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COINCIDENCE POINT RESULTS UNDER GERAGHTY-TYPE CONTRACTION

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Abstract. The main aim of this research article is to establish some coincidence point theorem for G-non-decreasing mappings under Geraghty-type contraction on partially ordered metric spaces. Furthermore, we derive some multidimensional results with the help of our unidimensional results. Our results improve and generalize various well-known results in the literature.

1. Introduction

As the Banach contraction principle is a powerful tool for solving many problems in applied mathematics and sciences, it has been improved and extended in many ways. In particular, Geraghty [9] proved in 1973, an interesting generalization of Banach contraction principle which had a lot of applications.

The concept of multidimensional fixed/coincidence point was introduced by Roldan et al. in [18] which is an extension of Berzig and Samet's notion given in [2]. For more details, one can consult [1, 3-8, 10-13, 15-23].

In this research paper, we establish some coincidence point theorem for G-non-decreasing mappings under Geraghty-type contraction on partially ordered metric spaces. With the help of our unidimensional results, we obtain some multidimensional results. The results we obtain generalize, extend and improve several classical and very recent related results in the literature of metric spaces.

2. Preliminaries

If X is a non-empty set, then we denote $X \times X \times ... \times X$ (n times) by X^n, where $n \in \mathbb{N}$ with $n \geq 2$. Let $\{A, B\}$ be a partition of the set $\Lambda_n = \{1, 2, ..., n\}$, that is, A

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and B are non-empty subsets of Λ_n such that $A \cup B = \Lambda_n$ and $A \cap B = \emptyset$. Denote

$$\Omega_{A,\ B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A,\ \sigma(B) \subseteq B\}$$

and

$$\Omega'_{A,\ B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B,\ \sigma(B) \subseteq A\}.$$

Hence, let σ_1, σ_2, ..., σ_n be n mappings from Λ_n into itself and let Υ be the $n-$tuple $(\sigma_1,\ \sigma_2,\ ...,\ \sigma_n)$.

Let $F : X^n \to X$ and $G : X \to X$ be two mappings. For simplicity, we denote $G(\varepsilon)$ by $G\varepsilon$ where $\varepsilon \in X$.

A partial order $\preceq$ on X can be extended to a partial order $\sqsubseteq$ on X^n in the following way. If $(X, \preceq)$ be a partially ordered space, ε, $\delta \in X$ and $i \in \Lambda_n$, we will use the following notations:

$$\varepsilon \preceq_i \delta \Rightarrow \begin{cases} \varepsilon \preceq \delta, & \text{if } i \in A, \\ \varepsilon \succeq \delta, & \text{if } i \in B. \end{cases} \tag{2.1}$$

Consider on the product space X^n the following partial order: for $Y = (\varepsilon_1,\ \varepsilon_2,\ ...,\ \varepsilon_i,\ ...,\ \varepsilon_n)$, $V = (\delta_1,\ \delta_2,\ ...,\ \delta_i,\ ...,\ \delta_n) \in X^n$,

$$Y \sqsubseteq V \Leftrightarrow \varepsilon_i \preceq_i \delta_i. \tag{2.2}$$

Note that $\sqsubseteq$ depends on A and B. We say that two points Y and V are comparable, if $Y \sqsubseteq V$ or $V \sqsubseteq Y$. Obviously, $(X^n, \sqsubseteq)$ is a partially ordered set.

Definition 2.1 ([15, 18, 20])**.** A point $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) \in X^n$ is called a $\Upsilon-$*coincidence point* of the mappings $F : X^n \to X$ and $G : X \to X$ if

$$F(\varepsilon_{\sigma_i(1)},\ \varepsilon_{\sigma_i(2)},\ ...,\ \varepsilon_{\sigma_i(n)}) = G\varepsilon_i,\ \text{for all } i \in \Lambda_n.$$

If G is the identity mapping on ε, then $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) \in X^n$ is called a $\Upsilon-$*fixed point* of the mapping F.

