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ONE SIDED APPROXIMATION OF UNBOUNDED FUNCTIONS FOR ALGEBRAIC POLYNOMIAL OPERATORS IN WEIGHTED $L_{p,\alpha}$-SPACES

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Abstract. The objective of this article is to acquire analogs for the degree of best one-sided approximation to investigate some Jackson's well-known theorems for best one-sided approximations in weighted $L_{p,\alpha}$-spaces. In addition, some operators that are used to approximate unbounded functions have been introduced as be algebraic polynomials in the same weighted spaces. Our main results are given in terms of degree of the best one-sided approximation in terms of averaged modulus of smoothness.

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1. Introduction

The theory of approximation deals with approximating the term complicated should be used here complex my confuse the reader to the of complex numbers. Problems concerning one-sided approximation have previously been considered by the authors [10, 11]. Researchers focus on approximating one-sided functions, including the better 1-sided approximation of unbounded functions, which is the subject of this paper.

Denote S_n as set of nth degree algebraic polynomials on $[0,1]$ with nodes at given points. i.e., $s \in S_n$ if $s \in C^{n-1}[0,1]$ and s is an algebraic function of degree n on the interval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, k$. The best one-sided approximation $\widetilde{E}_n(f)_{L_{p,\alpha}}$ in $L_{p,\alpha}$, to the function by algebraic polynomial in S_n on the interval

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$[0,1]$ given by:

$$\widetilde{E}_n\left(f\right)_{L_{p,\alpha}} = inf\left(\left(\int_0^1 \left(\left(S\left(x\right)-s\left(x\right)\right)w_\alpha\left(x\right)\right)^p w_\alpha\left(x\right)dx\right)^{\frac{1}{p}} : S, s \in S_n,\right.$$

$$\left. s(x) \leq f(x) \leq S(x), x \in [0,1]\right)$$

In 2008 [14], Oleksanor investigated one-sided weighted approximation by bounded polynomials in L_p-space on the real line and obtained results. In 2010 [15], Motornyi and Pas'ko examined the best one-sided approximation through a class of differentiable functions in L_p-space. In 2012 [13], Rensuoli and Yong studied the better m-term 1-sided multi approximation by trigonometric polynomials on some classes of Besov space in $L_p(T_d)$, $p \geq 1$. In 2014 [1], Adell et al. introduced operators for approximating Riemann integrable functions on $[0,1]$ using algebraic polynomials in L_p-spaces. In 2019 [5], Auad and Abdulsttar establish some results of best simultaneous approximation of unbounded functions in weighted space by given two different definitions and considered the relation between best simultaneous approximation. In 2021[6] Alaa and Mohamed, constructive characterization of modulus of smoothness are considered and the direct & converse algebraic polynomials approximation theorem in weighted spaces of unbounded functions in weighted space, finally in 2023 [3, 4] Alaa and etc. investigated of weighted space which contained the unbounded functions which is to be approximated by linear operators in terms some Well-known approximation tools such as the modulus of smoothness and K-functional.

In this paper, the weight function $w_\alpha\left(x\right) = e^{-a\prod_{i=1}^{d} x_i} \in W$ such that $W = \{w \mid w : [0,1] \to R^+\}$ is utilized which is non-negative measurable function on R^+.

2. Basic definitions

Definition 2.1. [12] Let $f \in L_{p,\alpha}(X), X = [u, \mathrm{v}]$ such that $p \in [1,\infty)$, , then the weighted space of all real respected unbounded functions f, such that $\int_u^{\mathrm{v}} \left|f\left(x\right)w_\alpha\left(x\right)\right|^p dx < \infty, \alpha > 0$ is defined by:

$$\|f\|_{p,\alpha} = \left[\int_u^{\mathrm{v}} \left|f\left(x\right)w_\alpha\left(x\right)\right|^p dx\right]^{\frac{1}{p}}, x \in X. \tag{1}$$

