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NOTE ON NEWTON-TYPE INEQUALITIES INVOLVING TEMPERED FRACTIONAL INTEGRALS

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Abstract. We propose a new method of investigation of an integral equality associated with tempered fractional integrals. In addition to this, several Newton-type inequalities are considered for differentiable convex functions by taking the modulus of the newly established identity. Moreover, we establish some Newton-type inequalities with the help of Hölder and power-mean inequality. Furthermore, several new results are presented by using special choices of obtained inequalities.

1. Introduction and Preliminaries

Simpson's second rule is possessed of the rule of three-point Newton-Cotes quadrature. Computations for three steps quadratic kernel are generally called Newton-type results. These results are known to be Newton-type inequalities in literature. The study of Newton-type inequalities has been extensively pursued by numerous mathematicians. For instance, in [6], Newton-type inequalities were established for the case of functions whose second derivatives are convex. Newton-type inequalities were investigated by post-quantum integrals in [13]. Noor et al. proved Newton-type inequalities connected with harmonic convex and p-harmonic convex functions in [18] and [19], respectively. Moreover, in paper [1], Newton-type inequalities were considered for the case of quantum differentiable convex functions. Furthermore, in paper [4], several error estimates of Newton-type quadrature formula were presented by bounded variation and Lipschitzian mappings. See [5, 8, 10, 17, 24] and the references cited therein for some recent results about Newton-type inequalities including convex differentiable functions.

Definition 1. [20] Consider that I is an interval of real numbers. Then, a function $f : I \to \mathbb{R}$ is said to be convex, if

$$f\left(tx + (1-t)\, y\right) \leq tf\left(x\right) + (1-t)\, f\left(y\right)$$

is valid for all $x, y \in I$ and all $t \in [0, 1]$.

Simpson-type inequalities are derived from Simpson's rules and take the following form of inequalities:

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i. Simpson's quadrature formula (Simpson's 1/3 rule):

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

ii. Newton-Cotes quadrature formula or Simpson's second formula (Simpson's 3/8 rule):

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right].$$

The popularity of fractional calculus has increased in recent years due to its broad range of applications in various fields of science. Given the significance of fractional calculus, various fractional integral operators can be considered. It can be obtained the bounds of new formulas by using the Hermite–Hadamard-type and Simpson-type inequalities. For instance, sundry Newton-type inequalities for the case of functions whose first derivative in absolute value at certain power are arithmetically-harmonically convex were acquired in [3]. In addition, some Newton-type inequalities were established by using Riemann-Liouville fractional integrals for differentiable convex functions and the authors also acquired several Riemann-Liouville fractional Newton-type inequalities for functions of bounded variation in [24]. Furthermore, some Newton-type inequalities were investigated and some applications for the case of special cases of real functions were also given in [6]. Please see [9, 10] for more detailed information and unexplained subjects.

Definition 2. [7,11] Let $f \in L_1[a,b]$, $a,b \in \mathbb{R}$ with $a<b$. The Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha>0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x>a \tag{1}$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x<b, \tag{2}$$

respectively. Here, Γ denotes the Gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du.$$

Let us note that $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

The advancement of tempered fractional calculus has expanded the scope of applications for the case of fractional calculus, making it a versatile tool in various fields of study. The definitions of fractional integration with weak singular and exponential kernels were firstly reported in Buschman's earlier work [2]. Please see [15, 21, 25] for more detailed information and unexplained subjects about the sundry definitions of the tempered fractional operators. In [16], Mohammed et al. were investigated several Hermite–Hadamard-type inequalities associated with the tempered fractional integrals for the case of convex functions which cover the previously published result such as Riemann integrals, Riemann-Liouville fractional integrals. To be more precise, the authors followed the Sarikaya et al. [22] and Sarikaya and Yildirim [23] techniques in order to prove several Hermite–Hadamard-type inequalities which including the tempered fractional integrals.

Definition 3. The incomplete gamma function, λ-incomplete gamma function are defined by

$$\Upsilon(\alpha, x) := \int_0^x t^{\alpha-1} e^{-t} dt$$

and

$$\Upsilon_\lambda(\alpha, x) := \int_0^x t^{\alpha-1} e^{-\lambda t} dt,$$

respectively. Here, $0 < \alpha < \infty$ and $\lambda \geq 0$.

The properties of the λ-incomplete gamma function are as follows:

Remark 1. [16] For real numbers $\alpha > 0$; $x, \lambda \geq 0$ and $a < b$, we have

i. $\Upsilon_{\lambda(b-a)}(\alpha, 1) = \int_0^1 t^{\alpha-1} e^{-\lambda(b-a)t} dt = \frac{1}{(b-a)^\alpha} \Upsilon_\lambda(\alpha, b-a)$,

ii. $\int_0^1 \Upsilon_{\lambda(b-a)}(\alpha, x)\, dx = \frac{\Upsilon_\lambda(\alpha, b-a)}{(b-a)^\alpha} - \frac{\Upsilon_\lambda(\alpha+1, b-a)}{(b-a)^{\alpha+1}}$.

