, contracting by y^j, multiplying by b^i both sides and using the fact that $b^2 = b_i b^i$, we get $b_{i0} b^i = \frac{b^2 c_0}{2}$. Now, using this in equation 4.30 gives $H_\alpha = 0$. Again, using 4.23 and 4.24 and using $b_{00} = 0$ and $b_{ij} B^i_\alpha B^j_\beta = 0$, we get $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = \frac{1}{2\alpha} h_{\alpha\beta}$. Thus using the equations 4.6, 4.7, 4.8, 4.12 and 4.20, we get

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = -\frac{c_0 b^2}{1 + n(n+1)} h_{\alpha\beta}. \tag{4.36}$$

Thus using the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ and equation 4.36, equation 4.19 reduces to

$$-\frac{c_0 b^2}{1 + n(n+1)} h_{\alpha\beta} + \sqrt{\frac{b^2}{1 + n(n+1)}} H_{\alpha\beta} = 0 \tag{4.37}$$

$$H_{\alpha\beta} = \frac{c_0 b}{\sqrt{1 + n(n+1)}} h_{\alpha\beta}.$$

□

Theorem 4.5. *Let (M, F) be a Finsler manifold, where $F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric and M^{n-1} be its associated hypersurface. Then the hypersurface M^{n-1} will be hyperplane of second kind if and only if $b_{ij} = e b_i b_j$.*

Proof. We know from Lemma 3.5, hypersurface M^{n-1} is a hyperplane of second kind if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Now we consider these two sufficient conditions one by one.

(1) If $H_{\alpha\beta} = 0$, then equation (4.37) becomes $c_0 = c_i y^i = 0$.
(2) Again, if $H_\alpha = 0$, then Lemma 3.3 and Lemma 3.4 imply $H_0 = 0$.
We have already shown above $H_0 = 0$ is equivalent to $2b_{ij} = b_i c_j + b_j c_i$.

Now we combine case 1 and case 2. For that, put the value of $c_i(x) = e(x) b_i(x)$ obtained in case 1 to equation obtained in case 2, we get $b_{ij} = e b_i b_j$. Thus we shown that the hypersurface M^{n-1} of the Finsler manifold (M, F) will be hyperplane of second kind iff $b_{ij} = e b_i b_j$. □

Theorem 4.6. *Let (M, F) be a Finsler manifold, where $F(\alpha, \beta) = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n \in N$, is a generalized square metric and M^{n-1} be its associated hypersurface. Then the hypersurface M^{n-1} will not be hyperplane of third kind.*

Proof. We know from sufficient conditions of Lemma 3.6 a hypersurface becomes a hyperplane of third kind if $H_\alpha = 0$, $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$. Now we consider these three sufficient conditions one by one.

(1) If $H_\alpha = 0$, then we get the condition $2b_{ij} = b_i e(x) b_j(x) + b_j e(x) b_i(x)$, which has already been proved above and is termed as the condition of hyperplane of first kind.
(2) If $H_{\alpha\beta} = 0$, then we get the condition $b_{ij} = e b_i b_j$, which has already been proved above and is termed as the condition of hyperplane of second kind.
(3) Now put $M_{\alpha\beta} = 0$ in Equation (4.15), we get

$$0 = \frac{(n+1)}{2\alpha}\left(\sqrt{\frac{b^2}{1+n(n+1)}}\right)h_{\alpha\beta},$$

which implies that no condition could be deduced to satisfy $M_{\alpha\beta} = 0$, i.e., it is impossible to find a condition under which a hypersurface becomes a hyperplane of third kind, as term on the R.H.S. of the above equation can never be zero. Finally, we shown that hypersurface M^{n-1} is not a hyperplane of third kind.

□

Corollary 4.7. *Let (M, F) be a Finsler manifold, where F may be any of the following Finsler metrics obtained by generalized square metric $F = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$, $n = 1, 2, 3, ...$*

(1) $F = \frac{(\alpha+\beta)^2}{\alpha}$ *(popularly known as square metric)*
(2) $F = \frac{(\alpha+\beta)^3}{\alpha^2}$
(3) $F = \frac{(\alpha+\beta)^4}{\alpha^3}$, *etc..*
Also let M^{n-1} be the corresponding hypersurfaces of the given Finsler manifold (M, F). Then, in either case, show that the hypersurface is a hyperplane of first kind, second kind and not of the third kind.