Definition 2.2. For $f \in L_{p,\alpha}(X)$, $X = [a,\ b], \delta > 0$, we might know the following notion:

$$\omega(f;\delta) = \sup\left\{\left|f\left(t\right)-f\left(x\right)\right|\right\}, |x-t| \leq \delta,$$

where x, t belong to the domain of definition of the function f, be the continuity modulus of f. The $L_{p,\alpha}$-modulus of continuity of f is defined as

$$\omega\left(f,\delta\right)_{L_{p,\alpha}} = \sup_{0<k<\delta}\left(\int_u^{\mathrm{v}} \left|\left(f\left(x\right)-f(t)\right)w_\alpha\left(x\right)\right|^p dx\right)^{\frac{1}{p}}.$$

The k^{th} average modulus of smoothness for $f \in L_{p,\alpha}(X)$ is given by

$$\tau_n\left(f, \delta\right) = \left\|\omega_n(f, ., \delta)\right\|_{L_{p,\alpha}},$$

where,

$$\omega_n(f, ., \delta)_{L_{p,\alpha}} = \sup_{0<k<\delta} \left\{\left\|\Delta_k^n f(.)\right\|\right\}, \delta > 0,$$

such that, the k^{th} symmetric difference of f is given by:

$$\Delta_k^n f\left(x\right) = \sum_{i=1}^{n} \left(-1\right)^{n-i} \binom{n}{i} f\left(x + ik\right), x, k \in X.$$

The k^{th} order module of smoothness of f is defined by

$$\omega_n(f, ., \delta)_{L_{p,\alpha}} = \sup_{0<k<\delta} \left(\int_u^v \left|\Delta_k^n f\left(x\right) w_\alpha\left(x\right)\right|^p dx\right)^{\frac{1}{p}}, \alpha > 0.$$

These moduli of continuity are well studied compared to moduli of variation.

Definition 2.3. [8] The modulus of variation of difference of a mapping f is the function $V(f,\ \delta)$ with scope the positive integers, clear by

$$V\left(f, \delta\right) = \sup_{\pi_n} \ \sum_{i=1}^{k} \left|f\left(u_i\right) - f(v_i)\right|,$$

where π_n is an arbitrary system of n disjoint subintervals (a_i, b_i) of $(0, 1)$.

Definition 2.4. The degree of best one-sided approximation of f is:

$$\widetilde{E}_n(f)_{p,\alpha} = \underbrace{\inf}_{p_n \in P_n} \left\{\left\|p_n - q_n\right\|_{p,\alpha} p_n(x) \leq f(x) \leq q_n(x)\right\}.$$

Where P_n is the space of all polynomials of degree n of one-variable. Also, the degree of best approximation of the function f is given by:

$$E_n(f)_{p,\alpha} = \underbrace{\inf}_{p_n \in P_n} \left\{\left\|f - p_n\right\|_{p,\alpha}\right\}$$

3. Auxiliary lemmas

In this section various properties of the modulus $\tau\ (f; \delta)_{L_{p,\alpha}}$ are offered, which are named as an auxiliary lemmas.

Lemma 3.1. [2] $\tau\ (f; \delta)_{L_{\infty,\alpha}} = \omega\ (f; \delta)$.
In relationship with this lemma, we see that the situation of uniform approximation of function fundamentally accords with 1-sided approximation in $L_{\infty,\alpha}$.

Lemma 3.2. [12] $\tau\ (f; \delta)_{L_{1,\alpha}} \xrightarrow[\delta \to 0]{0}$ *if and only if f is an unbounded function.*

Lemma 3.3. [12] *If f and g are unbounded functions, then*

$$\tau(f + g; \delta)_{L_{p,\alpha}} \leq \tau\ (f; \delta)_{L_{p,\alpha}} + \tau(g; \delta)_{L_{p,\alpha}}.$$

Lemma 3.4. *For any unbounded function f, we have the inequality*

$$\omega(f;\delta)_{L_{p,\alpha}([0,1])} \leq \ \tau(f;\delta)_{L_{p,\alpha}([0,1])}.$$

Proof.

