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DOMINATION IN BIPOLAR INTUITIONISTIC FUZZY GRAPHS

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Abstract. The intention of this paper is to acquaint domination, total domination on bipolar intuitionistic fuzzy graphs. Subsequently for bipolar intuitionistic fuzzy graphs the domination number and the total domination number are defined. Consequently we proved necessary and sufficient condition for a d-set to be minimal d-set, bounds for domination number and equality conditions for domination number and order.

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1. Introduction

In graph theory one of the vital concepts is domination and its growth is grand. Dominating sets in graph theory were introduced by Berge, Ore in 1962. In 1977 Cockayne, Hedetniemi introduced a domination number and independent domination number. In real time problems, eventhough the graphs are the best mathematical models for representing the relations between the nodes and their connections, the vagueness in the relationship cannot be expressed. After a long time criticism on fuzzy set, the theory of fuzzy mathematics boomed a research area in the last four decades. A. Somasundram and S. Somasundram [6] defined domination in fuzzy graph using effective edges in 1988.
Even some real time problems need much more generalization of the fuzzy set-theory. For that reason intuitionistic fuzzy sets, bipolar fuzzy sets and interval valued fuzzy sets are introduced. Recently many researchers extended and merged different fuzzy sets to apply for the real time applications. A few in that list are interval valued intuitionistic fuzzy, neutrosophic sets, picture fuzzy sets

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and bipolar intuitionistic fuzzy set(BIFS).
The combination of intuitionistic fuzzy sets and graph theory is a new concept named as intuitionistic fuzzy graph introduced by R. Parvathi and M. G. Karunambigai [3]. The intuitionistic fuzzy graph can be used in a model where membership and non-membership value are very important. The application area of intuitionistic fuzzy graphs are chemistry, economics, computer sciences, engineering, medicine, decision making problems etc. In 2010 R. Parvathi [4] introduced the notion of domination in intuitionistic fuzzy graphs.
The combination of bipolar fuzzy sets and graph theory is a new concept named as bipolar fuzzy graph introduced by Akram [1] in the year 2011. The bipolar fuzzy graph can be used in a problem if there are two similar but opposite type of qualitative variables like success and failure, safe and danger etc. The application area of bipolar fuzzy graphs is medicine, management, environment, computer, cognitive sciences, economics etc. M. G. Karunambigai [2] defined the domination in bipolar fuzzy graph(BFG) in 2013. Several domination parameters for BFG are introduced by V. Mohanaselvi and S. Sivamani (see [5]) from 2014.
By combining bipolar and intuitionistic fuzzy sets, in 2019 Sonia Mandal and Madhumanagal Pal [7] defined the bipolar intuitionistic fuzzy graph, best fits for some real time applications. By the motivation of above works, we initiate the domination in bipolar intuitionistic fuzzy graphs.

2. Preliminaries

Throughout this work a graph will represent a graph without loops.

Definition 2.1. A bipolar intuitionistic fuzzy graph is defined as $G = (A, B)$ where V is an underlying vertex set, $A = (\chi_1^+, \chi_1^-, \eta_1^+, \eta_1^-)$ is a bipolar intuitionistic fuzzy set on V, $B(\chi_2^+, \chi_2^-, \eta_2^+, \eta_2^-)$ is a bipolar intuitionistic fuzzy set in $E \subseteq V \times V \ni \chi_2^+(\alpha, \beta) \leq (\chi_1^+(\alpha) \wedge \chi_1^+(\beta))$, $\chi_2^-(\alpha, \beta) \geq (\chi_1^-(\alpha) \vee \chi_1^-(\beta))$, $\eta_2^+(\alpha, \beta) \leq (\eta_1^+(\alpha) \vee \eta_1^+(\beta))$ and $\eta_2^-(\alpha, \beta) \geq (\eta_1^-(\alpha) \wedge \eta_1^-(\beta)) \quad \forall\, (\alpha, \beta) \in E$.

The below definitions are initiated in this paper.

