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EXPANSIVE TYPE MAPPINGS IN DISLOCATED QUASI-METRIC SPACE WITH SOME FIXED POINT RESULTS AND APPLICATION

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Abstract. In this paper, we prove some new fixed point results for expansive type mappings in complete dislocated quasi-metric space. A common fixed point result is also established considering such mappings. Suitable examples are provided to demonstrate our results. The solution to a system of Fredholm integral equations is also established to show the applicability of our results.

1. Introduction

Metric fixed point theory is regarded as a cornerstone in analysis. The Banach contraction principle [3] applied to complete metric space marks as crucial and noteworthy advancement in the fixed point theory. Following this breakthrough, numerous researchers have contributed to the field of fixed point theory by establishing various theorems in their respective works [1, 6–8, 12, 13, 18].

In 1986, Matthews [14] introduced some concepts of metric domains in the context of domain theory, where the idea of dislocated metric space first appeared. Later in 2000, Hitzler and Seda [10] introduced the concept of dislocated metric space, in which the self-distance of a point is not necessarily zero. They also generalized the Banach contraction principle in this space. Dislocated metric space has a crucial role in topology, logical programming, electronic engineering and computer science etc. Zeyada et al. [24] presented the complete dislocated quasi-metric space and generalized the result of Hitzler [10] in dislocated quasi-metric space.

In the year 1984, Wang et al. [22] introduced the groundbreaking concept of expansive mapping, conducting a thorough exploration and unveiling intricate fixed point results within the realm of complete metric space. Expansive mappings have applications in dynamical system theory, chaos theory and nonlinear analysis. Following this, a number of researchers have done rigorous investigations, systematically expanding and elaborating fixed point theoretical outcomes in this particular domain [2, 5, 9, 16, 17, 19–21, 23].

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Motivated by this, in this paper, we derive some new fixed point results in the setting of dislocated quasi-metric space, pertaining to both single and paired self-mappings. Our results extend some existing results. The significance of our study is that the results are achieved without imposing the continuity requirement. Examples are provided and an application is given to illustrate practical implication of our findings.

2. Preliminaries

Hitzler et al. [10] presented the notion of dislocated metric space in the following manner.

DEFINITION 2.1. [10] Let X be a nonempty set and $d : X \times X \longrightarrow [0, \infty)$ be a distance function satisfying the following conditions:
(i) $d(x, y) = d(y, x) = 0 \implies x = y \quad \forall x, y \in X$,
(ii) $d(x, y) = d(y, x) \quad \forall x, y \in X$,
(iii) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$.
Then d is called a dislocated metric (d-metric) on X. If d satisfies (i) and (iii), then d is called dislocated quasi-metric (dq-metric) on X and the pair (X, d) is called dislocated quasi-metric space with dq-metric d. Every d-metric space is always dq-metric space, but converse is not true.

EXAMPLE 2.2. Let $X = \mathbb{R}$ and $d : X \times X \longrightarrow [0, \infty)$ be defined by

$$d(x, y) = max\,\{|x|, |y|\} \quad \forall\ x, y \in X.$$

Then X is a d-metric space and also a dq-metric space.
But if $d : X \times X \longrightarrow [0, \infty)$ is defined by

$$d(x, y) = |x| \quad \forall\ x, y \in X,$$

then X is a dq-metric space, but not a d-metric space.

DEFINITION 2.3. [24] A sequence $\{x_n\}$ in a dq-metric space (X, d) is called a Cauchy sequence if for every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$d(x_m, x_n) < \epsilon \quad \forall\ m, n \geq n_0.$$

$\{x_n\}$ is said to be dq-convergent to x if

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

Here, x is called a dq-limit of $\{x_n\}$.
(X, d) is called complete if every Cauchy sequence in X is dq-convergent.

DEFINITION 2.4. [24] Let (X, d_1) and (X, d_2) be two dq-metric spaces. Then the function $f : X \longrightarrow Y$ is said to be continuous if for each sequence $\{x_n\}$, which is d_1q-convergent to x_0 in X, the sequence $\{f(x_n)\}$ is d_2q-convergent to $f(x_0)$ in Y.