$$\begin{aligned}
\omega(f;\delta)_{L_{p,\alpha}} &= \sup_{0<h\leq\delta} \left(\int_0^1 \left|\left(f\left(x+h\right)-f\left(x\right)\right) w_\alpha\left(x\right)\right|^p dx\right)^{\frac{1}{p}} \\
&= \sup_{0<h\leq\delta} \left(\int_0^1 \left|\left(f\left(x+\frac{h}{2}\right)-f(x-\frac{h}{2})\right) w_\alpha\left(x\right)\right|^p dx\right)^{\frac{1}{p}} \\
&\leq \ \left(\int_0^1 \left(\sup_{0<h\leq\delta} \left|\left(f\left(x+\frac{h}{2}\right)-f(x-\frac{h}{2})\right) w_\alpha\left(x\right)\right|\right)^p dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_0^1 \left(\omega\left(f;x;\delta\right) w_\alpha\left(x\right)\right)^p dx\right)^{\frac{1}{p}} = \left\|\omega\left(f;\delta\right)\right\|_{L_{p,\alpha}} = \tau(f;\delta)_{L_{p,\alpha}}.
\end{aligned}$$

□

Lemma 3.5. [12] *For any unbounded function f and any $\lambda \geq 0$, we have*

$$\tau(f;\lambda\delta)_{L_{p,\alpha}} \leq (\lambda+1)\ \ \tau(f;\delta)_{L_{p,\alpha}}.$$

If k is an integer, then

$$\tau(f;k\delta)_{L_{p,\alpha}} \leq k\tau(f;\delta)_{L_{p,\alpha}}.$$

The verify of this lemma is fundamentally the similar as the proof by Dolzhonko and Seviast' yanov in [9] *for the situation $p = 1$.*

Lemma 3.6. *If f is a function unbounded on $[0,1]$, then*

$$\tau(f;n^{-1})_{L_{1,\alpha}} \leq 3n^{-1}\ \Delta_k^n(f;n),$$

where

$$\Delta_k^n\left(f;n\right) = \sup\sum_{i=2}^{n}\left|f\left(x_i\right)-f(x_{i-1})\right|,$$

Proof. Write

$$S\left(f;x;\delta\right) = \sup f\left(t\right); \left|t-x\right| \leq \frac{\delta}{2},$$

$$J\left(f;x;\delta\right) = \inf f\left(t\right); \left|t-x\right| \leq \frac{\delta}{2}.$$

It follows from the definition of $\tau(f;\delta)_{L_{1,\alpha}}$ that

$$\begin{aligned}
&\tau(f;n^{-1})_{L_{1,\alpha}} \\
&= \int_0^1 \omega\left(f;x;n^{-1}\right) w_\alpha\left(x\right) dx = \int_0^1 \left(S\left(f;x;n^{-1}\right)-J\left(f;x;n^{-1}\right)\right) w_\alpha\left(x\right) dx, \\
&= \sum_{i=1}^{n} \int_{\frac{(i-1)}{n}}^{\frac{i}{n}} \left(S\left(f;x;\ n^{-1}\right)-J\left(f;x;n^{-1}\right)\right) w_\alpha\left(x\right) dx,
\end{aligned}$$

$$\leq \frac{1}{n}\sum_{i=1}^{n}\left(S\left(f;\ \xi_i;n^{-1}\right)-J\left(f;\xi_i;n^{-1}\right)\right) \tag{2}$$

where $\xi_i \in [\frac{i}{n}, \frac{(i+1)}{n}]$. Let $\epsilon > 0$ and $\overline{\xi}_i$, $\underline{\xi}_i$ be such that

$$S\left(f;\xi_i;n^{-1}\right) \ \leq f\left(\overline{\xi_i}\right)+\frac{\epsilon}{(2n)},\ J\left(f;\ \xi_i;n^{-1}\right)\geq f\left(\ \underline{\xi}_i\ \right)-\frac{\epsilon}{(2n)}.$$