If we represent a mapping $\sigma : \Lambda_n \to \Lambda_n$ throughout its ordered image, that is, $\sigma = (\sigma(1),\ \sigma\ (2),\ ...,\ \sigma\ (n))$, then

(i) Gnana-Bhaskar and Lakshmikantham's coupled fixed points occur when $n = 2$, $\sigma_1 = (1,\ 2)$ and $\sigma_2 = (2,\ 1)$,

(ii) Berinde and Borcut's tripled fixed points are associated with $n = 3$, $\sigma_1 = (1,\ 2,\ 3)$, $\sigma_2 = (2,\ 1,\ 2)$ and $\sigma_3 = (3,\ 2,\ 1)$,

(iii) Karapinar's quadruple fixed points are considered when $n = 4$, $\sigma_1 = (1,\ 2,\ 3,\ 4)$, $\sigma_2 = (2,\ 3,\ 4,\ 1)$, $\sigma_3 = (3,\ 4,\ 1,\ 2)$ and $\sigma_4 = (4,\ 1,\ 2,\ 3)$.

These cases consider A as the odd numbers in $\{1, 2, ..., n\}$ and B as its even numbers. However, Berzig and Samet [2] use $A = \{1, 2, ..., m\}$, $B = \{m+1, ..., n\}$ and arbitrary mappings.

Definition 2.2 ([18])**.** Let $(X, \preceq)$ be a partially ordered space. We say that F has the *mixed* $(G, \preceq)$*-monotone property* if F is G-monotone non-decreasing in arguments of A and G-monotone non-increasing in arguments of B, that is, for all $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n, \delta, \tau \in X$ and all i,

$$G\delta \preceq G\tau \Rightarrow F(\varepsilon_1, ..., \varepsilon_{i-1}, \delta, \varepsilon_{i+1}, ..., \varepsilon_n) \preceq_i F(\varepsilon_1, ..., \varepsilon_{i-1}, \tau, \varepsilon_{i+1}, ..., \varepsilon_n).$$

Definition 2.3 ([20, 22])**.** Let (X, d) be a metric space and define $\Delta_n, \rho_n : X^n \times X^n \to [0, +\infty)$, for $Y = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$, $V = (\delta_1, \delta_2, ..., \delta_n) \in X^n$, by

$$\Delta_n(Y, V) = \frac{1}{n}\sum_{i=1}^{n} d(\varepsilon_i, \delta_i) \text{ and } \rho_n(Y, V) = \max_{1\leq i\leq n} d(\varepsilon_i, \delta_i).$$

Then Δ_n and ρ_n are metrics on X^n and (X, d) is complete if and only if (X^n, Δ_n) are complete. Similarly, (X, d) is complete if and only if (X^n, ρ_n) are complete. It is easy to see that

$$\begin{aligned} \Delta_n(Y^k, Y) &\to 0 \text{ (as } k\to\infty) \Leftrightarrow d(\varepsilon_i^k, \varepsilon_i) \to 0 \text{ (as } k\to\infty), \\ \text{and } \rho_n(Y^k, Y) &\to 0 \text{ (as } k\to\infty) \Leftrightarrow d(\varepsilon_i^k, \varepsilon_i) \to 0 \text{ (as } k\to\infty),\ i\in\Lambda_n, \end{aligned}$$

where $Y^k = (\varepsilon_1^k, \varepsilon_2^k, ..., \varepsilon_n^k)$ and $Y = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) \in X^n$.

Definition 2.4 ([18])**.** We will say that two mappings $F, G : X \to X$ are *commuting* if $GF\varepsilon = FG\varepsilon$ for all $\varepsilon \in X$. We will say that $F : X^n \to X$ and $G : X \to X$ are *commuting* if $GF(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) = F(G\varepsilon_1, G\varepsilon_2, ..., G\varepsilon_n)$ for all $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n \in X$.