LEMMA 2.5. [24] *Every subsequence of dq-convergent sequence to a point* x_0 *is dq-convergent to* x_0.

LEMMA 2.6. [24] *dq-limits in a dq-metric space are unique.*

LEMMA 2.7. [15] *If* x *is a limit of some sequence* $\{x_n\}$ *in a dq-metric space* (X, d), *then* $d(x, x) = 0$.

3. Main results

In this section, we formulate some fixed point results for onto expansive type mapping in a complete dq-metric space.

THEOREM 3.1. *Let (X, d) be a complete dq-metric space and T be an onto self-mapping on X such that*

$$(1) \qquad d(Tx, Ty) \geq \lambda\ min \left\{ \begin{array}{c} \alpha_1 d(x,y),\ \beta_1 \frac{d(Tx,x)d(Ty,y)}{d(x,y)} + \beta_2 d(x,y), \\ \gamma_1 d(Tx,x) + \gamma_2 d(Ty,y) + \gamma_3 d(x,y), \\ \delta_1 d(Tx,y) + \delta_2 d(Ty,x) + \delta_3 d(x,y), \end{array} \right\},$$

for all $x, y \in X$ with $d(x,y) \neq 0$, $\lambda > 1$, nonnegative real numbers $\alpha_1, \beta_i, \gamma_j, \delta_i$ for $i = 1, 2;\ j = 1, 2, 3$ and

$$\frac{1}{\lambda} < min \left\{ \alpha_1, \beta_2, \gamma_3, \delta_1 + \delta_3, \lambda\delta_2(\delta_1 + \delta_2) + \delta_3, \right\}, \delta_3 > 0.$$

Then T has a unique fixed point in X.

Proof. We take

$$\tau(x,y) = min \left\{ \begin{array}{c} \alpha_1 d(x,y),\ \beta_1 \frac{d(Tx,x)d(Ty,y)}{d(x,y)} + \beta_2 d(x,y), \\ \gamma_1 d(Tx,x) + \gamma_2 d(Ty,y) + \gamma_3 d(x,y), \\ \delta_1 d(Tx,y) + \delta_2 d(Ty,x) + \delta_3 d(x,y), \end{array} \right\}.$$

Let $x_0 \in X$. Since T is onto, there exists $x_1 \in X$ such that $x_0 = Tx_1$. Continuing in this way, we define a sequence $\{x_n\}$ in X with $x_{n-1} = Tx_n$, $n \in \mathbb{N}$.

The following cases will arise.
Case 1. If $\tau(x,y) = \alpha_1 d(x,y)$, then

$$(2) \qquad d(Tx, Ty) \geq \lambda\alpha_1 d(x,y) \quad \forall\ x, y \in X.$$

Now, using (2), we get

$$d(x_{n-1}, x_n) = d(Tx_n, Tx_{n+1}) \geq \lambda\alpha_1 d(x_n, x_{n+1}),$$

$$i.e.,\ d(x_n, x_{n+1}) \leq \frac{1}{\lambda\alpha_1} d(x_{n-1}, x_n).$$

Let $h = \frac{1}{\lambda\alpha_1}$. Then the above inequality becomes

$$d(x_n, x_{n+1}) \leq h\ d(x_{n-1}, x_n).$$

Also,

$$d(x_{n+1}, x_{n+2}) \leq h\ d(x_n, x_{n+1}) \leq h^2\ d(x_{n-1}, x_n).$$

From this, we get

$$d(x_n, x_{n+1}) \leq h^n\ d(x_0, x_1),\ n \in \mathbb{N}.$$

For $m > n$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq (h^n + h^{n+1} + \cdots + h^{m-1}) d(x_0, x_1) \\ &\leq \frac{h^n}{1-h} d(x_0, x_1). \end{aligned}$$