Then from 2 we have

$$\tau(f;n^{-1})_{L_{1,\alpha}} \leq \frac{1}{n}\sum_{i=1}^{n}\left|f\left(\overline{\xi}_i\right)-f\left(\underline{\xi}_i\right)\right|+\epsilon. \tag{3}$$

we split up the sum in 3 as follows:

$$\sum_{i=1}^{n}\left|f\left(\overline{\xi}_i\right)-f(\underline{\xi}_i)\right| = \sum_{j=0}^{2}\sum_{i}\left|f\left(\overline{\xi}_{3i-j}\right)-f(\underline{\xi}_{3i-j})\right|.$$

For $n \geq 4$,

$$\sum i\left|f\left(\overline{\xi}_{3i-j}\right)-f(\underline{\xi}_{3i-j})\right| \ \leq \Delta_k^n\left(f;n\right),$$

thus, the quantity of points situated in the last sum is less than $\left[\frac{2n}{3}\right]+2\ \leq n$, and

$$\max\left(\overline{\xi}_{3i-j},\underline{\xi}_{3i-j}\right) \leq \min\left(\overline{\xi}_{3i+3-j},\underline{\xi}_{3i+3-j}\right).$$

Thus

$$\sum_{i=1}^{n}\left|f\left(\overline{\xi}_i\right)-f(\underline{\xi}_i)\right| \ \leq 3A_k^n\left(f;n\right).$$

□

Lemma 3.7. [8] *If $V(f,\delta) < \infty$, then for any $\delta > 0$ we have the inequality*

$$\tau(f;\delta)_{L_{1,\alpha}} \leq \delta\ V(f,\delta)_{L_{1,\alpha}},$$

where $V(f,\delta)$ is the variation of the function f on the interval [0,1].

Lemma 3.8. *Let f be an absolutely continuous function. Then*

$$\tau(f;\delta)_{L_{p,\alpha}} \leq \delta\left\|f'\right\|_{L_{p,\alpha}}.$$

Proof. Since

$$f\left(x\right)-f\left(y\right)=\int_y^x f'\left(t\right)dt,$$

we obtain

$$\omega\left(f;\ x;\ \delta\right)=\max_{\substack{|t-x|\leq\delta/2\\ |t'-x|\leq\delta/2}}\left|f\left(t\right)-f\left(t'\right)\right| = \max_{\substack{|t-x|\leq\delta/2\\ |t'-x|\leq\delta/2}}\left|\int_t^{t'} f'\left(u\right)du\right|$$

$$\leq \int_{x-\delta/2}^{x+\delta/2}\left|f'\left(t\right)\right|dt = \int_{-\delta/2}^{\delta/2}\left|f'(x-u)\right|du.$$

Thus

$$\tau(f;\ \delta)_{L_{p,\alpha}} = \|\omega\left(f;\ x;\ \delta\right)\|_{L_{p,\alpha}} \leq \left\| \int_{-\,\delta/2}^{\delta/2} \left|f'\left(x-u\right)du\right| \right\|_{L_{p,\alpha}}$$

$$\leq \int_{-\,\delta/2}^{\delta/2} \|f'\|_{L_{p,\alpha}}\, du = \delta\, \|f'\|_{L_{p,\alpha}}\,.$$

□

Lemma 3.9. *Let f be unbounded function, $f \in L_{p,\alpha}\left([0,1]\right)$, $\int_0^1 f\left(t\right) w_\alpha\left(t\right) dt = 0$, and let there exist a polynomial $T \in T_n$ such that $T(x) \geq\ f(x)$ for $x \in [0,1]$. Then there exists a polynomial $R \in T_n$ such that $R(x) \geq \int_0^x f\left(t\right) w_\alpha\left(t\right) dt, x \in [0,1]$ and $\left(\int_0^1 \left|R\left(x\right) - \int_0^x f\left(t\right) w_\alpha\left(t\right) dt\right|^p dx\right)^{\frac{1}{p}} \leq \frac{c\eta}{n}$, where $\eta = \|f-T\|_{L_{p,\alpha}}\left([0,1]\right)$ As a lemma we've the following approximation, which we benefit from it in the future:*

Lemma 3.10. *If $f^{(r-1)}$ is an unconditionally continuous function, then*

$$\widetilde{E}_k\left(f\right)_{L_{p,\alpha}} \leq n^{-r} c_3 \left\|f^r\right\|_{L_{p,\alpha}}.$$

The proof of Lemma 3.9 and lemma similar with the Theorems 4.1 and 4.2 in [4].