Definition 2.5 ([16])**.** Let $(X, d, \preceq)$ be a partially ordered metric space and let $F : X^n \to X$ and $G : X \to X$ be two mappings. Let $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_n)$ be an $n-$tuple of mappings Λ_n into itself verifying $\sigma_i \in \Omega_{A, B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A, B}$ if $i \in B$. We will say that (F, G) is an $(O, \Upsilon)-$*compatible pair* if, for all $i \in \Lambda_n$,

$$\lim_{m\to\infty} d(GF(\varepsilon_m^{\sigma_i(1)}, \varepsilon_m^{\sigma_i(2)}, ..., \varepsilon_m^{\sigma_i(n)}), F(G\varepsilon_m^{\sigma_i(1)}, G\varepsilon_m^{\sigma_i(2)}, ..., G\varepsilon_m^{\sigma_i(n)})) = 0,$$

whenever $\{\varepsilon_m^1\}, \{\varepsilon_m^2\}, ..., \{\varepsilon_m^n\}$ are sequences in X such that $\{G\varepsilon_m^1\}, \{G\varepsilon_m^2\}, ..., \{G\varepsilon_m^n\}$ are $\preceq -$monotone and

$$\lim_{m\to\infty} F(\varepsilon_m^{\sigma_i(1)}, \varepsilon_m^{\sigma_i(2)}, ..., \varepsilon_m^{\sigma_i(n)}) = \lim_{n\to\infty} G\varepsilon_m^i \in X, \text{ for all } i\in\Lambda_n.$$

Lemma 2.1 ([1, 20, 22, 23]). *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and let $F : X^n \to X$ and $G : X \to X$ be two mappings. Let $\Upsilon = (\sigma_1,\ \sigma_2,\ ...,\ \sigma_n)$ be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,\ B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,\ B}$ if $i \in B$. Define F_Υ, $G_\Upsilon : X^n \to X^n$, for all $\varepsilon_1,\ \varepsilon_2,\ ...,\ \varepsilon_n \in X$, by*

$$F_\Upsilon(\varepsilon_1,\ \varepsilon_2,\ ...,\ \varepsilon_n) = \begin{pmatrix} F(\varepsilon_{\sigma_1(1)},\ \varepsilon_{\sigma_1(2)},\ ...,\ \varepsilon_{\sigma_1(n)}), \\ F(\varepsilon_{\sigma_2(1)},\ \varepsilon_{\sigma_2(2)},\ ...,\ \varepsilon_{\sigma_2(n)}), \\ ...,\ F(\varepsilon_{\sigma_n(1)},\ \varepsilon_{\sigma_n(2)},\ ...,\ \varepsilon_{\sigma_n(n)}) \end{pmatrix},$$

$$G_\Upsilon(\varepsilon_1,\ \varepsilon_2,\ ...,\ \varepsilon_n) = (G\varepsilon_1,\ G\varepsilon_2,\ ...,\ G\varepsilon_n). \tag{2.3}$$

(1) *If F has the mixed $(G,\ \preceq)-$monotone property, then F_Υ is monotone $(G,\ \sqsubseteq)-$non-decreasing.*

(2) *If F is $d-$continuous, then F_Υ is Δ_n-continuous and ρ_n-continuous.*

(3) *If G is $d-$continuous, then G_Υ is Δ_n-continuous and ρ_n-continuous.*

(4) *A point $(\varepsilon_1,\ \varepsilon_2,\ ...,\ \varepsilon_n) \in X^n$ is a $\Upsilon-$fixed point of F if and only if $(\varepsilon_1,\ \varepsilon_2,\ ...,\ \varepsilon_n)$ is a fixed point of F_Υ.*

(5) *A point $(\varepsilon_1,\ \varepsilon_2,\ ...,\ \varepsilon_n) \in X^n$ is a $\Upsilon-$coincidence point of F and G if and only if $(\varepsilon_1,\ \varepsilon_2,\ ...,\ \varepsilon_n)$ is a coincidence point of F_Υ and G_Υ.*

(6) *If $(X,\ d,\ \preceq)$ is regular, then $(X^n,\ \Delta_n,\ \sqsubseteq)$ and $(X^n,\ \rho_n,\ \sqsubseteq)$ are also regular.*

(7) *If F and G are $(O,\ \Upsilon)$-compatible, then F_Υ and G_Υ are O-compatible.*

(8) *If there exist $\varepsilon_0^1,\ \varepsilon_0^2,\ ...,\ \varepsilon_0^n \in X$ verifying $G\varepsilon_0^i \preceq_i F(\varepsilon_0^{\sigma_i(1)},\ \varepsilon_0^{\sigma_i(2)},\ ...,\ \varepsilon_0^{\sigma_i(n)})$, for all $i \in \Lambda_n$, then there exists $Y_0 = (\varepsilon_0^1,\ \varepsilon_0^2,\ ...,\ \varepsilon_0^n) \in X^n$ such that $G_\Upsilon(Y_0) \sqsubseteq F_\Upsilon(Y_0)$.*