Since $h = \frac{1}{\lambda\alpha_1} \in [0,1)$ for $\alpha_1 > \frac{1}{\lambda}$, $\{x_n\}$ is a Cauchy sequence in X and there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.
Since T is onto, we can find $p \in X$ such that $Tp = u$. Now for all $n \in \mathbb{N}$,

$$d(u, x_n) = d(Tp, Tx_{n+1}) \geq \lambda\alpha_1 d(p, x_{n+1}).$$

Since $d(u, .), d(p, .) : X \longrightarrow \mathbb{R}$ are continuous,

$$d(u, u) = \lim_{n\to\infty} d(u, x_n) \geq \lambda\alpha_1 \lim_{n\to\infty} d(p, x_{n+1}),$$

$$i.e., \;\; \lambda\alpha_1 d(p, u) \leq d(u, u).$$

Using Lemma 2.7, since $\alpha_1 > \frac{1}{\lambda}$, we get $d(p, u) = 0$ and similarly, $d(u, p) = 0$. Thus $d(p, u) = d(u, p) = 0$. So $u = p$.

To show the uniqueness, let u and v be two different fixed points of T in X. Then

$$d(Tu, Tv) \geq \lambda\alpha_1 d(u, v),$$

$$i.e., \;\; (\lambda\alpha_1 - 1)d(u, v) \leq 0,$$

which gives $d(u, v) = 0$. Similarly, $d(v, u) = 0$, and thus $u = v$.

Case 2: If $\tau(x, y) = \beta_1 \frac{d(Tx,x)d(Ty,y)}{d(x,y)} + \beta_2 d(x, y)$, then

$$d(Tx, Ty) \geq \lambda \left(\beta_1 \frac{d(Tx, x)d(Ty, y)}{d(x, y)} + \beta_2 d(x, y) \right). \tag{3}$$

Using (3), we get

$$\begin{aligned} d(x_{n-1}, x_n) &= d(Tx_n, Tx_{n+1}) \\ &\geq \lambda\beta_1 \frac{d(Tx_n, x_n)d(Tx_{n+1}, x_{n+1})}{d(x_n, x_{n+1})} + \lambda\beta_2 \; d(x_n, x_{n+1}) \\ &= \lambda\beta_1 \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_n, x_{n+1})} + \lambda\beta_2 \; d(x_n, x_{n+1}) \\ &= \lambda\beta_1 \; d(x_{n-1}, x_n) + \lambda\beta_2 \; d(x_n, x_{n+1}) \\ &\geq \lambda\beta_2 \; d(x_n, x_{n+1}). \end{aligned}$$

This implies that

$$d(x_n, x_{n+1}) \leq h \; d(x_{n-1}, x_n), \;\text{ where } h = \frac{1}{\lambda\beta_2} \in [0, 1).$$

Similar to the previous case, $\{x_n\}$ is a Cauchy sequence in X, which converges to some $u \in X$, which can be shown to be the unique fixed point of T.

Case 3: If $\tau(x, y) = \gamma_1 \; d(Tx, x) + \gamma_2 \; d(Ty, y) + \gamma_3 \; d(x, y)$, then

$$d(Tx, Ty) \geq \lambda \left[\gamma_1 \; d(Tx, x) + \gamma_2 \; d(Ty, y) + \gamma_3 \; d(x, y)\right] \quad \forall \; x, y \in X, \tag{4}$$

and using (4), we have

$$\begin{aligned} d(x_{n-1}, x_n) &\geq \lambda\gamma_1 \; d(x_{n-1}, x_n) + \lambda\gamma_2 \; d(x_n, x_{n+1}) + \lambda\gamma_3 \; d(x_n, x_{n+1}) \\ &\geq \lambda(\gamma_2 + \gamma_3)d(x_n, x_{n+1}), \end{aligned}$$

$$i.e., \;\; d(x_n, x_{n+1}) \leq h \; d(x_{n-1}, x_n), \quad \text{where } h = \frac{1}{\lambda(\gamma_2 + \gamma_3)} \in [0, 1).$$

So $\{x_n\}$ is a Cauchy sequence in X, converging to some $u \in X$, which is the unique fixed point of T.