4. Main results

We now strengthen 1-sided approximations of unbounded functions in the space $L_{p,\alpha}\left([0,1]\right)$ in terms average modulus of smoothness. The main objective of this paper is to obtain the following analogs by utilizing the modulus

$$\tau\ \left(f;\ \delta\right)_{L_{p,\alpha}} = \|\omega\ \ \left(f;\ x;\ \delta\right)\|_{L_{p,\alpha}}\,,$$

To establish and investigate some Jackson's types theorems for best one-sided approximations in weighted $L_{p,\alpha}$-spaces, where $\omega\ \left(f;\ x;\ \delta\right) = \sup\left\{\left|f\left(t\right) - f\left(t'\right)\right|\right\}$ $|t-x| \leq \frac{\delta}{2}, |t'-x| \leq \frac{\delta}{2}$, and t, t' belongs to the domain of definition of the function f. Although several problems concerning one-sided approximation by algebraic polynomials with certain properties have been studied.

We first prove this theorem.

Theorem 4.1. *Let the function f have integrable unbounded derivative f' on the interval $[0,1]$. Then for any $k \geq 1$, we have the inequality.*

$$\widetilde{E}_k\left(f\right)_{L_{p,\alpha}} \leq \left(k+1\right)\Delta_n \widetilde{E}_{k-1}\left(f'\right)_{L_{p,\alpha}}\,,$$

where, $\Delta_n = \max|x_i - x_{i-1}|, (1 \leq i \leq n)$.

Proof. (see [6]) Clearly we may assume that $f(0) = 0. Lets, l \in S_{k-1}$ Such that

$$s\left(x\right) \geq f'\left(x\right) \geq l\left(x\right)\ \ ,\ x \in [0,1]\,, \tag{4}$$

$$\|s-l\|_{L_{p,\alpha}([0,1])} \leq \widetilde{E}_{k-1}\left(f'\right)_{L_{p,\alpha}} + \epsilon,\ \epsilon > 0.$$

Set

$$\varphi_i(x) = \sum_{j=i}^{i+k} \frac{k(x_j - x)_+{}^{k-1}}{\omega_{t'}(x_j)},$$

$$(x-t)_+{}^{k-1} = \begin{cases} (x-t)^{k-1}, \ x \geq t \ , \\ 0, x \leq t, \end{cases}$$

$$i = 0, 1, \ldots, n-k \ ; \ \omega_i(x) = (x - x_i)(x - x_{i+1}) \ldots (x - x_{i+k}) \,.$$

Since,

$$\int_{-\infty}^{\infty} \varphi_i(x) \, w_\alpha(x) \, dx = 1.$$

Clearly, $\varphi_i \in S_{k-1}.Set$

$$A_i = \int_{x_i}^{x_{i+1}} (s(x) - f'(x)) \, w_\alpha(x) \, dx \geq 0, \ B_i = \int_{x_i}^{x_{i+1}} (f'(x) - l(x)) \, w_\alpha(x) \, dx \geq 0.$$

And consider that

$$s^*(x) = \int_0^x s(t) \, w_\alpha(t) \, dt - \sum_{i=0}^{n-k-1} A_i \int_0^x \varphi_{i+1}(t) w_\alpha(t) \, dt,$$

$$l^*(x) = \int_0^x l(t) \, w_\alpha(t) \, dt + \sum_{i=0}^{n-k-1} B_i \int_0^x \varphi_{i+1}(t) w_\alpha(t) \, dt,$$

we have $s^* \in S_k, l^* \in S_k$. If $x_{i_0} < x < x_{i_\varepsilon+1}$, then since $f(0) = 0$ we have

