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RIGHT-RADAU-TYPE INEQUALITIES FOR MULTIPLICATIVE DIFFERENTIABLE s-CONVEX FUNCTIONS†

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Abstract. In this study, a new identity is introduced for multiplicative differentiable functions, forming the foundation for a range of 2-point right-Radau-type inequalities applicable to multiplicative s-convex functions. These established results are then showcased through applications that underscore their relevance within the domain of special means.

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1. Introduction

A convex function is a mathematical function characterized by a distinct geometric property. Specifically, a function f is deemed convex on the interval $[a, b]$ if, for any two points x and y within $[a, b]$, and for any t within the interval $[0, 1]$, the following inequality is satisfied [33].

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f(y).$$

The fundamental inequality for convex functions is undoubtedly the Hermite-Hadamard inequality, which can be stated as follows: For every convex function f on the interval $[a, b]$ with $a < b$, we have

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a}\int_a^b f\left(x\right)dx \leq \tfrac{f(a)+f(b)}{2}. \tag{1}$$

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The concept of convexity holds a crucial and central role across numerous domains, maintaining a close association with the evolution of inequalities' theory. This concept serves as a significant tool in exploring properties of solutions to differential equations and in error estimation for quadrature formulas. The efficiency of this notion has led to the establishment of various forms of convexity, introduced as generalizations and variants, aimed at expanding the scope of study to encompass broader classes of functions. For further exploration of papers focusing on quadrature via diverse forms of convexity, we direct readers to references such as [8, 13, 14, 15, 16, 17, 23, 24, 25, 28, 36, 37] and the sources cited therein.

In 1967, Grossman and Katz [21] made a significant breakthrough by introducing the first non-Newtonian computation system known as geometric calculus. This marked a departure from the traditional calculus established by Newton and Leibniz in the 17th century. Over the subsequent years, they embarked on a journey of exploration and innovation, giving rise to a diverse family of non-Newtonian calculus. These novel type of calculus introduced profound modifications that diverged markedly from the classical calculus known today.

Referred to as the non-Newtonian calculus or multiplicative calculus, these variants adopt a distinct approach. Instead of the conventional addition and subtraction operations, the framework employs ordinary product and ratio operations. This transformation extends to the domain of positive real numbers, where the customary arithmetic operations take on new meanings as summation and exponential difference.

The non-Newtonian calculus has proven its value, especially when dealing with functions that exhibit exponential variability. Its unique perspective and altered operations provide researchers and mathematicians with a powerful tool for tackling problems involving exponential growth and decay, offering fresh insights and opening new avenues of exploration in various scientific and mathematical disciplines [7, 18, 38].

The most relevant type of convexity associated with this concept is multiplicative convexity, defined as follows:

Definition 1.1 ([33]). A function $f : I \to \mathbb{R}^+$ is termed log-convex or multiplicatively convex if for all $x, y \in I$ and all $t \in [0,1]$ we have

$$f\left(tx + (1-t)\,y\right) \leq \left[f\left(x\right)\right]^t \left[f(y)\right]^{1-t}.$$

In a recent study, Ali et al. [1] established the analogous version of the Hermite-Hadamard inequality for multiplicatively convex functions as follows:

Theorem 1.2. *Suppose f is a positive and multiplicative convex function over the interval $[a, b]$. In this context, the subsequent inequalities are valid:*

$$f\left(\tfrac{a+b}{2}\right) \leq \left(\int_a^b f\left(x\right)^{dx}\right)^{\frac{1}{b-a}} \leq \sqrt{f\left(a\right) f\left(b\right)}.$$

On of the most renowned generalizations of multiplicative convexity is the multiplicative s-convexity in the second sense which is defined as follows:

Definition 1.3 ([39])**.** A function $f : I \to \mathbb{R}^{+}$ is termed multiplicatively s-convex in the second sense for some fixed $s \in (0,1]$ if

$$f\left(tx+(1-t)\,y\right) \leq \left[f\left(x\right)\right]^{t^{s}} \left[f(y)\right]^{(1-t)^{s}}$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

In [32], Özçan, provided the Hermite-Hadamard inequality related to multiplicative s-convexity in the following manner:

Theorem 1.4. *Suppose f is a positive and multiplicative s-convex function in the second sense over the interval $[a,b]$. In this context, the subsequent inequalities are valid:*

$$\left[f\left(\tfrac{a+b}{2}\right)\right]^{2^{s-1}} \leq \left(\int_{a}^{b} f\left(x\right)^{dx}\right)^{\frac{1}{b-a}} \leq \left[f\left(a\right)f\left(b\right)\right]^{\frac{1}{s+1}}.$$

Regarding the outcomes obtained in [1] across diverse forms of multiplicative convexity, we direct interested readers to references such as [2, 3, 11, 20]. Subsequently, a number of researchers have delved into the exploration of diverse quadrature rules, with a specific focus on Newton-Cotes formulas. Notably, in [4], Ali et al. established Ostrowski and Simpson type inequalities for multiplicative differentiable functions. Berhail and Meftah contributed midpoint and trapezium inequalities for the same category of functions in [9]. Similarly, in [10], Boulares et al. discussed Bullen's inequalities, while Moumen et al. investigated Simpson's inequalities in [31]. Additionally, to gain deeper insights into multiplicative inequalities connected with Newton-Cotes rules, references such as [5, 12, 19, 22, 26, 27, 29, 30, 34, 35, 40] can be consulted.