Definition 2.2. For a BIFG, G order is defined by

$$\mu = \sum_{\alpha \in V} \left\{ \frac{1 + \chi_1^+(\alpha) + \chi_1^-(\alpha) + \eta_1^+(\alpha) + \eta_1^-(\alpha)}{2} \right\}.$$

For a BIFG, G size is defined by

$$\lambda = \sum_{(\alpha,\beta) \in E} \left\{ \frac{1 + \chi_2^+(\alpha, \beta) + \chi_2^-(\alpha, \beta) + \eta_2^+(\alpha, \beta) + \eta_2^-(\alpha, \beta)}{2} \right\}.$$

Definition 2.3. In a BIFG, G let $W \subseteq V$. The cardinality of W is defined by

$$\sum_{\alpha \in W} \left\{ \frac{1 + \chi_1^+(\alpha) + \chi_1^-(\alpha) + \eta_1^+(\alpha) + \eta_1^-(\alpha)}{2} \right\}.$$

In a BIFG, G let $F \subseteq E$. The cardinality of F is defined by

$$\sum_{(\alpha,\beta)\in F} \left\{ \frac{1+\chi_2^+(\alpha,\beta)+\chi_2^-(\alpha,\beta)+\eta_2^+(\alpha,\beta)+\eta_2^-(\alpha,\beta)}{2} \right\}.$$

In a BIFG, G let $\alpha \in V$. The cardinality of α is defined by

$$|\alpha| = \left\{ \frac{1+\chi_1^+(\alpha)+\chi_1^-(\alpha)+\eta_1^+(\alpha)+\eta_1^-(\alpha)}{2} \right\}.$$

In a BIFG, G let $e = (\alpha,\beta) \in E$. The cardinality of e is defined by

$$|e| = \left\{ \frac{1+\chi_2^+(\alpha,\beta)+\chi_2^-(\alpha,\beta)+\eta_2^+(\alpha,\beta)+\eta_2^-(\alpha,\beta)}{2} \right\}.$$

Example 2.1.

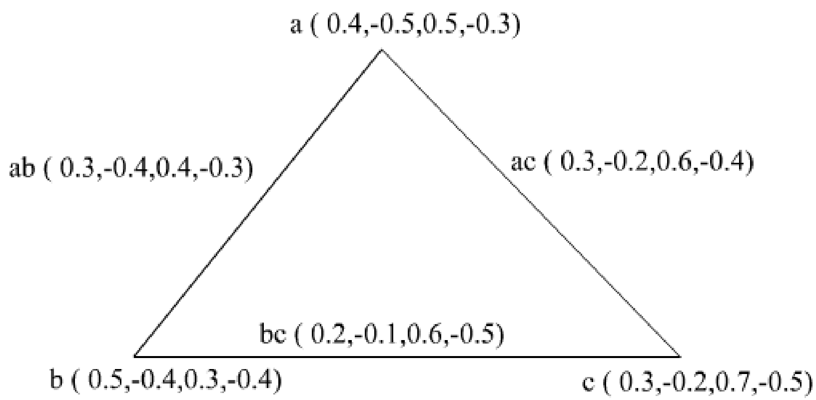

FIGURE 1

For the BIFG in figure **??**

Order=$\mu = \frac{1+0.4-0.5+0.5-0.3}{2} + \frac{1+0.5-0.4+0.3-0.4}{2} + \frac{1+0.3-0.2+0.7-0.5}{2} = 1.7$

Size=$\lambda = \frac{1+0.3-0.4+0.4-0.3}{2} + \frac{1+0.3-0.2+0.6-0.4}{2} + \frac{1+0.2-0.1+0.6-0.5}{2} = 1.75$

Let $W = \{a, b\}$, then $|W| = \frac{1+0.4-0.5+0.5-0.3}{2} + \frac{1+0.5-0.4+0.3-0.4}{2} = 1.05$

Let $F = \{ab, ac\}$, then $|F| = \frac{1+0.3-0.4+0.4-0.3}{2} + \frac{1+0.3-0.2+0.6-0.4}{2} = 1.15$