$$s^*(x) - \int_0^x f'(t) \, w_\alpha(t) \, dt = s^*(x) - f(x)$$

$$= \int_0^x (s(t) - f'(t)) \, w_\alpha(t) \, dt - \sum_{i=0}^{n-k-1} A_i \int_0^x \varphi_{i+1}(t) \, w_\alpha(t) \, dt$$

$$= \sum_{i=0}^{i_0-1} \int_{x_i}^{x_{i+1}} (s(t) - f'(t)) \, w_\alpha(t) \, dt + \int_{x_0}^x (s(t) - f'(t) \, w_\alpha(t) \, dt$$

$$- \sum_{i=0}^{i_0-k-1} A_i - \sum_{i=i_0-k}^{i_0-1} A_i \int_0^x \varphi_{i+1}(t) \, w_\alpha(t) \, dt$$

$$= \int_{x_0}^x (s(t) - f'(t)) \, w_\alpha(t) \, dt + \sum_{i=i_0-k}^{i_0-1} A_i \left(1 - \int_0^x \varphi_{i+1}(t) \, w_\alpha(t) \, dt \right) \geq 0. \quad (5)$$

$$\left(A_i \geq 0, 0 \leq \int_0^x \varphi_{i+1}(t) \, w_\alpha(t) \, dt \leq 1 \right).$$

We many prove analogously that $l^*(x) \leq f(x)$. Set $s(x) = 0, l(x) = 0$ for $x < 0$. Then

$$\|s^* - l^*\|_{L_{p,\alpha}[0,1]} = \left\{ \int_0^1 \left| \int_0^x (s(t) - l(t)) \, w_\alpha(t) \, dt \right. \right.$$

$$-\sum_{i=0}^{n-k-1}(A_i+B_i)\int_0^x \varphi_{i+1}(t)\,w_\alpha(t)\,dt\Big|^p dx\Bigg\}^{\frac{1}{p}}$$

$$\leq\left\{\int_0^1\left|\int_{x-(k+1)\Delta_n}^{x}(s(t)-l(t))\,w_\alpha(t)\,dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$=\left\{\int_0^1\left|\int_0^{(k+1)\Delta_n}(s(x-(k+1)\Delta_n+t)-l(x-(k+1)\Delta_n+t\ ))\,w_\alpha(t)\,dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$\leq\int_0^{(k+1)\Delta_n}\left\{\int_0^1|s(x-(k+1)\Delta_n+t)-l(x-(k+1)\Delta_n+t\ )|^p\ w_\alpha(x)\,dx\right\}^{\frac{1}{p}}dt$$

$$\leq\int_0^{(k+1)\Delta_n}\left(\widetilde{E}_{k-1}(f')_{L_{p,\alpha}}+\varepsilon\right)dt=(k+1)\,\Delta_n\ \left(\widetilde{E}_{k-1}(f')_{L_{p,\alpha}}+\varepsilon\right). \quad (6)$$

The relation of the theorem follows from inequality 5 and and inequality 6. □

Theorem 4.2. *If f is an unbounded function on the closed interval* $[0,1]$, *then*

$$\widetilde{E}_0(f)_{L_{p,\alpha}}\leq 2\tau\left(f;n^{-1}\right)_{L_{p,\alpha}},1\leq p\leq\infty.$$

Proof. Set

$$\begin{aligned} s(x)&=\sup_{t\in[x_{i-1},x_i]}f(t),x\in[x_{i-1},x_i)\,,\ s(1)=\lim_{x\to 1}s(x),\\ l(x)&=\inf_{t\in[x_{i-1},x_i]}f(t),x\in[x_{i-1},x_i)\,,\ l(1)=\lim_{x\to 1}l(x).\end{aligned} \quad (7)$$

It follows from 7 (see the notation of Lemma 3.6) that

$$\begin{aligned} f(x)&\leqq s(x)\leq S\left(f,\ x;\ 2n^{-1}\right),\\ f(x)&\geq l(x)\geq J\left(f,\ x\ ;\ 2n^{-1}\right).\end{aligned} \quad (8)$$