Gauss quadrature formulas stand out as a remarkable class of numerical integration techniques, showcasing distinct advantages over Newton-Cotes formulas. Gauss formulas, unlike their Newton-Cotes counterparts, strategically select their nodes and weights, resulting in enhanced accuracy for polynomial approximations. This precision is particularly pronounced when dealing with functions that exhibit rapid oscillations or sharp changes. The effectiveness of Gauss quadrature lies in its ability to achieve superior approximation quality using a limited number of evaluation points, thereby significantly reducing computational effort.

In the context of multiplicative calculus, while several variants of Newton-Cotes formulas have been explored, an intriguing aspect emerges when considering Gauss quadrature. Surprisingly, the literature has yet to delve into the investigation of Gauss quadrature formulas within the realm of multiplicative calculus. This presents an unexplored avenue for future research, inviting scholars to explore the potential of Gauss quadrature methods in the context of multiplicative calculus and uncover novel insights in numerical integration techniques.

Motivated by these considerations, our study takes a pioneering step by introducing a novel parametrized identity for multiplicative differentiable functions. Leveraging this identity, we proceed to establish a series of right-Radau-type inequalities tailored to multiplicative differentiable s-convex functions. At the end, some applications to special means are given.

2. Preliminaries

In this section, we initiate by revisiting several definitions, properties, and concepts related to differentiation and multiplicative integration. The comprehensive mathematical framework of multiplicative calculus was formally presented by Bashirov et al. [6].

Definition 2.1 ([6])**.** The multiplicative derivative of a positive function f, denoted as f^*, is defined as follows:

$$\frac{d^* f}{dt} = f^*(t) = \lim_{h \to 0} \left(\frac{f(t+h)}{f(t)} \right)^{\frac{1}{h}}.$$

Remark 2.1. Assuming that f takes positive values and is differentiable at t, the multiplicative derivative f^* is well-defined. The connection between f^* and the usual derivative f' is as follows:

$$f^*(t) = e^{(\ln f(t))'} = e^{\frac{f'(t)}{f(t)}}.$$

Let's outline several key properties of the multiplicative derivative.

Proposition 2.2 ([6])**.** *Assuming f and g are multiplicatively differentiable functions and α is an arbitrary constant, the functions αf, $f+g$, fg, f/g, and f^g all exhibit * differentiability. Moreover, the ensuing properties apply:*

- $(\alpha f)^*(t) = f^*(t)$,
- $(f+g)^*(t) = f^*(t)^{\frac{f(t)}{f(t)+g(t)}} g^*(t)^{\frac{g(t)}{f(t)+g(t)}}$,
- $(fg)^*(t) = f^*(t) g^*(t)$,
- $\left(\frac{f}{g}\right)^*(t) = \frac{f^*(t)}{g^*(t)}$,
- $(f^g)^*(t) = f^*(t)^{g(t)} f(t)^{g'(t)}$.

In their work [6], Bashirov et al. introduced the notion of the * integral, known as the multiplicative integral, denoted by $\int_a^b (f(t))^{dt}$.

The connection between the Riemann integral and the multiplicative integral can be described as follows:

Proposition 2.3 ([6])**.** *In the event that f is Riemann integrable over the interval $[a,b]$, it follows that f is also multiplicative integrable on the same interval $[a,b]$. Moreover, the following relation holds:*

$$\int_a^b (f(t))^{dt} = \exp\left\{ \int_a^b \ln(f(t))\, dt \right\}.$$

Furthermore, Bashirov et al. demonstrated that the multiplicative integral exhibits the subsequent characteristics:

Proposition 2.4 ([6]). *Suppose f is a positive function that is Riemann integrable over the interval $[a,b]$. In this scenario, it can be affirmed that f is also multiplicative integrable on the same interval $[a,b]$, and the following relation holds:*

- $\int_a^b ((f(t))^p)^{dt} = \left(\int_a^b (f(t))^{dt}\right)^p$,
- $\int_a^b (f(t)g(t))^{dt} = \int_a^b (f(t))^{dt} \int_a^b (g(t))^{dt}$,
- $\int_a^b \left(\frac{f(t)}{g(t)}\right)^{dt} = \frac{\int_a^b (f(t))^{dt}}{\int_a^b (g(t))^{dt}}$,
- $\int_a^b (f(t))^{dt} = \int_a^c (f(t))^{dt} \int_c^b (f(t))^{dt}$, $a < c < b$,
- $\int_a^a (f(t))^{dt} = 1$ *and* $\int_a^b (f(t))^{dt} = \left(\int_b^a (f(t))^{dt}\right)^{-1}$.