Case 4: If $\tau(x,y) = \delta_1\ d(Tx,y) + \delta_2\ d(Ty,x) + \delta_3\ d(x,y)$, then

$$d(Tx,Ty) \geq \lambda\,[\delta_1\ d(Tx,y) + \delta_2\ d(Ty,x) + \delta_3\ d(x,y)] \quad \forall\ x,y \in X. \tag{5}$$

Now,

$$\begin{aligned} d(x_{n-1},x_n) &\geq \lambda\delta_1\ d(Tx_n,x_{n+1}) + \lambda\delta_2\ d(Tx_{n+1},x_n) + \lambda\delta_3\ d(x_n,x_{n+1}) \\ &= \lambda\delta_1\ d(x_{n-1},x_{n+1}) + \lambda\delta_2\ d(x_n,x_n) + \lambda\delta_3\ d(x_n,x_{n+1}) \\ &\geq \lambda\delta_2\ d(x_n,x_n) + \lambda\delta_3\ d(x_n,x_{n+1}). \qquad (6) \end{aligned}$$

Also,

$$\begin{aligned} d(x_n,x_n) &= d(Tx_{n+1},Tx_{n+1}) \\ &\geq \lambda\delta_1\ d(Tx_{n+1},x_{n+1}) + \lambda\delta_2\ d(Tx_{n+1},x_{n+1}) + \lambda\delta_3\ d(x_{n+1},x_{n+1}) \\ &\geq \lambda(\delta_1+\delta_2)d(x_n,x_{n+1}). \qquad (7) \end{aligned}$$

Using (7) in (6), we get

$$d(x_{n-1},x_n) \geq \lambda^2\delta_2(\delta_1+\delta_2)d(x_n,x_{n+1}) + \lambda\delta_3\ d(x_n,x_{n+1}),$$

$$i.e.,\ d(x_n,x_{n+1}) \leq h\ d(x_{n-1},x_n), \quad \text{where } h = \frac{1}{\lambda^2\delta_2(\delta_1+\delta_2)+\lambda\delta_3} \in [0,1).$$

Thus $\{x_n\}$ is a Cauchy sequence in X and $\lim_{n\to\infty} x_n = u$, for some $u \in X$. Then u is the unique fixed point of T. □

Now, we demonstrate Theorem 3.1 by an example as follows:

EXAMPLE 3.2. Consider $X = \mathbb{R}$ and define a complete dq-metric $d : X \times X \longrightarrow [0,\infty)$ by $d(x,y) = |y| \quad \forall\ x,y \in X$.
We define an onto self-mapping T on X by

$$Tx = \begin{cases} 3x, & x \leq 3 \\ 2x, & x > 3. \end{cases} \tag{8}$$

Suppose that, $\lambda = \frac{3}{2}, \beta_1 = \gamma_1 = \delta_1 = \delta_2 = 0, \gamma_2 = \frac{1}{2}, \alpha_1 = \beta_2 = \gamma_3 = \delta_3 = 1$.

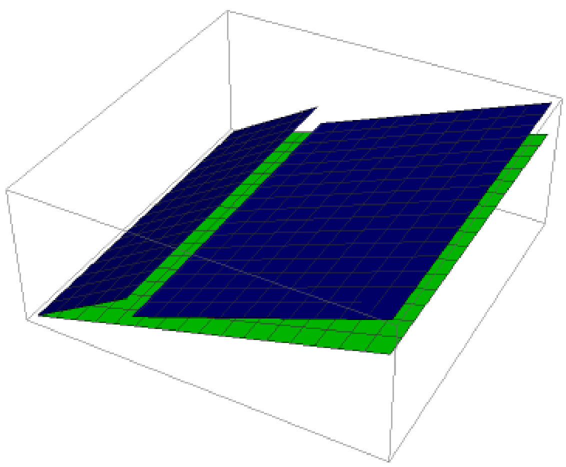

FIGURE 1.