Proof. By Theorems 4.4, 4.5 and 4.6 it can be easily deduce that the hypersurfaces M^{n-1} corresponding to different Finsler manifolds (M, F) are a hyperplane of first kind, second kind and not of the third kind. It is remarkable that corresponding author has already been published a paper [12] on part (1) of this corollary. □

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References

1. V.K. Chaubey and A.K. Mishra, *Hypersurfaces of a Finsler space with a special (α, β)-metric*, J. Contemp. Mathemat. Anal. **52** (2017), 1-7.
2. V.K. Chaubey and B.K. Tripathi, *Finslerian Hypersurfaces of a Finsler Spaces with (α, β)-metric*, Journal of Dynamical Systems and Geometric Theories **12** (2014), 19-27.
3. G.C. Chaubey, G. Shanker and V. Pandey, *On a hypersurface of a Finsler space with the special metric, $\alpha + \frac{\beta^{n+1}}{(\alpha-\beta)^n}$*, J.T.S. **7** (2013), 39-47.
4. M. Kumar Gupta and S. Sharma, *On Hypersurface of a Finsler space subjected to h-Matsumoto change*, https://arxiv.org/pdf/2205.03520.pdf. **322** (2013), 401-409.
5. I.Y. Lee, H.Y. Park and Y.D. Lee, *On a hypersurface of a special Finsler space with a metric $\alpha + \frac{\beta^2}{\alpha}$*, Korean J. Math. Sciences **322** (2013), 401-409.
6. M. Kitayama, *On Finslerian hypersurfaces given by β change*, Balkan J. of Geometry and It's Applications **7** (2002), 49-55.
7. John M. Lee, *Introduction to Smooth Manifolds*, Springer, New York, 2013.
8. M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaiseisha Press, Otsu, Japan, 2013.
9. M. Matsumoto, *The induced and intrinsic Finsler connection of a hypersurface and Finslerian projective geometry*, J. Math. Kyoto Univ. **25** (1985), 107-144.
10. M. Matsumoto, *Theory of Finsler spaces with* (α, β)*-metric*, Rep. on Math. Phys. **31** (1992), 43-83.
11. P.N. Panday and G.P. Yadav, *Hypersurface of a Finsler space with Randers change of special (α, β)-metric*, Bulletin of the Transilvania University of Brasov. Series III: Mathematics and Computer Science **31** (2018), 195-206.
12. M. Rafee and A. Kumar, and G.C. Chaubey, *On the Hypersurface of a Finsler space with the square metric*, Int. J. Pure. Appl. Math. **118** (2018), 723-733.
13. G. Randers, *On an Asymmetrical Metric in the Four-Space of General Relativity*, Phys. Rev. **2** (1941), 195-199.
14. G. Shanker and R. Yadav, *On the hypersurface of a second approximate Matsumoto metric* $\alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^2}{\alpha^2}$, International J. of Contemp. Math. Sciences **7** (2012), 115-124.
15. G. Shanker and Vijeta singh, *On the hypersurface of a Finsler space with Randers change of generalized* $(\alpha, \beta) - metric$, International Journal of pure and applied Mathematics **105** (2015), 223-234.
16. C. Shibata, *On invariant tensors of β-change of Finsler metrics*, J. Math. Kyoto Univ. **24** (1984), 163-188.
17. H.S. Shukla and O.P. Pandey and A.K. Mishra, *Hypersurface of a Finsler space with a special (α, β)-metric*, South Asian **24** (1984), 163-188.
18. U.P. Singh and Bindu Kumari, *On a Matsumoto space*, International Journal of pure and applied Mathematics **4** (2001), 521-531.
19. Brijesh Kumar Tripathi, *Hypersurfaces of a Finsler space with exponential form of (α, β)-metric*, Annals of the University of Craiova-Mathematics and Computer Science Series **47** (2020), 132-140.
20. H. Wosoughi, *On a Hypersurface of a Special Finsler Space with an Exponential β α,-Metric*, Int. J. Contemp. Math. Sciences **6** (2011), 1969-1980.

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