Theorem 2.5 (Multiplicative Integration by Parts [6]). *Let f be a multiplicative differentiable function, and let g be a differentiable function on $[a,b]$. In this context, it can be established that the function f^g is multiplicative integrable on $[a,b]$, and the following holds:*

$$\int_a^b \left(f^*(t)^{g(t)}\right)^{dt} = \frac{f(b)^{g(b)}}{f(a)^{g(a)}} \times \frac{1}{\int_a^b \left(f(t)^{g'(t)}\right)^{dt}}.$$

Lemma 2.6 ([3]). *Consider a multiplicative differentiable function $f : [a,b] \to \mathbb{R}^+$, and let $g : [a,b] \to \mathbb{R}$ as well as $h : J \subset \mathbb{R} \to \mathbb{R}$ be two differentiable functions. Under these conditions, the following relation holds:*

$$\int_a^b \left(f^*(h(t))^{h'(t)g(t)}\right)^{dt} = \frac{f(h(b))^{g(b)}}{f(h(a))^{g(a)}} \times \frac{1}{\int_a^b \left(f(h(t))^{g'(t)}\right)^{dt}}.$$

3. Main results

In pursuit of substantiating our theoretical findings, it becomes imperative to invoke the subsequent lemma.

Lemma 3.1. *Assume that $f : [a,b] \to \mathbb{R}^+$ is a multiplicative differentiable function over $[a,b]$. If f^* is multiplicative integrable on the interval $[a,b]$, then the* *following identity, pertaining to the 2-point right-Radau rule, holds for multiplicative integrals:*

$$\left(f\left(\tfrac{2a+b}{3}\right)\right)^{\frac{3}{2}}\left(f\left(b\right)\right)^{\frac{1}{2}}\left(\int_a^b f\left(u\right)^{du}\right)^{\frac{2}{a-b}}$$

$$=\left(\int_0^1\left(\left(f^{*}\left(\left(1-t\right)a+t\tfrac{2a+b}{3}\right)\right)^{2t}\right)^{dt}\right)^{\frac{b-a}{9}}\left(\int_0^1\left(\left(f^{*}\left(\left(1-t\right)\tfrac{2a+b}{3}+tb\right)\right)^{8t-5}\right)^{dt}\right)^{\frac{b-a}{9}}.$$

Proof. Let

$$I_1=\left(\int_0^1\left(\left(f^{*}\left(\left(1-t\right)a+t\tfrac{2a+b}{3}\right)\right)^{2t}\right)^{dt}\right)^{\frac{b-a}{9}}$$

and

$$I_2=\left(\int_0^1\left(\left(f^{*}\left(\left(1-t\right)\tfrac{2a+b}{3}+tb\right)\right)^{8t-5}\right)^{dt}\right)^{\frac{b-a}{9}}.$$

Using Lemma 2.6, I_1 gives

$$\begin{aligned}
I_1&=\left(\int_0^1\left(\left(f^{*}\left(\left(1-t\right)a+t\tfrac{2a+b}{3}\right)\right)^{2t}\right)^{dt}\right)^{\frac{b-a}{9}}\\
&=\int_0^1\left(\left(f^{*}\left(\left(1-t\right)a+t\tfrac{2a+b}{3}\right)\right)^{\frac{2(b-a)}{9}t}\right)^{dt}\\
&=\int_0^1\left(\left(f^{*}\left(\left(1-t\right)a+t\tfrac{2a+b}{3}\right)\right)^{\frac{b-a}{3}\frac{2}{3}t}\right)^{dt}\\
&=\frac{\left(f\left(\frac{2a+b}{3}\right)\right)^{\frac{2}{3}}}{\left(f(a)\right)^{0}}\frac{1}{\int_0^1\left(\left(f\left((1-t)a+t\frac{2a+b}{3}\right)\right)^{\frac{2}{3}}\right)^{dt}}\\
&=\left(f\left(\tfrac{2a+b}{3}\right)\right)^{\frac{2}{3}}\frac{1}{\exp\left\{\int_0^1\frac{2}{3}\ln f\left((1-t)a+t\frac{2a+b}{3}\right)dt\right\}}\\
&=\left(f\left(\tfrac{2a+b}{3}\right)\right)^{\frac{2}{3}}\frac{1}{\exp\left\{\int_a^{\frac{2a+b}{3}}\left(\frac{2}{(b-a)}\ln f(u)\right)du\right\}}\\
&=\left(f\left(\tfrac{2a+b}{3}\right)\right)^{\frac{2}{3}}\left(\int_a^{\frac{2a+b}{3}}f\left(u\right)^{du}\right)^{\frac{2}{a-b}}.
\end{aligned}\tag{2}$$

Likewise

$$\begin{aligned}
I_2 &= \left(\int_0^1 \left(\left(f^*\left((1-t)\tfrac{2a+b}{3}+tb\right)\right)^{8t-5}\right)^{dt}\right)^{\frac{b-a}{9}} \\
&= \left(\int_0^1 \left(\left(f^*\left((1-t)\tfrac{2a+b}{3}+tb\right)\right)^{\frac{b-a}{9}(8t-5)}\right)^{dt}\right) \\
&= \int_0^1 \left(\left(f^*\left((1-t)\tfrac{2a+b}{3}+tb\right)\right)^{\frac{2(b-a)}{3}\left(\frac{8t-5}{6}\right)}\right)^{dt} \\
&= \frac{(f(b))^{\frac{1}{2}}}{\left(f\left(\frac{2a+b}{3}\right)\right)^{-\frac{5}{6}}} \frac{1}{\int_0^1 \left(\left(f\left((1-t)\frac{2a+b}{3}+tb\right)\right)^{\frac{4}{3}}\right)^{dt}} \\
&= (f(b))^{\frac{1}{2}} \left(f\left(\tfrac{2a+b}{3}\right)\right)^{\frac{5}{6}} \frac{1}{\exp\left\{\int_0^1 \frac{4}{3}\ln f\left((1-t)\frac{2a+b}{3}+tb\right)dt\right\}} \\
&= (f(b))^{\frac{1}{2}} \left(f\left(\tfrac{2a+b}{3}\right)\right)^{\frac{5}{6}} \frac{1}{\exp\left\{\int_{\frac{2a+b}{3}}^{b} \left(\frac{2}{(b-a)}\ln f(u)\right)du\right\}} \\
&= (f(b))^{\frac{1}{2}} \left(f\left(\tfrac{2a+b}{3}\right)\right)^{\frac{5}{6}} \left(\int_{\frac{2a+b}{3}}^{b} f(u)^{du}\right)^{\frac{2}{a-b}}. \qquad (3)
\end{aligned}$$