Let $\alpha = a$, then $|\alpha| = \frac{1+0.4-0.5+0.5-0.3}{2} = 0.55$

Let $e = bc$, then $|e| = \frac{1+0.2-0.1+0.6-0.5}{2} = 0.6$

Definition 2.4. In BIFG, G for an edge $e=(\alpha,\beta)$, if $\chi_2^+(\alpha,\beta)=(\chi_1^+(\alpha)\wedge\chi_1^+(\beta))$, $\chi_2^-(\alpha,\beta)=(\chi_1^-(\alpha)\vee\chi_1^-(\beta))$, $\eta_2^+(\alpha,\beta)=(\eta_1^+(\alpha)\vee\eta_1^+(\beta))$ and $\eta_2^-(\alpha,\beta)=(\eta_1^-(\alpha)\wedge\eta_1^-(\beta))$, for all $\alpha,\beta\in V$, then $e=(\alpha,\beta)$ is called an effective edge.

Definition 2.5. Let $\alpha\in V, e\in E$ in G. A neighborhood of α is defined as $N(\alpha)=\{\beta:\beta\in V \;\&\; (\alpha,\beta)$ is an effective edge in $G\}$ and
$N(e)=\{$All effective edges adjacent to $e\}$.

Definition 2.6. Let $\alpha\in V$. It is called an isolated vertex if $\chi_2^+(\alpha,\beta)=0$, $\chi_2^-(\alpha,\beta)=0$, $\eta_2^+(\alpha,\beta)=0$, and $\eta_2^-(\alpha,\beta)=0$.

Definition 2.7. Let $A=\{\alpha\in V$ such that (α,β) is an effective edge$\}$. The neighborhood degree on $\alpha\in V$ is defined by

$$D_1(\alpha)=\sum_{\alpha\in N(\alpha)}\left\{\frac{1+\chi_1^+(\alpha)+\chi_1^-(\alpha)+\eta_1^+(\alpha)+\eta_1^-(\alpha)}{2}\right\}.$$

The effective degree on $\alpha\in V$ is

$$D_2(\alpha)=\sum_{(\alpha,\beta)\in A}\left\{\frac{1+\chi_2^+(\alpha,\beta)+\chi_2^-(\alpha,\beta)+\eta_2^+(\alpha,\beta)+\eta_2^-(\alpha,\beta)}{2}\right\}.$$

The least $D_1(\alpha)$ of G is $\delta_1=\wedge\{D_1(\alpha)|\alpha\in V\}$, the greatest $D_2(\alpha)$ of G is $\Delta_1=\vee\{D_1(\alpha)|\alpha\in V\}$, the least effective degree on G is $\delta_2=\wedge\{D_2(\alpha)|\alpha\in V\}$, the greatest effective degree on G is $\Delta_2=\vee\{D_2(\alpha)|\alpha\in V\}$.

Example 2.2.

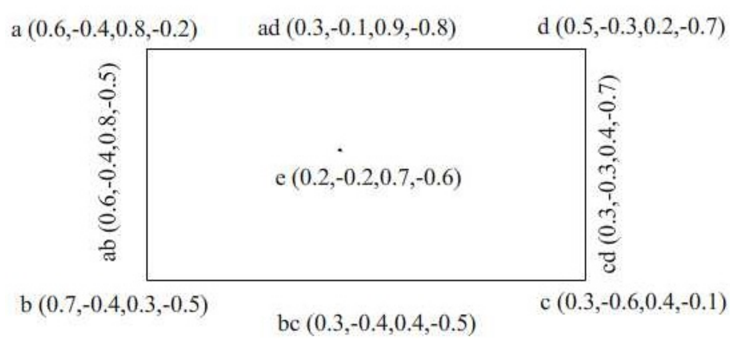

FIGURE 2

Here

- The edges ab, dc are effective edges
- $N(a)=\{b\}, N(b)=\{a,c\}, N(c)=\{b,d\}, N(d)=\{c\}$. Then $D_1(a)=0.5, D_1(b)=1.4, D_1(c)=0.9, D_1(d)=0.5$ with $\delta_1=0.5$ and $\Delta_1=1.4$

- $N(ab) = \{bc\}, N(bc) = \{ab, cd\}, N(cd) = \{bc\}$. Then $D_2(ab) = 0.4, D_2(bc) = 1.1, D_2(cd) = 0.4$ with $\delta_2 = 0.4$ and $\Delta_2 = 1.1$.
- The vertex e is an isolated vertex.