Definition 4. [12, 14] The fractional tempered integral operators $\mathcal{J}_{a+}^{(\alpha,\lambda)} f$ and $\mathcal{J}_{b-}^{(a,\lambda)} f$ of order $\alpha > 0$ and $\lambda \geq 0$ are presented by

$$\mathcal{J}_{a+}^{(\alpha,\lambda)} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} e^{-\lambda(x-t)} f(t) dt, \quad x \in [a, b] \tag{3}$$

and

$$\mathcal{J}_{b-}^{(\alpha,\lambda)} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} e^{-\lambda(t-x)} f(t) dt, \quad x \in [a, b], \tag{4}$$

respectively. Here, f belongs to $L_1[a, b]$.

Let us consider $\lambda = 0$. Then, the fractional integral in (3) and (4) reduces to the Riemann-Liouville fractional integral in (1) and (2), respectively.

The structure of the paper is divided into four parts, starting with an overview and introduction and preliminaries. The fundamental definitions of fractional calculus and other relevant research in this discipline are given in above. An integral equality will be proved in Section 2. Moreover, we investigate Newton-type inequalities including tempered fractional operators. In Section 3, several results will be given by using special choices of obtained inequalities. Finally, in Section 4, we will give several ideas for the further directions of research.

2. Principal outcomes

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If f' belongs to $L[a, b]$, then the following identity holds:*

$$\frac{3^{\alpha-1}\Gamma(\alpha)}{(b-a)^\alpha \Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha, 1)} \left[\mathcal{J}_{a+}^{(\alpha,\lambda)} f\left(\frac{2a+b}{3}\right) + \mathcal{J}_{\frac{2a+b}{3}+}^{(\alpha,\lambda)} f\left(\frac{a+2b}{3}\right) + \mathcal{J}_{\frac{a+2b}{3}+}^{(\alpha,\lambda)} f(b) \right]$$
$$- \frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right]$$

$$= \frac{b-a}{9\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}\left[I_1+I_2+I_3\right]. \tag{5}$$

Here, $\Gamma(\alpha)$ *is Euler Gamma function and*

$$\begin{cases} I_1=\int\limits_0^1\left\{\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{5}{8}\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right\}f'\left(ta+(1-t)\left(\frac{2a+b}{3}\right)\right)dt,\\ I_2=\int\limits_0^1\left\{\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right\}f'\left(t\left(\frac{2a+b}{3}\right)+(1-t)\left(\frac{a+2b}{3}\right)\right)dt,\\ I_3=\int\limits_0^1\left\{\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{3}{8}\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right\}f'\left(t\left(\frac{a+2b}{3}\right)+(1-t)\,b\right)dt. \end{cases}$$

Proof. By using integration by parts and changing variables with $x=ta+(1-t)\left(\frac{2a+b}{3}\right)$, we readily obtain

$$\begin{aligned} I_1 &= \int\limits_0^1\left\{\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{5}{8}\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right\}f'\left(ta+(1-t)\left(\frac{2a+b}{3}\right)\right)dt\\ &= -\frac{3}{b-a}\left\{\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{5}{8}\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right\}f\left(ta+(1-t)\left(\frac{2a+b}{3}\right)\right)\Bigg|_0^1\\ &\quad+\frac{3}{b-a}\int\limits_0^1 t^{\alpha-1}e^{-\lambda\left(\frac{b-a}{3}\right)t}f\left(ta+(1-t)\left(\frac{2a+b}{3}\right)\right)dt\\ &= -\frac{9\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}{8(b-a)}f(a)-\frac{15\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}{8(b-a)}f\left(\frac{2a+b}{3}\right)\\ &\quad+\left(\frac{3}{b-a}\right)^{1+\alpha}\int\limits_a^{\frac{2a+b}{3}}\left(\frac{2a+b}{3}-x\right)^{\alpha-1}e^{-\lambda\left(\frac{2a+b}{3}-x\right)}f(x)\,dx\\ &= -\frac{9\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}{8(b-a)}f(a)-\frac{15\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}{8(b-a)}f\left(\frac{2a+b}{3}\right)\\ &\quad+\left(\frac{3}{b-a}\right)^{1+\alpha}\Gamma(\alpha)\,\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{2a+b}{3}\right). \end{aligned}$$