By multiplying (2) with (3), we arrive at the sought-after outcome. This concludes the proof. □

Theorem 3.2. *Under the assumptions of Lemma 3.1. If f is incresing on $[a,b]$ and f^* is multiplicative s-convex on $[a,b]$, then we have*

$$\begin{aligned}
&\left|\left(f\left(\tfrac{2a+b}{3}\right)\right)^{\frac{3}{2}} (f(b))^{\frac{1}{2}} \left(\int_a^b f(u)^{du}\right)^{\frac{2}{a-b}}\right| \\
&\leq \left((f^*(a))\left(f^*\left(\tfrac{2a+b}{3}\right)\right)^{\frac{7s+4}{2}+3\left(\frac{3}{8}\right)^{s+1}} f^*(b)^{\frac{3s-2}{2}+5\left(\frac{5}{8}\right)^{s+1}}\right)^{\frac{2(b-a)}{9(s+1)(s+2)}}.
\end{aligned}$$

Proof. Building upon Lemma 3.1, the characteristics of multiplicative integration, and the multiplicative s-convexity of f^*, we deduce that:

$$\left|\left(f\left(\tfrac{2a+b}{3}\right)\right)^{\frac{3}{2}} (f(b))^{\frac{1}{2}} \left(\int_a^b f(u)^{du}\right)^{\frac{2}{a-b}}\right|$$

$$\leq \exp\left\{\frac{b-a}{9}\int\limits_0^1 \left|\ln\left(f^*\left((1-t)a+t\tfrac{2a+b}{3}\right)\right)^{2t}\right| dt\right\}$$

$$\times \exp\left\{\frac{b-a}{9}\int\limits_0^1 \left|\ln\left(\left(f^*\left((1-t)\tfrac{2a+b}{3}+tb\right)\right)^{8t-5}\right)\right| dt\right\}$$

$$= \exp\left\{\frac{b-a}{9}\int\limits_0^1 2t\left|\ln f^*\left((1-t)a+t\tfrac{2a+b}{3}\right)\right| dt\right\}$$

$$\times \exp\left\{\frac{b-a}{9}\int\limits_0^1 |8t-5|\left|\ln f^*\left((1-t)\tfrac{2a+b}{3}+tb\right)\right| dt\right\}$$

$$\leq \exp\left\{\frac{b-a}{9}\int\limits_0^1 2t\left|\ln\left(f^*(a)\right)^{(1-t)^s}\left(f^*\left(\tfrac{2a+b}{3}\right)\right)^{t^s}\right| dt\right\}$$

$$\times \exp\left\{\frac{b-a}{9}\int\limits_0^1 |8t-5|\left|\ln\left(f^*\left(\tfrac{2a+b}{3}\right)\right)^{(1-t)^s}\left(f^*(b)\right)^{t^s}\right| dt\right\}$$

$$= \exp\left\{\frac{b-a}{9}\int\limits_0^1 2t\left((1-t)^s\ln\left(f^*(a)\right)+t^s\ln f^*\left(\tfrac{2a+b}{3}\right)\right) dt\right\}$$

$$\times \exp\left\{\frac{b-a}{9}\int\limits_0^1 |8t-5|\left((1-t)^s\ln\left(f^*\left(\tfrac{2a+b}{3}\right)\right)+t^s\ln f^*(b)\right) dt\right\}$$

$$= \exp\left\{\frac{b-a}{9}\left(\frac{2}{(s+1)(s+2)}\ln f^*(a)+\frac{2}{s+2}\ln f^*\left(\tfrac{2a+b}{3}\right)\right)\right\}$$

$$\times \exp\left\{\frac{b-a}{9}\left(\left(\frac{5s+2}{(s+1)(s+2)}+\frac{6}{(s+1)(s+2)}\left(\tfrac{3}{8}\right)^{s+1}\right)\ln f^*\left(\tfrac{2a+b}{3}\right)\right.\right.$$

$$\left.\left.+\left(\frac{3s-2}{(s+1)(s+2)}+\frac{10}{(s+1)(s+2)}\left(\tfrac{5}{8}\right)^{s+1}\right)\ln f^*(b)\right)\right\}$$

$$= \left(\left(f^*(a)\right)\left(f^*\left(\tfrac{2a+b}{3}\right)\right)^{\frac{7s+4}{2}+3\left(\frac{3}{8}\right)^{s+1}} f^*(b)^{\frac{3s-2}{2}+5\left(\frac{5}{8}\right)^{s+1}}\right)^{\frac{2(b-a)}{9(s+1)(s+2)}},$$