Since $s,\ l\ \in S_0$ and

$$\omega(f,x;\ \delta)=S(f,\ x;\ \delta)-J(f,\ x;\ \delta),$$

using Lemma 3.5, from 8 we have

$$\widetilde{E}_0(f)_{L_{p,\alpha}}\leq\left\{\int_0^1|(s(x)-l(x))\,w_\alpha(x)|^p\,dx\right\}^{\frac{1}{p}}$$

$$\begin{aligned}&\leq\left\{\int_0^1\left|\left(S\left(f,x;\ 2n^{-1}\right)-J\left(f,x;\ 2n^{-1}\right)\right)w_\alpha(x)\right|^p dx\right\}^{\frac{1}{p}}\\ &=\tau\left(f;2n^{-1}\right)_{L_{p,\alpha}}\leqq 2\tau\left(f;\ n^{-1}\right)_{L_{p,\alpha}}.\end{aligned}$$

□

Theorem 4.3. *Let f have integrable unbounded k-th derivative $f^{(k)}$ on the interval* $[0,1]$. $f \in L_{p,\alpha}([0,1])$ *with* $1 \le p \le \infty$. *Then*

$$\widetilde{E}_k(f)_{L_{p,\alpha}} \le 2(k+1)!\left(n^{-1}\right)^k \tau\left(f^{(k)};\ n^{-1}\right)_{L_{p,\alpha}}.$$

Proof. In fact, by Theorem 4.1 k-times in succession and applying Theorem 4.2 to $f^{(k)}$, we obtain

$$\widetilde{E}_k(f)_{L_{p,\alpha}} \le (k+1)\, n^{-1} \widetilde{E}_{k-1}(f')_{L_{p,\alpha}} \le \dots$$

$$\le (k+1)!\left(n^{-1}\right)^k \widetilde{E}_0\left(f^{(k)}\right)_{L_{p,\alpha}} 2(k+1)!\left(n^{-1}\right)^k \tau\left(f^k; n^{-1}\right)_{L_{p,\alpha}}.$$

From the properties of the average module $\tau(f;\delta)_{L_{p,\alpha}}$ and Theorem 4.3, we have the following. □

Corollary 4.4. *Let* $f \in L_{p,\alpha}(X);1 \le p \le \infty$ *with k-th derivative* f^k.*Then*

(1) $\widetilde{E}_k(f)_c \le 2(k+1)!\left(n^{-1}\right)^k \omega\left(f^{(k)}; n^{-1}\right)$,

(2) *(Freud -Popov Theorem* [8] *)*

$$\widetilde{E}_k(f)_{L_{1,\alpha}} \leqq 2(k+1)!\left(n^{-1}\right)^{k+1} V(f^{(k)},\delta).$$

Corollary 4.5. *(Babenko-Ligun Theorem* [7]*). if* $\left\|f^{(k+1)}\right\|_{L_{p,\alpha}} < \infty$, *then*

$$\widetilde{E}_k(f)_{L_{p,\alpha}} \le c_3(k)\left\|f^{(k+1)}\right\|_{L_{p,\alpha}} n^{-(k+1)}.$$

Corollary 4.6. *Let f be an unbounded function. Then*

$$\widetilde{E}_n^T(f)_{L_{p,\alpha}} \le c\tau\left(f; n^{-1}\right)_{L_{p,\alpha}}, 1 \le p \le \infty,$$

where c is an absolute constant.

Proof. Set $x_i = 2in^{-1}, i = 0,\dots,2n, y_i = {}^{(x_{i-1}+x_i)}/_2, i = 1,\dots,2n, y_{2n+1} = y_1$ and define the functions s_n and J_n as follows:

$$S_n(x) = \begin{cases} \sup_{t\in[x_{i-1},x_i]} for \quad x = y_i,\ i = 1,\dots,2n, \\ \max\{S_n(y_i), S_n(y_{n+1})\} \quad for \quad x = x_i\ , i = 1,\dots,2n, \\ S_n(0) = S_n(1) \text{ linear and continuous for } x \in [x_{i-1}, y_i] and\ x \in [y_i, x_i], \\ i = 1,\dots,2n \quad . \end{cases}$$

$$J_n(x) = \begin{cases} \inf_{t\in[x_{i-1},x_i]} for \quad x = y_i,\ i = 1,\dots,2n, \\ \min\{J_n(y_i), J_n(y_{n+1})\} \quad for \quad x = x_i\ , i = 1,\dots,2n, \\ J_n(0) = J_n(1) \text{ linear and continuous for } x \in [x_{i-1}, y_i] and\ x \in [y_i, x_i], \\ i = 1,\dots,2n \quad . \end{cases}$$