In a similar manner, changing variables with $x=t\left(\frac{2a+b}{3}\right)+(1-t)\left(\frac{a+2b}{3}\right)$ and $x=t\left(\frac{a+2b}{3}\right)+(1-t)\,b$, we get

$$\begin{aligned} I_2 &= \int\limits_0^1\left\{\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right\}f'\left(t\left(\frac{2a+b}{3}\right)+(1-t)\left(\frac{a+2b}{3}\right)\right)dt\\ &= -\frac{3}{b-a}\left\{\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right\}f\left(t\left(\frac{2a+b}{3}\right)+(1-t)\left(\frac{a+2b}{3}\right)\right)\Bigg|_0^1\\ &\quad+\frac{3}{b-a}\int\limits_0^1 t^{\alpha-1}e^{-\lambda\left(\frac{b-a}{3}\right)t}f\left(t\left(\frac{2a+b}{3}\right)+(1-t)\left(\frac{a+2b}{3}\right)\right)dt \end{aligned}$$

$$= -\frac{3\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}{2(b-a)} f\left(\frac{2a+b}{3}\right) - \frac{3\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}{2(b-a)} f\left(\frac{a+2b}{3}\right)$$
$$+\left(\frac{3}{b-a}\right)^{1+\alpha} \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \left(\frac{a+2b}{3}-x\right)^{\alpha-1} e^{-\lambda\left(\frac{a+2b}{3}-x\right)} f(x)\,dx$$
$$= -\frac{3\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}{2(b-a)} f\left(\frac{2a+b}{3}\right) - \frac{3\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}{2(b-a)} f\left(\frac{a+2b}{3}\right)$$
$$+\left(\frac{3}{b-a}\right)^{1+\alpha} \Gamma(\alpha)\, \mathcal{J}^{(\alpha,\lambda)}_{\frac{2a+b}{3}+} f\left(\frac{a+2b}{3}\right)$$

and

$$I_3 = \int_0^1 \left\{ \Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t) - \frac{3}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1) \right\} f'\left(t\left(\frac{a+2b}{3}\right) + (1-t)\,b\right) dt$$
$$= -\frac{3}{b-a} \left\{ \Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t) - \frac{3}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1) \right\} f\left(t\left(\frac{a+2b}{3}\right) + (1-t)\,b\right)\Bigg|_0^1$$
$$+\frac{3}{b-a}\int_0^1 t^{\alpha-1} e^{-\lambda\left(\frac{b-a}{3}\right)t} f\left(t\left(\frac{a+2b}{3}\right) + (1-t)\,b\right) dt$$
$$= -\frac{15\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}{8(b-a)} f\left(\frac{a+2b}{3}\right) - \frac{9\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}{8(b-a)} f(b)$$
$$+\left(\frac{3}{b-a}\right)^{1+\alpha} \int_{\frac{a+2b}{3}}^{b} (b-x)^{\alpha-1} e^{-\lambda(b-x)} f(x)\,dx$$
$$= -\frac{15\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}{8(b-a)} f\left(\frac{a+2b}{3}\right) - \frac{9\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}{8(b-a)} f(b)$$
$$+\left(\frac{3}{b-a}\right)^{1+\alpha} \Gamma(\alpha)\, \mathcal{J}^{(\alpha,\lambda)}_{\frac{a+2b}{3}+} f(b).$$

Finally, if we multiply $I_1+I_2+I_3$ by $\frac{b-a}{9\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}$, then we have (5). This finishes the proof of Lemma 1. □

THEOREM 1. *Suppose that all the assumptions of Lemma 1 hold. Moreover, let $|f'|$ be a convex function on $[a,b]$. Then, we have the following Newton-type inequality*

$$\Bigg| \frac{3^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)} \left[\mathcal{J}^{(\alpha,\lambda)}_{a+} f\left(\frac{2a+b}{3}\right) + \mathcal{J}^{(\alpha,\lambda)}_{\frac{2a+b}{3}+} f\left(\frac{a+2b}{3}\right) + \mathcal{J}^{(\alpha,\lambda)}_{\frac{a+2b}{3}+} f(b) \right]$$
$$-\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \Bigg|$$
$$\leq \frac{b-a}{27\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)} \left\{ (2A_2(\alpha,\lambda) + A_1(\alpha,\lambda) + A_4(\alpha,\lambda) + A_3(\alpha,\lambda) + A_5(\alpha,\lambda))\,|f'(a)| \right.$$

(6)

$$+\left(A_2(\alpha,\lambda)-A_1(\alpha,\lambda)+2A_4(\alpha,\lambda)-A_3(\alpha,\lambda)+3A_6(\alpha,\lambda)-A_5(\alpha,\lambda)\right)|f'(b)|\right\}.$$

Here,

$$\begin{cases} A_1(\alpha,\lambda)=\int_0^1 t\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|dt,\\ A_2(\alpha,\lambda)=\int_0^1 \left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|dt,\\ A_3(\alpha,\lambda)=\int_0^1 t\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|dt,\\ A_4(\alpha,\lambda)=\int_0^1 \left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|dt,\\ A_5(\alpha,\lambda)=\int_0^1 t\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|dt,\\ A_6(\alpha,\lambda)=\int_0^1 \left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|dt. \end{cases}$$