where we have used

$$\int\limits_0^1 2t(1-t)^s dt = \frac{2}{(s+1)(s+2)}, \qquad (4)$$

$$\int\limits_0^1 2t^{s+1}dt = \frac{2}{s+2}, \qquad (5)$$

$$\int_0^1 |8t-5|\,(1-t)^s\,dt = \frac{5s+2}{(s+1)(s+2)} + \frac{6}{(s+1)(s+2)}\left(\frac{3}{8}\right)^{s+1} \tag{6}$$

and

$$\int_0^1 |8t-5|\,t^s dt = \frac{3s-2}{(s+1)(s+2)} + \frac{10}{(s+1)(s+2)}\left(\frac{5}{8}\right)^{s+1}. \tag{7}$$

The proof is finished. □

Corollary 3.3. *In Theorem 3.2, taking $s=1$, we get*

$$\left|\left(f\left(\frac{2a+b}{3}\right)\right)^{\frac{3}{2}}\left(f(b)\right)^{\frac{1}{2}}\left(\int_a^b f(u)^{du}\right)^{\frac{2}{a-b}}\right|$$
$$\le \left(\left(f^*(a)\right)^{64}\left(f^*\left(\frac{2a+b}{3}\right)\right)^{379}\left(f^*(b)\right)^{157}\right)^{\frac{b-a}{1728}}.$$

Corollary 3.4. *If the function $f^* \le M$, Corollary 3.3 becomes*

$$\left|\left(f\left(\frac{2a+b}{3}\right)\right)^{\frac{3}{2}}\left(f(b)\right)^{\frac{1}{2}}\left(\int_a^b f(u)^{du}\right)^{\frac{2}{a-b}}\right| \le M^{\frac{25(b-a)}{72}}.$$

Example 3.5. Let's examine the function $f:[0,1]\to\mathbb{R}^+$ defined as $f(u) = 2^{u^{s+1}}$, where $s\in(0,1]$. Consequently, the multiplicative derivative $f^*(y) = 2^{(s+1)y^s}$ demonstrates multiplicative s-convexity on $[0,1]$.

Drawing from Theorem 3.2, we acquire:

$$2^{\frac{1+3^s}{2\times 3^s}-\frac{2}{s+2}} \le 2^{\left(\frac{1}{3^s}\left(\frac{7s+4}{2}+3\left(\frac{3}{8}\right)^{s+1}\right)+\left(\frac{3s-2}{2}+5\left(\frac{5}{8}\right)^{s+1}\right)\right)\frac{2}{9(s+2)}}.$$

The two terms of the aforementioned inequality are elucidated in Table 1 and depicted in Figure 1.

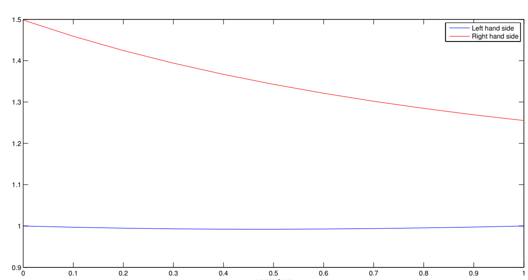

FIGURE 1. $s\in(0,1]$

s	LHS	RHS
0.10	0.9970	1.4591
0.20	0.9947	1.4246
0.30	0.9931	1.3940
0.40	0.9923	1.3669
0.50	0.9922	1.3427
0.60	0.9927	1.3212
0.70	0.9938	1.3018
0.80	0.9954	1.2846
0.90	0.9975	1.2691
1.00	1.0000	1.2552

Table 1. Numerical validation of Theorem 3.2

Given these representations, it becomes evident that the right-hand side is greater than the left-hand side for any $s \in (0,1]$, thereby corroborating the validity of the outcome derived from Theorem 3.2.

Theorem 3.6. *Under the assumptions of Lemma 3.1. If f is incresing on $[a,b]$ and $(\ln f^*)^q$ is s-convex on $[a,b]$, where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\left|\left(f\left(\frac{2a+b}{3}\right)\right)^{\frac{3}{2}}\left(f(b)\right)^{\frac{1}{2}}\left(\int_a^b f(u)^{du}\right)^{\frac{2}{a-b}}\right|$$

$$\leq\left(f^*(a)^2\left(f^*\left(\frac{2a+b}{3}\right)\right)^{2+\left(\frac{5^{p+1}+3^{p+1}}{8}\right)^{\frac{1}{p}}}\left(f^*(b)\right)^{\left(\frac{5^{p+1}+3^{p+1}}{8}\right)^{\frac{1}{p}}}\right)^{\frac{b-a}{9}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}}.$$