Definition 2.8. A BIFG, G is said to be complete if $\forall\, \alpha, \beta \in V$, $\chi_2^+(\alpha,\beta) = (\chi_1^+(\alpha) \wedge \chi_1^+(\beta))$, $\chi_2^-(\alpha,\beta) = (\chi_1^-(\alpha) \vee \chi_1^-(\beta))$, $\eta_2^+(\alpha,\beta) = (\eta_1^+(\alpha) \vee \eta_1^+(\beta))$ and $\eta_2^-(\alpha,\beta) = (\eta_1^-(\alpha) \wedge \eta_1^-(\beta))$.

Example 2.3.

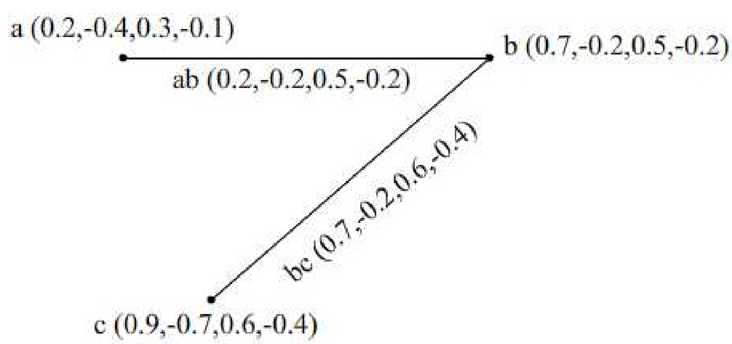

FIGURE 3

Since all the edges are effective edges G is complete.

Definition 2.9. In a BIFG, G consider $\alpha, \beta \in V$. If there exists an effective edge between $\alpha, \beta \in G$ then α is dominating β. A vertex set $D \subseteq V$ is said to be a dominating set (d-set) in G if $\forall\, \alpha \in V \backslash D$ there exists $\beta \in D$ such that α dominates β.

Definition 2.10. If D is a d-set of G, any $D_1 \subseteq D$ is not a d-set of G then D is called minimal d-set. The least cardinality minimal d-set is called domination number, denoted by γ.

Example 2.4. In Figure 3.4 the dominating set is $D = \{v_3, v_5\}$. Because every edge in D dominates an element in $V \backslash D = \{v_1, v_2, v_4\}$. Now by the definition of domination number, $\gamma = \left[\dfrac{1+0.2-0.4+0.3-0.5}{2} + \dfrac{1+0.6-0.6+0.3-0.3}{2}\right] = 0.8$.

Definition 2.11. The complement of a BIFG, $G = (V, E)$ is a BIFG, $\overline{G} = (\overline{V}, \overline{E})$, where

(1) $\overline{V} = V$

(2) For all $\alpha \in V, \overline{\chi_1^+(\alpha)} = \chi_1^+(\alpha)$, $\overline{\chi_1^-(\alpha)} = \chi_1^-(\alpha)$, $\overline{\eta_1^+(\alpha)} = \eta_1^+(\alpha)$, and $\overline{\eta_1^-(\alpha)} = \eta_1^-(\alpha)$