Proof. By using Lemma 1, integration by parts and convexity of $|f'|$, we obtain

$$\begin{aligned}
&\left|\frac{3^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{2a+b}{3}\right)+\mathcal{J}_{\frac{2a+b}{3}+}^{(\alpha,\lambda)}f\left(\frac{a+2b}{3}\right)+\mathcal{J}_{\frac{a+2b}{3}+}^{(\alpha,\lambda)}f(b)\right]\right.\\
&\left.-\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\right|\\
&\leq\frac{b-a}{9\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}\left\{\left|\int_0^1\left\{\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right\}f'\left(ta+(1-t)\left(\frac{2a+b}{3}\right)\right)dt\right|\right.\\
&+\left|\int_0^1\left\{\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right\}f'\left(t\left(\frac{2a+b}{3}\right)+(1-t)\left(\frac{a+2b}{3}\right)\right)dt\right|\\
&\left.+\left|\int_0^1\left\{\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right\}f'\left(t\left(\frac{a+2b}{3}\right)+(1-t)b\right)dt\right|\right\}\\
&\leq\frac{b-a}{9\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}\left\{\int_0^1\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|\left|f'\left(\left(\frac{2+t}{3}\right)a+\left(\frac{1-t}{3}\right)b\right)\right|dt\right.\\
&+\int_0^1\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|\left|f'\left(\left(\frac{1+t}{3}\right)a+\left(\frac{2-t}{3}\right)b\right)\right|dt\\
&\left.+\int_0^1\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|\left|f'\left(\frac{t}{3}a+\left(\frac{3-t}{3}\right)b\right)\right|dt\right\}\\
&\leq\frac{b-a}{9\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}\left\{|f'(a)|\int_0^1\left(\frac{2+t}{3}\right)\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|dt\right.
\end{aligned}$$

$$+\left|f'(b)\right|\int_0^1\left(\frac{1-t}{3}\right)\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|dt$$

$$+\left|f'(a)\right|\int_0^1\left(\frac{1+t}{3}\right)\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|dt$$

$$+\left|f'(b)\right|\int_0^1\left(\frac{2-t}{3}\right)\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|dt$$

$$+\left|f'(a)\right|\int_0^1\frac{t}{3}\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|dt$$

$$\left.+\left|f'(b)\right|\int_0^1\left(\frac{3-t}{3}\right)\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|dt\right\}$$

$$=\frac{b-a}{27\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}\left\{\left(2A_2(\alpha,\lambda)+A_1(\alpha,\lambda)+A_4(\alpha,\lambda)+A_3(\alpha,\lambda)+A_5(\alpha,\lambda)\right)\left|f'(a)\right|\right.$$
$$\left.+\left(A_2(\alpha,\lambda)-A_1(\alpha,\lambda)+2A_4(\alpha,\lambda)-A_3(\alpha,\lambda)+3A_6(\alpha,\lambda)-A_5(\alpha,\lambda)\right)\left|f'(b)\right|\right\}.$$

With this calculation, the proof of Theorem 1 ends. □

THEOREM 2. *Assume that all the assumptions of Lemma 1 hold. Assume also that* $|f'|^q$ *is a convex function on* $[a,b]$ *where* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p,q>1$. *Then, the following Newton-type inequality holds:*

$$\left|\frac{3^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{2a+b}{3}\right)+\mathcal{J}_{\frac{2a+b}{3}+}^{(\alpha,\lambda)}f\left(\frac{a+2b}{3}\right)+\mathcal{J}_{\frac{a+2b}{3}+}^{(\alpha,\lambda)}f(b)\right]\right.$$
$$\left.-\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\right|$$
$$\leq\frac{b-a}{9\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}\left\{A_7^{\frac{1}{p}}(\alpha,\lambda,p)\left[\frac{5\left|f'(a)\right|^q+\left|f'(b)\right|^q}{6}\right]^{\frac{1}{q}}\right.$$
$$\left.+A_8^{\frac{1}{p}}(\alpha,\lambda,p)\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}+A_9^{\frac{1}{p}}(\alpha,\lambda,p)\left[\frac{\left|f'(a)\right|^q+5\left|f'(b)\right|^q}{6}\right]^{\frac{1}{q}}\right\},\tag{7}$$

where

$$\begin{cases}A_7(\alpha,\lambda,p)=\int_0^1\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|^p dt,\\ A_8(\alpha,\lambda,p)=\int_0^1\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|^p dt,\\ A_9(\alpha,\lambda,p)=\int_0^1\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|^p dt.\end{cases}$$

Proof. Firstly, let us consider Lemma 1. Then, it yields

$$\Bigg|\frac{3^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{2a+b}{3}\right)+\mathcal{J}_{\frac{2a+b}{3}+}^{(\alpha,\lambda)}f\left(\frac{a+2b}{3}\right)+\mathcal{J}_{\frac{a+2b}{3}+}^{(\alpha,\lambda)}f(b)\right]$$

$$-\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\Bigg|$$

$$\leq\frac{b-a}{9\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)}\left\{\int_0^1\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|\left|f'\left(\left(\frac{2+t}{3}\right)a+\left(\frac{1-t}{3}\right)b\right)\right|dt\right.$$

$$+\int_0^1\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|\left|f'\left(\left(\frac{1+t}{3}\right)a+\left(\frac{2-t}{3}\right)b\right)\right|dt$$

(8)

$$\left.+\int_0^1\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|\left|f'\left(\frac{t}{3}a+\left(\frac{3-t}{3}\right)b\right)\right|dt\right\}.$$