The Figure 1 illustrates the condition (1) of Theorem 3.1, with blue surface representing the left part of the condition and green surface representing the right part of the condition. Thus all the conditions of Theorem 3.1 are satisfied. So T has a unique fixed point, which is clearly 0 here.

Corollary 3.3. *Let (X,d) be a complete dq-metric space and T be an onto self-mapping on X such that*

$$d(Tx,Ty) \geq \lambda\, d(x,y),$$

for all $x,y \in X$ with $\lambda > 1$. Then T has a unique fixed point in X.

Proof. Putting $\beta_1 = \gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 0$ and $\alpha_1 = \beta_2 = \gamma_3 = \delta_3 = 1$ in Theorem 3.1, we get this result. □

Remark 3.4. Corollary 3.3 extends the result of [22] in the framework of dq-metric space.

Corollary 3.5. *Let (X,d) be a complete dq-metric space and T be an onto self-mapping on X such that*

$$d(Tx,Ty) \geq \lambda\ min \left\{ \begin{array}{c} d(x,y),\ \frac{d(Tx,x)d(Ty,y)}{d(x,y)} + d(x,y), \\ d(Tx,x) + d(Ty,y) + d(x,y), \\ d(Tx,y) + d(Ty,x) + d(x,y), \end{array} \right\},$$

for all $x,y \in X$ with $d(x,y) \neq 0$, $\lambda > 1$. Then T has a unique fixed point in X.

Proof. By taking $\alpha_1 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \gamma_3 = \delta_1 = \delta_2 = \delta_3 = 1$ in Theorem 3.1, the result follows easily. □

We apply Corollary 3.5 to prove the next result.

Theorem 3.6. *Let (X,d) be a complete dq-metric space and T be an onto self-mapping on X such that*

$$\begin{aligned} d(Tx,Ty) \geq{}& a_1 d(x,y) + a_2 \left[\frac{d(Tx,x)d(Ty,y)}{d(x,y)} + d(x,y)\right] \\ &+ a_3\left[d(Tx,x) + d(Ty,y) + d(x,y)\right] + a_4\left[d(Tx,y) + d(Ty,x) + d(x,y)\right], \end{aligned} \tag{9}$$

for all $x,y \in X$ with $d(x,y) \neq 0$, where a_1, a_2, a_3, a_4 are non-negative real numbers satisfying $a_1 + a_2 + a_3 + a_4 > 1$. Then the self-mapping T has a unique fixed point in X.

Proof. Let

$$\tau(x,y) = min \left\{ \begin{array}{c} d(x,y),\ \frac{d(Tx,x)d(Ty,y)}{d(x,y)} + d(x,y), \\ d(Tx,x) + d(Ty,y) + d(x,y), \\ d(Tx,y) + d(Ty,x) + d(x,y), \end{array} \right\}. \tag{10}$$

Using (9), we get

$$\begin{aligned} d(Tx,Ty) &\geq a_1\ \tau(x,y) + a_2\ \tau(x,y) + a_3\ \tau(x,y) + a_4\ \tau(x,y) \\ &\geq (a_1 + a_2 + a_3 + a_4)\ \tau(x,y). \end{aligned}$$

Let $\lambda = a_1 + a_2 + a_3 + a_4$. Then the above inequality becomes

$$d(Tx,Ty) \geq \lambda\ \tau(x,y) \quad \text{with } \lambda > 1.$$

Hence Corollary 3.5 concludes the proof. □

In the following, we deduce a common fixed point theorem to a pair of onto expansive type self-mappings.