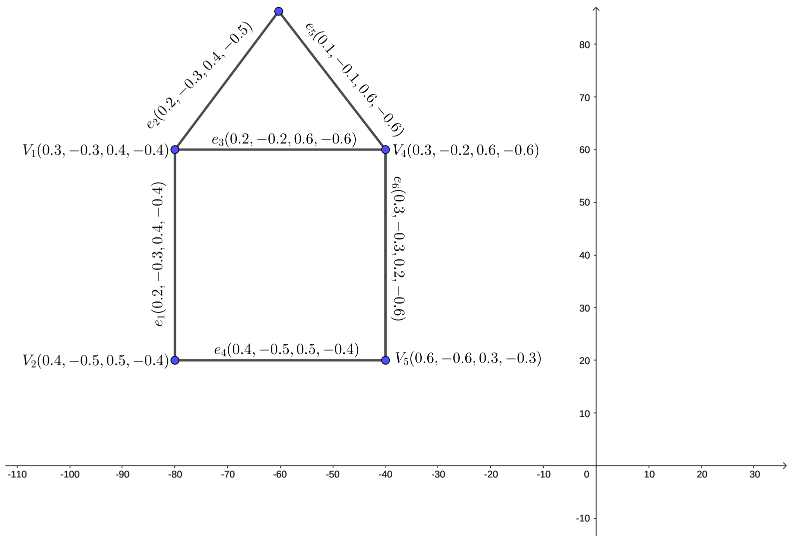

FIGURE 4

(3) For all $e = (\alpha, \beta) \in E$,
$\overline{\chi_2^+(\alpha,\beta)} = (\chi_1^+(\alpha) \wedge \chi_1^+(\beta)) - \chi_2^+(\alpha,\beta)$, $\overline{\chi_2^-(\alpha,\beta)} = (\chi_1^-(\alpha) \vee \chi_1^-(\beta)) - \chi_2^-(\alpha,\beta)$, $\overline{\eta_2^+(\alpha,\beta)} = \eta_2^+(\alpha,\beta) - (\eta_1^+(\alpha) \vee \eta_1^+(\beta))$ and $\overline{\eta_2^-(\alpha,\beta)} = \eta_2^-(\alpha,\beta) - (\eta_1^-(\alpha) \wedge \eta_1^-(\beta))$.

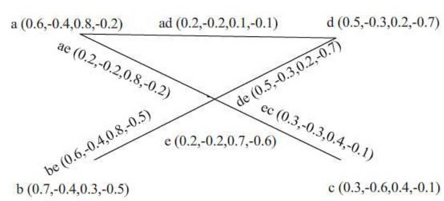

FIGURE 5

The above graph is the complement of BIFG in Figure **??**.

Remark. Assume G be a BIFG.

(1) Domination in G is symmetric on V.
(2) $N(\alpha) = \{\beta \in V\}$ such that α, β are dominating each other.
(3) If no effective edges are in G, then clearly V is the only d-set in G and $\gamma = p$.
(4) If $N(\alpha) = \phi$, then $\alpha \in V$ is contained in every d-set of G.
(5) If G is complete BIFG, then $\gamma = \wedge\{|\alpha|, \alpha \in V\}$ and $D = \{\alpha\}$.

3. Main Results

Theorem 3.1. *In a BIFG, G a d-set D is a minimal d-set if and only if* $\forall\, \alpha \in D$ *at least one of the below constraints hold.*

(1) $D \cap N(\alpha) = \phi$

(2) $\exists\, \beta \in V - D$ *for which* $D \cap N(\beta) = \{\alpha\}$.

Proof. If a minimal d-set of G is D and $\alpha \in D$, then $D - \{\alpha\}$ will not be a d-set of G. So $\exists\, \beta \in V - (D - \{\alpha\})$ such that it is dominated by no element in $D - \{\alpha\}$. If $\alpha = \beta$, then $N(\alpha) \cup D = \phi$. Otherwise β is dominated by D not by $D - \{\alpha\}$. Then β is the single vertex adjacent to α in D. Hence $D \cap N(\beta) = \{\alpha\}$. The converse is evident.

The above constraints (1) and (2) are necessary for a minimal $d-$set. In example 2.2, the $d-$set is $\{a, c, d, e\}$. Let $\alpha = d$, then $N(d) = \{c\}$ with $D \cup N(\alpha) \neq \phi$. Hence $\{a, c, d, e\}$ is not a minimal $d - set$.

Theorem 3.2. *Assume that G is a BIFG without isolated vertices. Then for all minimal d-set D,* $V \backslash D$ *is also a d-set.*

Proof. Assume that D is a minimal d-set and $\alpha \in D$. As no isolated vertex is in G, $\exists$ vertex $\beta \in N(\alpha)$. Hence by Theorem 3.1, it results that $\alpha \in V - D$. Hence $\forall\, v \in D$, an element in $V \backslash D$ dominates D. Hence $V \backslash D$ is a d-set.

In this theorem, our assumption is necessary. In example 2.2 the $d-$ set is $\{a, d, e\}$, then $V \backslash D = \{b, c\}$ is not a $d-$ set. Here, G is a BIFG with isolated vertex.