Now, we consider the integrals on the right side of (8). Using the convexity of $|f'|^q$ and Hölder inequality, we get

$$\int_0^1\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|\left|f'\left(\left(\frac{2+t}{3}\right)a+\left(\frac{1-t}{3}\right)b\right)\right|dt$$

$$\leq\left(\int_0^1\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\left(\frac{2+t}{3}\right)a+\left(\frac{1-t}{3}\right)b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq A_7^{\frac{1}{p}}(\alpha,\lambda,p)\left(\int_0^1\left(\frac{2+t}{3}\left|f'(a)\right|^q+\frac{1-t}{3}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$

(9)

$$=A_7^{\frac{1}{p}}(\alpha,\lambda,p)\left[\frac{5\left|f'(a)\right|^q+\left|f'(b)\right|^q}{6}\right]^{\frac{1}{q}}.$$

Similar to foregoing process, it follows

$$\int_0^1\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|\left|f'\left(\left(\frac{1+t}{3}\right)a+\left(\frac{2-t}{3}\right)b\right)\right|dt$$

$$\leq\left(\int_0^1\left|\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\left(\frac{1+t}{3}\right)a+\left(\frac{2-t}{3}\right)b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq A_8^{\frac{1}{p}}(\alpha,\lambda,p)\left(\int_0^1\left(\frac{1+t}{3}\left|f'(a)\right|^q+\frac{2-t}{3}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$

(10)

$$=A_8^{\frac{1}{p}}(\alpha,\lambda,p)\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}$$

and

$$\int_0^1 \left| \Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t) - \frac{3}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1) \right| \left| f'\left(\frac{t}{3}a + \left(\frac{3-t}{3}\right)b\right) \right| dt$$

$$\leq \left(\int_0^1 \left| \Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,t) - \frac{3}{8}\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1) \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(\frac{t}{3}a + \left(\frac{3-t}{3}\right)b\right) \right|^q dt \right)^{\frac{1}{q}}$$

$$\leq A_9^{\frac{1}{p}}(\alpha,\lambda,p) \left(\int_0^1 \left(\frac{t}{3}|f'(a)|^q + \frac{3-t}{3}|f'(b)|^q \right) dt \right)^{\frac{1}{q}}$$

(11)

$$= A_9^{\frac{1}{p}}(\alpha,\lambda,p) \left[\frac{|f'(a)|^q + 5|f'(b)|^q}{6} \right]^{\frac{1}{q}}.$$

If we consider from (9) to (11) in (8), then we have

$$\left| \frac{3^{\alpha-1}\Gamma(\alpha)}{(b-a)^\alpha \Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)} \left[\mathcal{J}_{a+}^{(\alpha,\lambda)} f\left(\frac{2a+b}{3}\right) + \mathcal{J}_{\frac{2a+b}{3}+}^{(\alpha,\lambda)} f\left(\frac{a+2b}{3}\right) + \mathcal{J}_{\frac{a+2b}{3}+}^{(\alpha,\lambda)} f(b) \right] \right.$$
$$\left. - \frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right|$$
$$\leq \frac{b-a}{9\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)} \left\{ A_7^{\frac{1}{p}}(\alpha,\lambda,p) \left[\frac{5|f'(a)|^q + |f'(b)|^q}{6} \right]^{\frac{1}{q}} \right.$$
$$\left. + A_8^{\frac{1}{p}}(\alpha,\lambda,p) \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} + A_9^{\frac{1}{p}}(\alpha,\lambda,p) \left[\frac{|f'(a)|^q + 5|f'(b)|^q}{6} \right]^{\frac{1}{q}} \right\}.$$

This ends the proof of Theorem 2. □

THEOREM 3. *Let us consider that all the assumptions of Lemma 1 hold. Let us also consider that $|f'|^q$ is convex on $[a,b]$ where $q \geq 1$. Then, we have the following Newton-type inequality*

$$\left| \frac{3^{\alpha-1}\Gamma(\alpha)}{(b-a)^\alpha \Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)} \left[\mathcal{J}_{a+}^{(\alpha,\lambda)} f\left(\frac{2a+b}{3}\right) + \mathcal{J}_{\frac{2a+b}{3}+}^{(\alpha,\lambda)} f\left(\frac{a+2b}{3}\right) + \mathcal{J}_{\frac{a+2b}{3}+}^{(\alpha,\lambda)} f(b) \right] \right.$$
$$\left. - \frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right|$$
$$\leq \frac{b-a}{9\,\Upsilon_{\lambda\left(\frac{b-a}{3}\right)}(\alpha,1)} \left\{ A_2^{1-\frac{1}{q}}(\alpha,\lambda) \left[\left(\frac{2A_2(\alpha,\lambda) + A_1(\alpha,\lambda)}{3} \right) |f'(a)|^q \right. \right.$$
$$\left. + \left(\frac{A_2(\alpha,\lambda) - A_1(\alpha,\lambda)}{3} \right) |f'(b)|^q \right]^{\frac{1}{q}}$$
$$+ A_4^{1-\frac{1}{q}}(\alpha,\lambda) \left[\left(\frac{A_4(\alpha,\lambda) + A_3(\alpha,\lambda)}{3} \right) |f'(a)|^q + \left(\frac{2A_4(\alpha,\lambda) - A_3(\alpha,\lambda)}{3} \right) |f'(b)|^q \right]^{\frac{1}{q}}$$