3. MAIN RESULTS

In [14], Kadelburg et al. introduced, denoted by Θ, the class of all functions $\theta : [0,\ +\infty) \to [0,\ 1)$ satisfying that for any sequence $\{s_n\}$ of non-negative real numbers, $\theta(s_n) \to 1$ implies that $s_n \to 0$.

Theorem 3.1. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and $F,\ G : X \to X$ be two mappings satisfying*

(*i*) *F is $(G,\ \preceq)$-non-decreasing and $F(X) \subseteq G(X)$,*

(*ii*) *there exists $\varepsilon_0 \in X$ such that $G\varepsilon_0 \preceq F\varepsilon_0$,*

(*iii*) *there exists $\theta \in \Theta$ such that*

$$d(F\varepsilon,\ F\delta) \leq \theta(d(G\varepsilon,\ G\delta))d(G\varepsilon,\ G\delta),$$

for all $\varepsilon,\ \delta \in X$ where $G\varepsilon \preceq G\delta$. Also assume that one of the following conditions holds.

(a) $(X,\ d)$ is complete, F and G are continuous and the pair $(F,\ G)$ is O-compatible,

(b) $(G(X),\ d)$ is complete and $(X,\ d,\ \preceq)$ is non-decreasing-regular,

(c) $(X,\ d)$ is complete, G is continuous and monotone non-decreasing, the pair $(F,\ G)$ is O-compatible and $(X,\ d,\ \preceq)$ is non-decreasing-regular.

Then F and G have a coincidence point.

Proof. Let $\varepsilon_0 \in X$ be arbitrary. By (i), we have $F(X) \subseteq G(X)$, and so there exists $\varepsilon_1 \in X$ such that $F\varepsilon_0 = G\varepsilon_1$. Then, by (ii), $G\varepsilon_0 \preceq F\varepsilon_0 = G\varepsilon_1$. Since F is $(G,\ \preceq)$-non-decreasing, $F\varepsilon_0 \preceq F\varepsilon_1$. Repeating this argument, we get a sequence $\{\varepsilon_n\}_{n\geq 0}$ such that $\{G\varepsilon_n\}$ is $\preceq$-non-decreasing, $G\varepsilon_{n+1} = F\varepsilon_n \preceq F\varepsilon_{n+1} = G\varepsilon_{n+2}$ and

$$G\varepsilon_{n+1} = F\varepsilon_n, \text{ for all } n \geq 0. \tag{3.1}$$

Let $\zeta_n = d(G\varepsilon_n,\ G\varepsilon_{n+1})$ for all $n \geq 0$. By using contractive condition (iii), we have

$$\begin{aligned} d(G\varepsilon_{n+1},\ G\varepsilon_{n+2}) &= d(F\varepsilon_n,\ F\varepsilon_{n+1}) \\ &\leq \theta(d(G\varepsilon_n,\ G\varepsilon_{n+1}))d(G\varepsilon_n,\ G\varepsilon_{n+1}). \end{aligned} \tag{3.2}$$

It follows from the fact $\theta < 1$ that

$$d(G\varepsilon_{n+1},\ G\varepsilon_{n+2}) < d(G\varepsilon_n,\ G\varepsilon_{n+1}), \text{ that is, } \zeta_{n+1} < \zeta_n \text{ for all } n \geq 0.$$

Thus the sequence $\{\zeta_n\}_{n\geq 0}$ is decreasing. Hence there exists $\zeta \geq 0$ such that

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} d(G\varepsilon_n,\ G\varepsilon_{n+1}) = \zeta. \tag{3.3}$$

Suppose $\zeta > 0$. Then, from (3.2), we obtain that

$$\frac{d(G\varepsilon_{n+1},\ G\varepsilon_{n+2})}{d(G\varepsilon_n,\ G\varepsilon_{n+1})} \leq \theta(d(G\varepsilon_n,\ G\varepsilon_{n+1})) < 1.$$