Proof. Building upon Lemma 3.1, s-convexity of $(\ln f^*)^q$ among with Hölder inequality, we deduce that:

$$\left|\left(f\left(\frac{2a+b}{3}\right)\right)^{\frac{3}{2}}\left(f(b)\right)^{\frac{1}{2}}\left(\int_a^b f(u)^{du}\right)^{\frac{2}{a-b}}\right|$$

$$\leq \exp\left\{\frac{b-a}{9}\int_0^1\left|\ln\left(f^*\left((1-t)a+t\frac{2a+b}{3}\right)\right)^{2t}\right|dt\right\}$$

$$\times \exp\left\{\frac{b-a}{9}\int_0^1\left|\ln\left(\left(f^*\left((1-t)\frac{2a+b}{3}+tb\right)\right)^{8t-5}\right)\right|dt\right\}$$

$$=\exp\left\{\frac{b-a}{9}\int_0^1 2t\left|\ln f^*\left((1-t)a+t\frac{2a+b}{3}\right)\right|dt\right\}$$

$$\times \exp\left\{\frac{b-a}{9}\int_0^1|8t-5|\left|\ln f^*\left((1-t)\frac{2a+b}{3}+tb\right)\right|dt\right\}$$

$$\leq \exp\left\{\frac{b-a}{9}\left(\int_0^1 (2t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\ln f^*\left((1-t)\,a+t\,\frac{2a+b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\times\exp\left\{\frac{b-a}{9}\left(\int_0^1 |8t-5|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\ln f^*\left((1-t)\,\frac{2a+b}{3}+tb\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq \exp\left\{\frac{b-a}{9}\left(\frac{2^p}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left((1-t)^s\left(\ln f^*(a)\right)^q+t^s\left(\ln f^*\left(\frac{2a+b}{3}\right)\right)^q\right)dt\right)^{\frac{1}{q}}\right\}$$

$$\times\exp\left\{\frac{b-a}{9}\left(\frac{5^{p+1}+3^{p+1}}{8(p+1)}\right)^{\frac{1}{p}}\left(\int_0^1 \left((1-t)^s\left(\ln f^*\left(\frac{2a+b}{3}\right)\right)^q+t^s\left(\ln\left(f^*(b)\right)\right)^q\right)dt\right)^{\frac{1}{q}}\right\}$$

$$=\exp\left\{\frac{b-a}{9}\left(\frac{2^p}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\left(\ln f^*(a)\right)^q+\left(\ln f^*\left(\frac{2a+b}{3}\right)\right)^q\right)^{\frac{1}{q}}\right\}$$

$$\times\exp\left\{\frac{b-a}{9}\left(\frac{5^{p+1}+3^{p+1}}{8(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\left(\ln f^*\left(\frac{2a+b}{3}\right)\right)^q+\left(\ln f^*(b)\right)^q\right)^{\frac{1}{q}}\right\}$$

$$\leq \exp\left\{\frac{b-a}{9}\left(\frac{2^p}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\ln f^*(a)+\ln f^*\left(\frac{2a+b}{3}\right)\right)\right\}$$

$$\times\exp\left\{\frac{b-a}{9}\left(\frac{5^{p+1}+3^{p+1}}{8(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\ln f^*\left(\frac{2a+b}{3}\right)+\ln f^*(b)\right)\right\}$$

$$=\left(f^*(a)^2\left(f^*\left(\frac{2a+b}{3}\right)\right)^{2+\left(\frac{5^{p+1}+3^{p+1}}{8}\right)^{\frac{1}{p}}}\left(f^*(b)\right)^{\left(\frac{5^{p+1}+3^{p+1}}{8}\right)^{\frac{1}{p}}}\right)^{\frac{b-a}{9}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}},$$

where we have used the fact that $A^q+B^q\leq (A+B)^q$ for $A\geq 0, B\geq 0$ with $q\geq 1$. The proof is finished. □

Corollary 3.7. *In Theorem 3.6, taking $s=1$, we obtain*

$$\left|\left(f\left(\frac{2a+b}{3}\right)\right)^{\frac{3}{2}}\left(f(b)\right)^{\frac{1}{2}}\left(\int_a^b f(u)^{du}\right)^{\frac{2}{a-b}}\right|$$

$$\leq\left(f^*(a)^2\left(f^*\left(\frac{2a+b}{3}\right)\right)^{2+\left(\frac{5^{p+1}+3^{p+1}}{8}\right)^{\frac{1}{p}}}\left(f^*(b)\right)^{\left(\frac{5^{p+1}+3^{p+1}}{8}\right)^{\frac{1}{p}}}\right)^{\frac{b-a}{9}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}}.$$

Corollary 3.8. *In Corollary 3.7, if the function $f^*\leq M$, we obtain*

$$\left|\left(f\left(\frac{2a+b}{3}\right)\right)^{\frac{3}{2}}\left(f(b)\right)^{\frac{1}{2}}\left(\int_a^b f(u)^{du}\right)^{\frac{2}{a-b}}\right|\leq M^{\frac{b-a}{9}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(2+\left(\frac{5^{p+1}+3^{p+1}}{8}\right)^{\frac{1}{p}}\right)2^{1-\frac{1}{q}}}.$$