(12)
$$+A_6^{1-\frac{1}{q}}(\alpha,\lambda)\left[\frac{A_5(\alpha,\lambda)}{3}|f'(a)|^q+\left(\frac{3A_6(\alpha,\lambda)-A_5(\alpha,\lambda)}{3}\right)|f'(b)|^q\right]^{\frac{1}{q}}\Bigg\}.$$

Here, $A_1(\alpha,\lambda)$*,* $A_2(\alpha,\lambda)$*,* $A_3(\alpha,\lambda)$*,* $A_4(\alpha,\lambda)$*,* $A_5(\alpha,\lambda)$*, and* $A_6(\alpha,\lambda)$ *are defined in Theorem 1.*

Proof. If we consider the convexity of $|f'|^q$ and power mean inequality, then we obtain

$$\int_0^1\left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|\left|f'\left(\left(\frac{2+t}{3}\right)a+\left(\frac{1-t}{3}\right)b\right)\right|dt$$
$$\leq\left(\int_0^1\left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|\left|f'\left(\left(\frac{2+t}{3}\right)a+\left(\frac{1-t}{3}\right)b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$=A_2^{1-\frac{1}{q}}(\alpha,\lambda)\left(\int_0^1\left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|\left|f'\left(\left(\frac{2+t}{3}\right)a+\left(\frac{1-t}{3}\right)b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq A_2^{1-\frac{1}{q}}(\alpha,\lambda)\left(\int_0^1\left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|\left[\frac{2+t}{3}|f'(a)|^q+\frac{1-t}{3}|f'(b)|^q\right]dt\right)^{\frac{1}{q}}$$

(13)
$$=A_2^{1-\frac{1}{q}}(\alpha,\lambda)\left[\left(\frac{2A_2(\alpha,\lambda)+A_1(\alpha,\lambda)}{3}\right)|f'(a)|^q+\left(\frac{A_2(\alpha,\lambda)-A_1(\alpha,\lambda)}{3}\right)|f'(b)|^q\right]^{\frac{1}{q}}$$

and similarly

$$\int_0^1\left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|\left|f'\left(\left(\frac{1+t}{3}\right)a+\left(\frac{2-t}{3}\right)b\right)\right|dt$$
$$\leq\left(\int_0^1\left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|\left|f'\left(\left(\frac{1+t}{3}\right)a+\left(\frac{2-t}{3}\right)b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$=A_4^{1-\frac{1}{q}}(\alpha,\lambda)\left(\int_0^1\left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|\left|f'\left(\left(\frac{1+t}{3}\right)a+\left(\frac{2-t}{3}\right)b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq A_4^{1-\frac{1}{q}}(\alpha,\lambda)\left(\int_0^1\left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{1}{2}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|\left[\frac{1+t}{3}|f'(a)|^q+\frac{2-t}{3}|f'(b)|^q\right]dt\right)^{\frac{1}{q}}$$

(14)

$$= A_4^{1-\frac{1}{q}}(\alpha,\lambda)\left[\left(\frac{A_4(\alpha,\lambda)+A_3(\alpha,\lambda)}{3}\right)|f'(a)|^q+\left(\frac{2A_4(\alpha,\lambda)-A_3(\alpha,\lambda)}{3}\right)|f'(b)|^q\right]^{\frac{1}{q}}$$

and

$$\int_0^1 \left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|\left|f'\left(\frac{t}{3}a+\left(\frac{3-t}{3}\right)b\right)\right|dt$$

$$\leq \left(\int_0^1 \left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|\left|f'\left(\frac{t}{3}a+\left(\frac{3-t}{3}\right)b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$= A_6^{1-\frac{1}{q}}(\alpha,\lambda)\left(\int_0^1 \left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|\left|f'\left(\frac{t}{3}a+\left(\frac{3-t}{3}\right)b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq A_6^{1-\frac{1}{q}}(\alpha,\lambda)\left(\int_0^1 \left|\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda(\frac{b-a}{3})}(\alpha,1)\right|\left[\frac{t}{3}|f'(a)|^q+\frac{3-t}{3}|f'(b)|^q\right]dt\right)^{\frac{1}{q}}$$

(15)

$$= A_6^{1-\frac{1}{q}}(\alpha,\lambda)\left[\frac{A_5(\alpha,\lambda)}{3}|f'(a)|^q+\left(\frac{3A_6(\alpha,\lambda)-A_5(\alpha,\lambda)}{3}\right)|f'(b)|^q\right]^{\frac{1}{q}}.$$