Theorem 3.3. *Assume a domination number, an order for G respectively* γ, μ *and a domination number, an order for* $\overline{G}$ *respectively* γ', μ'. *Then for each G,* $\gamma + \gamma' \leq \mu + \mu'$. *If no effective edges are in G then equality holds.*

Proof. The inequality immediately follows. The domination number $\gamma = \mu$ if and only if $\chi_2^+(\alpha, \beta) < (\chi_1^+(\alpha) \wedge \chi_1^+(\beta))$, $\chi_2^-(\alpha, \beta) > (\chi_1^-(\alpha) \vee \chi_1^-(\beta))$, $\eta_2^+(\alpha, \beta) < (\eta_1^+(\alpha) \vee \eta_1^+(\beta))$ and $\eta_2^-(\alpha, \beta) < (\eta_1^-(\alpha) \wedge \eta_1^-(\beta))$, $\forall\ \alpha, \beta \in V$. Also, $\gamma' = \mu'$ if and only if $(\chi_1^+(\alpha) \wedge \chi_1^+(\beta)) - \chi_2^+(\alpha, \beta) < (\chi_1^+(\alpha) \wedge \chi_1^+(\beta))$, $(\chi_1^-(\alpha) \vee \chi_1^-(\beta)) - \chi_2^-(\alpha, \beta) > (\chi_1^-(\alpha) \vee \chi_1^-(\beta))$, $(\eta_1^+(\alpha) \vee \eta_1^+(\beta)) - \eta_2^+(\alpha, \beta) < (\eta_1^+(\alpha) \vee \eta_1^+(\beta))$, and $(\eta_1^+ - (\alpha) \wedge \eta_1^-(\beta)) - \eta_2^-(\alpha, \beta) > (\eta_1^-(\alpha) \wedge \eta_1^-(\beta))$, $\forall\, \alpha, \beta \in V$. Hence, $0 < \chi_2^+(\alpha, \beta)$, $0 < \eta_2^+(\alpha, \beta)$, $\chi_2^-(\alpha, \beta) < 0$, $\eta_2^-(\alpha, \beta) < 0$. Combining these, the proof is complete.

Corollary.

(1) If no isolated vertices are in G it is true that $\gamma \leq \dfrac{\mu}{2}$.

(2) If no isolated vertices are in G and $\overline{G}$ then $\gamma + \gamma'$ if and only if $\gamma = \gamma' = \dfrac{\mu}{2}$.

Theorem 3.4. *For any BIFG, G it is true that* $\gamma \leq p - \Delta_1$.

Proof. Let a neighborhood degree $\alpha \in D$ is Δ_1, $V \backslash N(\alpha)$ is a d-set on G. Hence, $\gamma \leq |V - N(\alpha)| = \mu - \Delta_1$.

Theorem 3.5. *For any BIFG, G $\gamma = \mu$ and only if $\alpha \in V$ on G has a different neighbor.*

Proof. If in G neighbor of each vertex is different, then clearly D has a different td-set on G. Hence, $\gamma = \mu$. On the contrary, assume that $\gamma = \mu$. Suppose for α there are two neighbor β_1 and β_2, clearly $V - \{\alpha\}$ is $td-$set on G. Thus $\gamma = \mu$ contradict to our assumption. Hence each vertex on G has a different neighbors.

Theorem 3.6. *If $\gamma = \mu$ then the number of elements in the vertex set in BIFG, G is a multiple of two.*

Proof. Assume G be a BIFG and has $2n+1$ number of vertices. Since G has no loops, for each vertex v_1 there exists a unique neighboring vertex v_2 distinct from v_1. Thus there is an n number of distinct pairs of vertices such that in each pair one vertex is neighbor of the other vertex. Finally there is a single vertex which does not have a unique neighbor. This leads to a contradiction. Thus G must have an even number of vertices.

4. Conclusion

In the vital concepts list in fuzzy graph theory, domination is in top and its growth is spectacular. Here domination, total domination and independent sets are introduced. Some of the findings are also presented.

Future Work. Studies of various other dominations in BIFG will be focused in future.

Conflicts of interest : The authors declare no conflict of interest.

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