Theorem 3.9. *Under the assumptions of Lemma 3.1. If f is incresing on $[a,b]$ and $(\ln f^*)^q$ is s-convex on $[a,b]$ for $q>1$, then we have*

$$\left|\left(f\left(\frac{2a+b}{3}\right)\right)^{\frac{3}{2}}(f(b))^{\frac{1}{2}}\left(\int_a^b f(u)^{du}\right)^{\frac{2}{a-b}}\right|$$
$$\leq\left((f^*(a))^{\left(\frac{2}{(s+1)(s+2)}\right)^{\frac{1}{q}}}\left(f^*\left(\frac{2a+b}{3}\right)\right)^{\left(\frac{2}{s+2}\right)^{\frac{1}{q}}+\left(\frac{17}{8}\right)^{1-\frac{1}{q}}\left(\frac{5s+2}{(s+1)(s+2)}+\frac{6}{(s+1)(s+2)}\left(\frac{3}{8}\right)^{s+1}\right)^{\frac{1}{q}}}\right.$$
$$\left.(f^*(b))^{\left(\frac{17}{8}\right)^{1-\frac{1}{q}}\left(\frac{3s-2}{(s+1)(s+2)}+\frac{10}{(s+1)(s+2)}\left(\frac{5}{8}\right)^{s+1}\right)^{\frac{1}{q}}}\right)^{\frac{b-a}{9}}.$$

Proof. Building upon Lemma 3.1, s-convexity of $(\ln f^*)^q$ among with power mean inequality, we deduce that:

$$\left|\left(f\left(\frac{2a+b}{3}\right)\right)^{\frac{3}{2}}(f(b))^{\frac{1}{2}}\left(\int_a^b f(u)^{du}\right)^{\frac{2}{a-b}}\right|$$
$$\leq\exp\left\{\frac{b-a}{9}\int_0^1\left|\ln\left(f^*\left((1-t)a+t\frac{2a+b}{3}\right)\right)^{2t}\right|dt\right\}$$
$$\times\exp\left\{\frac{b-a}{9}\int_0^1\left|\ln\left(\left(f^*\left((1-t)\frac{2a+b}{3}+tb\right)\right)^{8t-5}\right)\right|dt\right\}$$
$$=\exp\left\{\frac{b-a}{9}\int_0^1 2t\left|\ln f^*\left((1-t)a+t\frac{2a+b}{3}\right)\right|dt\right\}$$
$$\times\exp\left\{\frac{b-a}{9}\int_0^1|8t-5|\left|\ln f^*\left((1-t)\frac{2a+b}{3}+tb\right)\right|dt\right\}$$
$$\leq\exp\left\{\frac{b-a}{9}\left(\int_0^1 2tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 2t\left|\ln f^*\left((1-t)a+t\frac{2a+b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$
$$\times\exp\left\{\frac{b-a}{9}\left(\int_0^1|8t-5|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|8t-5|\left|\ln f^*\left((1-t)\frac{2a+b}{3}+tb\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$
$$\leq\exp\left\{\frac{b-a}{9}\left(\int_0^1 2t\left((1-t)^s(\ln f^*(a))^q+t^s\left(\ln f^*\left(\frac{2a+b}{3}\right)\right)^q\right)dt\right)^{\frac{1}{q}}\right\}$$
$$\times\exp\left\{\frac{b-a}{9}\left(\frac{17}{8}\right)^{1-\frac{1}{q}}\left(\int_0^1|8t-5|\left((1-t)^s\left(\ln f^*\left(\frac{2a+b}{3}\right)\right)^q+t^s(\ln(f^*(b)))^q\right)dt\right)^{\frac{1}{q}}\right\}$$
$$=\exp\left\{\frac{b-a}{9}\left(\frac{2}{(s+1)(s+2)}(\ln f^*(a))^q+\frac{2}{s+2}\left(\ln f^*\left(\frac{2a+b}{3}\right)\right)^q\right)^{\frac{1}{q}}\right\}$$
$$\times\exp\left\{\frac{b-a}{9}\left(\frac{17}{8}\right)^{1-\frac{1}{q}}\left(\left(\frac{5s+2}{(s+1)(s+2)}+\frac{6}{(s+1)(s+2)}\left(\frac{3}{8}\right)^{s+1}\right)\left(\ln f^*\left(\frac{2a+b}{3}\right)\right)^q\right.\right.$$

$$
\begin{aligned}
&+\left(\tfrac{3s-2}{(s+1)(s+2)}+\tfrac{10}{(s+1)(s+2)}\left(\tfrac{5}{8}\right)^{s+1}\right)\left(\ln f^{*}(b)\right)^{q}\Big)^{\frac{1}{q}}\Big\} \\
&\leq \exp\left\{\tfrac{b-a}{9}\left(\left(\tfrac{2}{(s+1)(s+2)}\right)^{\frac{1}{q}}\ln f^{*}(a)+\left(\tfrac{2}{s+2}\right)^{\frac{1}{q}}\ln f^{*}\left(\tfrac{2a+b}{3}\right)\right)\right\} \\
&\quad\times\exp\left\{\tfrac{b-a}{9}\left(\tfrac{17}{8}\right)^{1-\frac{1}{q}}\left(\left(\tfrac{5s+2}{(s+1)(s+2)}+\tfrac{6}{(s+1)(s+2)}\left(\tfrac{3}{8}\right)^{s+1}\right)^{\frac{1}{q}}\ln f^{*}\left(\tfrac{2a+b}{3}\right)\right.\right. \\
&\quad\left.\left.+\left(\tfrac{3s-2}{(s+1)(s+2)}+\tfrac{10}{(s+1)(s+2)}\left(\tfrac{5}{8}\right)^{s+1}\right)^{\frac{1}{q}}\ln f^{*}(b)\right)\right\} \\
&=\left(\left(f^{*}(a)\right)^{\left(\frac{2}{(s+1)(s+2)}\right)^{\frac{1}{q}}}\left(f^{*}\left(\tfrac{2a+b}{3}\right)\right)^{\left(\frac{2}{s+2}\right)^{\frac{1}{q}}+\left(\frac{17}{8}\right)^{1-\frac{1}{q}}\left(\frac{5s+2}{(s+1)(s+2)}+\frac{6}{(s+1)(s+2)}\left(\frac{3}{8}\right)^{s+1}\right)^{\frac{1}{q}}}\right. \\
&\quad\left.\left(f^{*}(b)\right)^{\left(\frac{17}{8}\right)^{1-\frac{1}{q}}\left(\frac{3s-2}{(s+1)(s+2)}+\frac{10}{(s+1)(s+2)}\left(\frac{5}{8}\right)^{s+1}\right)^{\frac{1}{q}}}\right)^{\frac{b-a}{9}},
\end{aligned}
$$