If we consider (13)-(15) in (8), then the proof Theorem 3 is completed. □

3. Special cases

Remark 2. If we assign $\lambda = 0$ in (5), then the equality becomes to

$$\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a^+}^\alpha f\left(\frac{2a+b}{3}\right)+J_{\left(\frac{2a+b}{3}\right)^+}^\alpha f\left(\frac{a+2b}{3}\right)+J_{\left(\frac{a+2b}{3}\right)^+}^\alpha f(b)\right]$$
$$-\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]$$
$$=\frac{(b-a)\alpha}{9}\left[I_1^*+I_2^*+I_3^*\right],$$

where

$$\begin{cases} I_1^* = \frac{1}{\alpha}\int_0^1\left(t^\alpha-\frac{5}{8}\right)f'\left(ta+(1-t)\left(\frac{2a+b}{3}\right)\right)dt, \\ I_2^* = \frac{1}{\alpha}\int_0^1\left(t^\alpha-\frac{1}{2}\right)f'\left(t\left(\frac{2a+b}{3}\right)+(1-t)\left(\frac{a+2b}{3}\right)\right)dt, \\ I_3^* = \frac{1}{\alpha}\int_0^1\left(t^\alpha-\frac{3}{8}\right)f'\left(t\left(\frac{a+2b}{3}\right)+(1-t)b\right)dt. \end{cases}$$

This coincides with [24, Lemma 1].

Remark 3. If we select $\lambda = 0$ in (6), then we obtain

$$\left| \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f\left(\frac{2a+b}{3}\right) + J_{\left(\frac{2a+b}{3}\right)^+}^{\alpha} f\left(\frac{a+2b}{3}\right) + J_{\left(\frac{a+2b}{3}\right)^+}^{\alpha} f(b) \right] \right.$$
$$\left. -\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right|$$
$$\leq \frac{(b-a)\alpha}{27} \left[\left(2A_2(\alpha,0) + A_1(\alpha,0) + A_4(\alpha,0) + A_3(\alpha,0) + A_5(\alpha,0)\right)|f'(a)| \right.$$
$$\left. + \left(A_2(\alpha,0) - A_1(\alpha,0) + 2A_4(\alpha,0) - A_3(\alpha,0) + 3A_6(\alpha,0) - A_5(\alpha,0)\right)|f'(b)| \right].$$

Here,

$$\begin{cases} A_1(\alpha,0) = \frac{1}{\alpha}\int\limits_0^1 t\left|t^{\alpha} - \frac{5}{8}\right| dt = \frac{1}{\alpha}\left[\frac{\alpha}{\alpha+2}\left(\frac{5}{8}\right)^{1+\frac{2}{\alpha}} + \frac{1}{\alpha+2} - \frac{5}{16}\right], \\ A_2(\alpha,0) = \frac{1}{\alpha}\int\limits_0^1 \left|t^{\alpha} - \frac{5}{8}\right| dt = \frac{1}{\alpha}\left[\frac{2\alpha}{\alpha+1}\left(\frac{5}{8}\right)^{1+\frac{1}{\alpha}} + \frac{1}{\alpha+1} - \frac{5}{8}\right], \\ A_3(\alpha,0) = \frac{1}{\alpha}\int\limits_0^1 t\left|t^{\alpha} - \frac{1}{2}\right| dt = \frac{1}{\alpha}\left[\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}} + \frac{1}{\alpha+2} - \frac{1}{4}\right], \\ A_4(\alpha,0) = \frac{1}{\alpha}\int\limits_0^1 \left|t^{\alpha} - \frac{1}{2}\right| dt = \frac{1}{\alpha}\left[\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}} + \frac{1}{\alpha+1} - \frac{1}{2}\right], \\ A_5(\alpha,0) = \frac{1}{\alpha}\int\limits_0^1 t\left|t^{\alpha} - \frac{3}{8}\right| dt = \frac{1}{\alpha}\left[\frac{\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{1+\frac{2}{\alpha}} + \frac{1}{\alpha+2} - \frac{3}{16}\right], \\ A_6(\alpha,0) = \frac{1}{\alpha}\int\limits_0^1 \left|t^{\alpha} - \frac{3}{8}\right| dt = \frac{1}{\alpha}\left[\frac{2\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{1+\frac{1}{\alpha}} + \frac{1}{\alpha+1} - \frac{3}{8}\right]. \end{cases}$$

This was established by Sitthiwirattham et al. [24, Theorem 4].

Remark 4. Consider $\lambda = 0$ and $\alpha = 1$ in (6). Then, (6) reduces to

$$\left| \frac{1}{b-a}\int\limits_a^b f(t)\,dt - \frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right|$$
$$\leq \frac{25(b-a)}{576}\left(|f'(a)| + |f'(b)|\right),$$

which was given in [24, Remark 3].