Clearly, we have

$$J_n(x) \leqq f(x) \le S_n(x),\ x \in [0,1]. \tag{9}$$

The derivatives $S_n'(x)$ and $J_n'(x)$ of S_n and J_n exist at each point of the interval [0,1]. Except the point $x_i, i = 0, \ldots, 2n, y_i, i = 1, \ldots, 2n$ moreover, using the definitions of the functions S_n and J_n, we immediately have

$$\begin{aligned} |S_n'(x)| &\leq n\omega\left(f, x; 2n^{-1}\right), \quad x \neq x_i,\ y_i, \\ |J_n'(x)| &\leq n\omega\left(f, x; 2n^{-1}\right), \quad x \neq x_i,\ y_i, \end{aligned} \tag{10}$$

e.g., if $x \in (y_i, x_i)$, then as S_n is linear we have

$$|S_n'(x)| \leqq n\left|S_n(y_{i+1}) - S_n(y_i)\right| \leq n\omega(f, x; 2n^{-1})),$$

and moreover,

$$0 \leq S_n(x) - J_n(x) \leq \omega(f, x; n^{-1}). \tag{11}$$

It follows from 10 that

$$\begin{aligned} \|S_n'(x)\|_{L_{p,\alpha}([0,1])} &\leq n\tau\left(f; 2n^{-1}\right)_{L_{p,\alpha}}, \\ \|J_n'(x)\|_{L_{p,\alpha}([0,1])} &\leq n\tau\left(f; 2n^{-1}\right)_{L_{p,\alpha}}, \end{aligned} \tag{12}$$

Moreover, 11 gives

$$\|S_n - J_n\|_{L_{p,\alpha}([0,1])} \leq \tau\left(f; n^{-1}\right)_{L_{p,\alpha}}. \tag{13}$$

Using Lemma 3.10, for $r = 1$, we obtain from 12

$$\widetilde{E}_n^T(S_n)_{L_{p,\alpha}} \leq c(1)\,\tau\left(f; 2n^{-1}\right)_{L_{p,\alpha}};\ \widetilde{E}_n^T(J_n)_{L_{p,\alpha}} \leq c(1)\,\tau\left(f; 2n^{-1}\right)_{L_{p,\alpha}}. \tag{14}$$

The following inequality is obvious:

$$\widetilde{E}_n^T(f)_{L_{p,\alpha}} \leq \widetilde{E}_n^T(S_n)_{L_{p,\alpha}} + \|S_n - J_n\|_{L_{p,\alpha}} + \widetilde{E}_n^T(J_n)_{L_{p,\alpha}}. \tag{15}$$

Using Lemma 3.5, from 13-15 we obtain

$$\widetilde{E}_n^T(f)_{L_{p,\alpha}} \leq 2c(1)\,\tau\left(f; 2n^{-1}\right)_{L_{p,\alpha}} + \tau\left(f; 2n^{-1}\right)_{L_{p,\alpha}} \leq c\tau\left(f; n^{-1}\right)_{L_{p,\alpha}}.$$

□

5. Conclusions

Average moduli $\tau_k(f;\delta)_{L_{p,\alpha}}$ can be defined in analogy to kth continuity moduli $\omega_k(f;\delta)_{L_{p,\alpha}}$. Also, we obtained by us for the average moduli $\tau_k(f;\delta)_{L_{p,\alpha}}$ are a generalizations for some results in the literature, equivalent to the general situation of direct theorem for $\omega_k(f;\delta)_{L_{p,\alpha}}$, obtained by Stechkin in the literature.

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