Taking the limit as $n \to \infty$, we get

$$\theta(d(G\varepsilon_n,\ G\varepsilon_{n+1})) \to 1 \text{ as } n \to \infty.$$

Using the properties of function θ, we have

$$\zeta_n = d(G\varepsilon_n,\ G\varepsilon_{n+1}) \to 0 \text{ as } n \to \infty.$$

which contradicts the fact that $\zeta > 0$. Hence, by (3.2), we get

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} d(G\varepsilon_n,\ G\varepsilon_{n+1}) = 0. \tag{3.4}$$

We now prove that $\{G\varepsilon_n\}_{n\geq 0}$ is a Cauchy sequence in $(X,\ d)$. Suppose, to the contrary, that the sequence $\{G\varepsilon_n\}_{n\geq 0}$ is not a Cauchy sequence. Then there exists an $\eta > 0$ for which we can find subsequences $\{\varepsilon_{n(k)}\}$, $\{\varepsilon_{m(k)}\}$ of $\{\varepsilon_n\}_{n\geq 0}$ with $n(k) > m(k) \geq k$ such that

$$d(G\varepsilon_{n(k)},\ G\varepsilon_{m(k)}) \geq \eta. \tag{3.5}$$

We can choose $n(k)$ to be the smallest positive integer satisfying (3.5). Then

$$d(G\varepsilon_{n(k)-1},\ G\varepsilon_{m(k)}) < \eta. \tag{3.6}$$

By using (3.5), (3.6) and triangle inequality, we have

$$\begin{aligned} \eta &\leq \omega_k = d(G\varepsilon_{n(k)},\ G\varepsilon_{m(k)}) \\ &\leq d(G\varepsilon_{n(k)},\ G\varepsilon_{n(k)-1}) + d(G\varepsilon_{n(k)-1},\ G\varepsilon_{m(k)}) \\ &< d(G\varepsilon_{n(k)},\ G\varepsilon_{n(k)-1}) + \eta. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3.4), we get

$$\lim_{k\to\infty} \omega_k = \lim_{k\to\infty} d(G\varepsilon_{n(k)},\ G\varepsilon_{m(k)}) = \eta. \tag{3.7}$$

By using triangle inequality, we have

$$\begin{aligned} \omega_k &= d(G\varepsilon_{n(k)},\ G\varepsilon_{m(k)}) \\ &\leq d(G\varepsilon_{n(k)},\ G\varepsilon_{n(k)+1}) + d(G\varepsilon_{n(k)+1},\ G\varepsilon_{m(k)+1}) + d(G\varepsilon_{m(k)+1},\ G\varepsilon_{m(k)}) \\ &\leq \zeta_{n(k)} + \zeta_{m(k)} + d(F\varepsilon_{n(k)},\ F\varepsilon_{m(k)}) \\ &\leq \zeta_{n(k)} + \zeta_{m(k)} + \theta(d(G\varepsilon_{n(k)},\ G\varepsilon_{m(k)}))d(G\varepsilon_{n(k)},\ G\varepsilon_{m(k)}) \\ &\leq \zeta_{n(k)} + \zeta_{m(k)} + \omega_k. \end{aligned}$$

This shows that

$$\omega_k \leq \zeta_{n(k)} + \zeta_{m(k)} + \theta(\omega_k)\omega_k \leq \zeta_{n(k)} + \zeta_{m(k)} + \omega_k.$$

Taking the limit as $n \to \infty$ in the above inequality, by using (3.4) and (3.7), we get

$$\theta(\omega_k) \to 1.$$

Using the properties of function θ, we obtain

$$\omega_k = d(G\varepsilon_{n(k)},\ G\varepsilon_{m(k)}) \to 0 \text{ as } k \to \infty,$$

which implies that

$$\lim_{k\to\infty} \omega_k = \lim_{k\to\infty} d(G\varepsilon_{n(k)},\ G\varepsilon_{m(k)}) = 0,$$

which contradicts the fact that $\eta > 0$. Therefore, $\{G\varepsilon_n\}_{n\geq 0}$ is a Cauchy sequence in X.

Now, we claim that F and G have a coincidence point for the cases (a)–(c).