Remark 5. For $\lambda = 0$ in the inequality (7), we have

$$\left| \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f\left(\frac{2a+b}{3}\right) + J_{\left(\frac{2a+b}{3}\right)^+}^{\alpha} f\left(\frac{a+2b}{3}\right) + J_{\left(\frac{a+2b}{3}\right)^+}^{\alpha} f(b) \right] \right.$$
$$\left. -\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right|$$
$$\leq \frac{(b-a)\alpha}{9}\left\{ \left(A_7(\alpha,0,p)\right)^{\frac{1}{p}} \left(\frac{5|f'(a)|^q + |f'(b)|^q}{6}\right)^{\frac{1}{q}} \right.$$
$$\left. + \left(A_8(\alpha,0,p)\right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}} + \left(A_9(\alpha,0,p)\right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + 5|f'(b)|^q}{6}\right)^{\frac{1}{q}} \right\},$$

where

$$\begin{cases} A_7(\alpha,0,p)=\int\limits_0^1\left|\frac{t^\alpha}{\alpha}-\frac{5}{8\alpha}\right|^p dt, \\ A_8(\alpha,0,p)=\int\limits_0^1\left|\frac{t^\alpha}{\alpha}-\frac{1}{2\alpha}\right|^p dt, \\ A_9(\alpha,0,p)=\int\limits_0^1\left|\frac{t^\alpha}{\alpha}-\frac{3}{8\alpha}\right|^p dt. \end{cases}$$

This was established by Sitthiwirattham et al. [24].

REMARK 6. Consider $\lambda=0$ and $\alpha=1$ in (7). Then, (7) coincides with

$$\begin{aligned} &\left|\frac{1}{b-a}\int\limits_a^b f(t)\,dt-\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\right| \\ &\leq \frac{b-a}{9}\left\{\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+5|f'(b)|^q}{6}\right)^{\frac{1}{q}}\right. \\ &\quad+\left(\frac{1}{2^p(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} \\ &\quad\left.+\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{5|f'(a)|^q+|f'(b)|^q}{6}\right)^{\frac{1}{q}}\right\}, \end{aligned}$$

which was presented in [24, Remark 5].

REMARK 7. For $\lambda=0$, the inequality (12) reduces to

$$\begin{aligned} &\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a+}^\alpha f\left(\frac{2a+b}{3}\right)+J_{\left(\frac{2a+b}{3}\right)^+}^\alpha f\left(\frac{a+2b}{3}\right)+J_{\left(\frac{a+2b}{3}\right)^+}^\alpha f(b)\right]\right. \\ &\left.-\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\right| \\ &\leq \frac{(b-a)\,\alpha}{9}\left\{(A_2(\alpha,0))^{1-\frac{1}{q}}\left(\left(\frac{2A_2(\alpha,0)+A_1(\alpha,0)}{3}\right)|f'(a)|^q\right.\right. \\ &\left.+\left(\frac{A_2(\alpha,0)-A_1(\alpha,0)}{3}\right)|f'(b)|^q\right)^{\frac{1}{q}} \\ &+(A_4(\alpha,0))^{1-\frac{1}{q}}\left(\left(\frac{A_4(\alpha,0)+A_3(\alpha,0)}{3}\right)|f'(a)|^q+\left(\frac{2A_4(\alpha,0)-A_3(\alpha,0)}{3}\right)|f'(b)|^q\right)^{\frac{1}{q}} \\ &\left.+(A_6(\alpha,0))^{1-\frac{1}{q}}\left(\frac{A_5(\alpha,0)}{3}|f'(a)|^q+\left(\frac{3A_6(\alpha,0)-A_5(\alpha,0)}{3}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Here, $A_1(\alpha,0)$, $A_2(\alpha,0)$, $A_3(\alpha,0)$, $A_4(\alpha,0)$, $A_5(\alpha,0)$, and $A_6(\alpha,0)$ are defined in Remark 3. For the proof, we refer to [24, Theorem 5].

REMARK 8. Consider $\lambda=0$ and $\alpha=1$ in (12). Then, (12) coincides with

$$\left|\frac{1}{b-a}\int\limits_a^b f(t)\,dt-\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\right|$$

$$\leq \frac{b-a}{36}\left[\left(\frac{17}{16}\right)^{1-\frac{1}{q}}\left(\frac{251\left|f'(a)\right|^{q}+973\left|f'(b)\right|^{q}}{1152}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{17}{16}\right)^{1-\frac{1}{q}}\left(\frac{973\left|f'(a)\right|^{q}+251\left|f'(b)\right|^{q}}{1152}\right)^{\frac{1}{q}}\right],$$

which was presented in [24, Remark 4].

4. Conclusions

Some new versions of Newton-type inequalities are considered for the case of differentiable convex functions by using tempered fractional integrals. More precisely, several Newton-type inequalities for the case of differentiable convex functions are constructed by using the Hölder and power-mean inequality. Moreover, more results are presented by using special choices of obtained inequalities.

The study of Newton-type inequalities by tempered fractional integrals has the potential to lead to new ideas and strategies for mathematicians in the future. Moreover, one can try to generalize our results by using a various version of convex function classes or another type fractional integral operators. Finally, one can obtain these type of inequalities by tempered fractional integrals for the case of convex functions by using quantum calculus.

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