where we have used (4)-(7) and the fact that $A^{q}+B^{q}\leq(A+B)^{q}$ for $A\geq 0, B\geq 0$ with $q\geq 1$. The proof is finished. □

Corollary 3.10. *In Theorem 3.9, taking $s=1$, we obtain*

$$
\left|\left(f\left(\tfrac{2a+b}{3}\right)\right)^{\frac{3}{2}}\left(f(b)\right)^{\frac{1}{2}}\left(\int_{a}^{b}f(u)^{du}\right)^{\frac{2}{a-b}}\right|
$$
$$
\leq\left(\left(f^{*}(a)\right)\left(f^{*}\left(\tfrac{2a+b}{3}\right)\right)^{2^{\frac{1}{q}}+\frac{17}{8}\left(\frac{251}{136}\right)^{\frac{1}{q}}}\left(f^{*}(b)\right)^{\frac{17}{8}\left(\frac{157}{136}\right)^{\frac{1}{q}}}\right)^{\left(\frac{1}{3}\right)^{\frac{1}{q}}\frac{b-a}{9}}.
$$

4. Applications to special means

We will now examine various means for arbitrary real numbers a, b, c:
The arithmetic mean: $A(a,b)=\frac{a+b}{2}$ and $A(a,b,c)=\frac{a+b+c}{3}$.
The geometric mean: $G(a,b)=\sqrt{ab}$, $ab\geq 0$.
The harmonic mean: $H(a,b)=\frac{2}{\frac{1}{a}+\frac{1}{b}}$ and $H(a,b,c)=\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$, $a,b,c>0$.
The logarithmic mean: $L(a,b)=\frac{b-a}{\ln b-\ln a}$, $a,b>0$ and $a\neq b$.

The p-logarithmic mean: $L_{p}(a,b)=\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a,b>0, a\neq b$ and $p\in\mathbb{R}\setminus\{-1,0\}$.

Proposition 4.1. *Consider a and b in $\mathbb{R}$ such that $0<a<b$. In this case, we can establish the following:*

$$
\exp\left\{\frac{3A^{2}(a,a,b)+b^{2}}{2}-2L_{2}^{2}(a,b)\right\}\leq\exp\left\{\frac{192A(a,b,b)+379A(a,a,b)+29b}{864}(b-a)\right\}.
$$

Proof. The conclusion can be deduced from Corollary 3.3, which is applied to the function $f(t)=\exp\{t^{2}\}$ yielding $f^{*}(t)=\exp\{2t\}$. Furthermore, the relation $\left(\int_{a}^{b}f(u)^{du}\right)^{\frac{1}{a-b}}=\exp\{-L_{2}^{2}(a,b)\}$ becomes evident in this context. □

Proposition 4.2. *Under the assumptions of Proposition 4.1, we have*

$$\exp\left\{\frac{3}{2}H\left(a,b,b\right)+\frac{1}{2b}-2G^{2}\left(a,b\right)L^{-1}\left(a,b\right)\right\}$$
$$\leq\left(\exp\left\{-2A\left(b^{2},H^{2}\left(a,b,b\right)\right)-\sqrt{19}A\left(H^{2}\left(a,b,b\right),a^{2}\right)\right\}\right)^{\frac{(b-a)\sqrt{6}}{27ab}}.$$

Proof. The conclusion can be inferred from Corollary 3.7 with $p=q=2$, applied to the function $f(t)=\exp\left\{\frac{1}{t}\right\}$ over the interval $\left[\frac{1}{b},\frac{1}{a}\right]$, where $f^{*}(t)=\exp\left\{-\frac{1}{t^{2}}\right\}$ and $\left(\int_{\frac{1}{b}}^{\frac{1}{a}}f(u)^{du}\right)^{\frac{1}{\frac{1}{b}-\frac{1}{a}}}=\exp\left\{-G^{2}(a,b)L^{-1}(a,b)\right\}$. □

5. Conclusion

In conclusion, this study has introduced a groundbreaking identity for multiplicative differentiable functions, providing a solid foundation for the establishment of a series of 2-point right-Radau-type inequalities tailored for multiplicative s-convex functions.

To summarize, the results presented herein deepen our understanding of multiplicative integral inequalities and offer potential avenues for further research and practical applications. As the mathematical community continues to explore these concepts, the findings from this study are likely to contribute significantly to the ongoing developments in the field.

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