Suppose (a) holds, that is, $(X,\, d)$ is complete, F and G are continuous and the pair $(F,\, G)$ is $O-$compatible. Since $(X,\, d)$ is complete, there exists $\varepsilon \in X$ such that $\{G\varepsilon_n\} \to \varepsilon$ and it follows from (3.1) that $\{F\varepsilon_n\} \to \varepsilon$, since F and G are continuous, $\{FG\varepsilon_n\} \to F\varepsilon$ and $\{GG\varepsilon_n\} \to G\varepsilon$. Since the pair $(F,\, G)$ is O-compatible, we conclude that

$$d(G\varepsilon,\ F\varepsilon) = \lim_{n\to\infty} d(GG\varepsilon_{n+1},\ FG\varepsilon_n) = \lim_{n\to\infty} d(GF\varepsilon_n,\ FG\varepsilon_n) = 0,$$

that is, ε is a coincidence point of F and G.

Now suppose (b) holds, that is, $(G(X),\, d)$ is complete and $(X,\, d,\, \preceq)$ is non-decreasing-regular. Since $\{G\varepsilon_n\}_{n\geq 0}$ is a Cauchy sequence in the complete space $(G(X),\, d)$, there exists $\delta \in G(X)$ such that $\{G\varepsilon_n\} \to \delta$. Let $\varepsilon \in X$ be any point such that $\delta = G\varepsilon$. Then $\{G\varepsilon_n\} \to G\varepsilon$. Since $(X,\, d,\, \preceq)$ is non-decreasing-regular and $\{G\varepsilon_n\}$ is $\preceq$-non-decreasing and converging to $G\varepsilon$, we have $G\varepsilon_n \preceq G\varepsilon$ for all $n \geq 0$. By using the contractive condition (iii),

$$d(G\varepsilon_{n+1},\ F\varepsilon) = d(F\varepsilon_n,\ F\varepsilon) \leq \theta(d(G\varepsilon_n,\ G\varepsilon))d(G\varepsilon_n,\ G\varepsilon).$$

It follows from the fact $\theta < 1$ that

$$d(G\varepsilon_{n+1},\ F\varepsilon) < d(G\varepsilon_n,\ G\varepsilon).$$

Letting $n \to \infty$ in the above inequality and using $G\varepsilon_n \to G\varepsilon$, we get $d(G\varepsilon,\, F\varepsilon) = 0$, that is, ε is a coincidence point of F and G.

Suppose now that (c) holds, that is, $(X,\, d)$ is complete, G is continuous and monotone non-decreasing, the pair $(F,\, G)$ is $O-$compatible and $(X,\, d,\, \preceq)$ is non-decreasing-regular. Since $(X,\, d)$ is complete, there exists $\varepsilon \in X$ such that $\{G\varepsilon_n\} \to \varepsilon$ and it follows from (3.1) that $\{F\varepsilon_n\} \to \varepsilon$, since G is continuous, $\{GG\varepsilon_n\} \to G\varepsilon$. Since the pair $(F,\, G)$ is $O-$compatible, we have

$$\lim_{n\to\infty} d(GG\varepsilon_{n+1},\ FG\varepsilon_n) = \lim_{n\to\infty} d(GF\varepsilon_n,\ FG\varepsilon_n) = 0.$$

The above fact and $\{GG\varepsilon_n\} \to G\varepsilon$ together imply that $\{FG\varepsilon_n\} \to G\varepsilon$.

Since $(X,\, d,\, \preceq)$ is non-decreasing-regular and $\{G\varepsilon_n\}$ is $\preceq$-non-decreasing and converging to ε, we have that $G\varepsilon_n \preceq \varepsilon$, which, by the monotonicity of G, implies $GG\varepsilon_n \preceq G\varepsilon$. Then by using the contractive condition (3.1), we get

$$d(FG\varepsilon_n,\ F\varepsilon) \leq \theta(d(GG\varepsilon_n,\ G\varepsilon))d(GG\varepsilon_n,\ G\varepsilon),$$

which, by the fact $\theta < 1$, implies

$$d(FG\varepsilon_n,\ F\varepsilon) < d(GG\varepsilon_n,\ G\varepsilon).$$

Taking the limit as $n \to \infty$ in the above inequality, by using the facts $\{GG\varepsilon_n\} \to G\varepsilon$ and $\{FG\varepsilon_n\} \to G\varepsilon$, we get $d(G\varepsilon,\, F\varepsilon) = 0$, that is, ε is a coincidence point of F and G. □

Putting $\theta(s) = k$ with $k \in [0,\, 1)$ for all $s \in [0,\, \infty)$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.2. *Let $(X,\, d,\, \preceq)$ be a partially ordered metric space and let $F,\, G : X \to X$ be two mappings satisfying conditions (i) and (ii) of Theorem 3.1 and assume that there exists $k \in [0,\, 1)$ such that*

$$d(F\varepsilon,\ F\delta) \leq kd(G\varepsilon,\ G\delta),$$

for all $\varepsilon,\, \delta \in X$ where $G\varepsilon \preceq G\delta$. Also assume that one of the conditions $(a) - (c)$ of Theorem 3.1 holds. Then F and G have a coincidence point.

Example 3.1. Suppose that $X = \mathbb{R}$, equipped with the usual metric $d : X^2 \to [0,$ b$+\infty)$ with the natural ordering of real numbers $\leq$. Let $F,\, G : X \to X$ be defined as

$$F\varepsilon = \ln\left(1 + \varepsilon^2\right) \text{ and } G\varepsilon = \varepsilon^2, \text{ for all } \varepsilon \in X.$$

Define $\theta : [0,\, +\infty) \to [0,\, 1)$ as follows

$$\theta(s) = \begin{cases} \dfrac{\ln(1+s)}{s}, & s > 0, \\ 0, & s = 0. \end{cases}$$

First, we shall show that the contractive condition of Theorem 3.1 holds for the mappings F and G. Let $\varepsilon,\, \delta \in X$ be such that $G\varepsilon \preceq G\delta$. Then we have

$$\begin{aligned} d(F\varepsilon,\ F\delta) &= |F\varepsilon - F\delta| \\ &= \left|\ln\left(1+\varepsilon^2\right) - \ln\left(1+\delta^2\right)\right| \\ &= \left|\ln\frac{1+\varepsilon^2}{1+\delta^2}\right| \\ &= \left|\ln\left(1 + \frac{\varepsilon^2 - \delta^2}{1+\delta^2}\right)\right| \\ &\leq \ln\left(1 + \left|\varepsilon^2 - \delta^2\right|\right) \\ &\leq \ln\left(1 + |G\varepsilon - G\delta|\right) \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\ln(1+|G\varepsilon - G\delta|)}{|G\varepsilon - G\delta|} \times |G\varepsilon - G\delta| \\
&\leq \frac{\ln(1+d(G\varepsilon,\ G\delta))}{d(G\varepsilon,\ G\delta)} \times d(G\varepsilon,\ G\delta) \\
&\leq \theta(d(G\varepsilon,\ G\delta))d(G\varepsilon,\ G\delta).
\end{aligned}
$$

This shows that the contractive condition of Theorem 3.1 holds with the function θ. In addition, all the other conditions of Theorem 3.1 are satisfied and $z = 0$ is a coincidence point of F and G.

4. Multidimensional Coincidence Point Results

Next we give an $n-$dimensional fixed point theorem for mixed monotone mappings. For brevity, $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$, $(\delta_1, \delta_2, ..., \delta_n)$ and $(\varepsilon_0^1, \varepsilon_0^2, ..., \varepsilon_0^n)$ will be denoted by Y, V and Y_0 respectively.

Theorem 4.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^n \to X$ and $G : X \to X$ be two mappings and $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_n)$ be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,\ B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,\ B}$ if $i \in B$. Suppose that the following properties are fulfilled:*

(i) $F(X^n) \subseteq G(X)$,

(ii) F has the mixed G-monotone property,

(iii) there exist $\varepsilon_0^1, \varepsilon_0^2, ..., \varepsilon_0^n \in X$ verifying $G\varepsilon_0^i \preceq_i F(\varepsilon_0^{\sigma_i(1)}, \varepsilon_0^{\sigma_i(2)}, ..., \varepsilon_0^{\sigma_i(n)})$, for all $i \in \Lambda_n$,

(iv) there exists $\theta \in \Theta$ such that

$$
\begin{aligned}
&d(F(\varepsilon_1,\ \varepsilon_2,\ ...,\ \varepsilon_n),\ F(\delta_1,\ \delta_2,\ ...,\ \delta_n)) \\
&\leq \theta\left(\max_{1\leq i\leq n} d(G\varepsilon_i,\ G\delta_i)\right)\left(\max_{1\leq i\leq n} d(G\varepsilon_i,\ G\delta_i)\right),
\end{aligned}
\tag{4.1}
$$

for all $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n, \delta_1, \delta_2, ..., \delta_n \in X$ with $\varepsilon_i \preceq_i \delta_i$, for $i \in \Lambda_n$. Also assume that one of the following conditions holds,

(a) (X, d) is complete, F and G are continuous and the pair (F, G) is (O, Υ)-compatible,

(b) $(G(X), d)$ is complete and $(X, d, \preceq)$ is non-decreasing-regular,

(c) (X, d) is complete, G is continuous and monotone non-decreasing, the pair (F, G) is $(O, \Upsilon)-$compatible and $(X, d, \preceq)$ is non-decreasing-regular.

Then F and G have a $\Upsilon-$coincidence point.

Proof. For fixed $i \in A$, we have $G\varepsilon_{\sigma_i(t)} \preceq_t G\delta_{\sigma_i(t)}$ for $t \in \Lambda_n$. From (4.1), we have

$$\begin{aligned} & d(F(\varepsilon_{\sigma_i(1)},\ \varepsilon_{\sigma_i(2)},\ ...,\ \varepsilon_{\sigma_i(n)}),\ F(\delta_{\sigma_i(1)},\ \delta_{\sigma_i(2)},\ ...,\ \delta_{\sigma_i(n)})) \\ \leq\ & \theta\left(\max_{1\leq i\leq n} d(G\varepsilon_i,\ G\delta_i)\right)\left(\max_{1\leq i\leq n} d(G\varepsilon_i,\ G\delta_i)\right), \end{aligned} \tag{4.2}$$

for all $i \in A$. Similarly, for fixed $i \in B$, we have $G\delta_{\sigma_i(t)} \succeq_t Gv_{\sigma_i(t)}$ for $t \in \Lambda_n$. It follows from (4.1) that

$$\begin{aligned} & d(F(\delta_{\sigma_i(1)},\ \delta_{\sigma_i(2)},\ ...,\ \delta_{\sigma_i(n)}),\ F(\varepsilon_{\sigma_i(1)},\ \varepsilon_{\sigma_i(2)},\ ...,\ \varepsilon_{\sigma_i(n)})) \\ =\ & d(F(\varepsilon_{\sigma_i(1)},\ \varepsilon_{\sigma_i(2)},\ ...,\ \varepsilon_{\sigma_i(n)}),\ F(\delta_{\sigma_i(1)},\ \delta_{\sigma_i(2)},\ ...,\ \delta_{\sigma_i(n)})) \\ \leq\ & \theta\left(\max_{1\leq i\leq n} d(G\varepsilon_i,\ G\delta_i)\right)\left(\max_{1\leq i\leq n} d(G\varepsilon_i,\ G\delta_i)\right), \end{aligned} \tag{4.3}$$

for all $i \in B$. By (2.2), (2.3), (4.2) and (4.3), we have

$$\rho_n(F_\Upsilon(Y),\ F_\Upsilon(V)) \leq \theta(\rho_n(G_\Upsilon(Y),\ G_\Upsilon(V)))\rho_n(G_\Upsilon(Y),\ G_\Upsilon(V)),$$

for all $Y,\ V \in X^n$ with $G_\Upsilon(Y) \sqsubseteq G_\Upsilon(V)$. It is only necessary to apply Theorem 3.1 to the mappings $F = F_\Upsilon$ and $G = G_\Upsilon$ in the ordered metric space $(X^n,\ \rho_n,\ \sqsubseteq)$ by taking all items of Lemma 2.1. □

In a similar way, we may state the result analog of Corollary 2.2.

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