THEOREM 3.7. *Let (X, d) be a complete dq-metric space and S, T be two onto self-mappings on X such that*

$$d(Sx, Ty) \geq \lambda\ min \left\{ \begin{array}{c} \alpha_1\ d(x,y),\ \beta_1\ \frac{d(Sx,x)d(Ty,y)}{d(x,y)} + \beta_2\ d(x,y), \\ \gamma_1\ d(Sx,x) + \gamma_2\ d(Ty,y) + \gamma_3\ d(x,y), \\ \delta_1\ d(Sx,y) + \delta_2\ d(Ty,x) + \delta_3\ d(x,y), \end{array} \right\}, \tag{11}$$

for all $x, y \in X$ with $d(x,y) \neq 0$, $\lambda > 1$, nonnegative real numbers $\alpha_1, \beta_i, \gamma_j, \delta_i$, for $i = 1,2; j = 1,2,3$ and

$$\frac{1}{\lambda} < min\{\alpha_1, \beta_2, \gamma_3, \delta_3\}.$$

Then S and T have a unique common fixed point in X.

Proof. Let

$$\tau(x,y) = min \left\{ \begin{array}{c} \alpha_1\ d(x,y),\ \beta_1\ \frac{d(Sx,x)d(Ty,y)}{d(x,y)} + \beta_2\ d(x,y), \\ \gamma_1\ d(Sx,x) + \gamma_2\ d(Ty,y) + \gamma_3\ d(x,y), \\ \delta_1\ d(Sx,y) + \delta_2\ d(Ty,x) + \delta_3\ d(x,y), \end{array} \right\}.$$

For $x_0 \in X$, since S, T are onto, there exist $x_1, x_2 \in X$ such that $x_0 = Sx_1, x_1 = Tx_2$. Continuing this process, we define a sequence $\{x_n\}$ by

$$\begin{aligned} &Sx_1 = x_0, \ldots, Sx_{2n-1} = x_{2n-2}, \\ &Tx_2 = x_1, \ldots, Tx_{2n} = x_{2n-1}, \quad n \in \mathbb{N}. \end{aligned} \tag{12}$$

We consider the following cases:
Case 1: If $\tau(x,y) = \alpha_1\ d(x,y)$, then

$$d(Sx, Ty) \geq \lambda\alpha_1\ d(x,y) \quad \forall\ x, y \in X. \tag{13}$$

Using (12) and (13), we get

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(Sx_{2n+1}, Tx_{2n+2}) \geq \lambda\alpha_1 d(x_{2n+1}, x_{2n+2}), \\ i.e.,\ d(x_{2n+1}, x_{2n+2}) &\leq \frac{1}{\lambda\alpha_1} d(x_{2n}, x_{2n+1}) \\ &= h\ d(x_{2n}, x_{2n+1}), \quad \text{where } h = \frac{1}{\lambda\alpha_1}. \end{aligned}$$

Also,

$$d(x_{2n}, x_{2n+1}) \leq h\ d(x_{2n-1}, x_{2n}).$$

So

$$d(x_{2n+1}, x_{2n+2}) \leq h^2\ d(x_{2n-1}, x_{2n}).$$

Thus we get

$$d(x_n, x_{n+1}) \leq h^n\ d(x_0, x_1),\ n \in \mathbb{N}.$$

For $m > n$,

$$d(x_n, x_m) \leq \frac{h^n}{1-h} d(x_0, x_1).$$

Since $h = \frac{1}{\lambda\alpha_1} \in [0,1)$, $\{x_n\}$ is a Cauchy sequence in X and there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Since S, T are onto mappings, we can find $p, q \in X$ such that $Sp = Tq = u$.
For all $n \in \mathbb{N}$,

$$d(x_{2n}, u) = d(Sx_{2n+1}, Tq) \geq \lambda\alpha_1 \ d(x_{2n+1}, q).$$

Using continuity of $d(., u), d(., q) : X \longrightarrow \mathbb{R}$ and Lemma 2.7, we get $d(u, q) = 0$. Similarly $d(q, u) = 0$. Thus $d(u, q) = d(q, u) = 0$, which implies that $u = q$. Similarly, $u = p$.

For the uniqueness, let u and v be two different fixed points of S and T in X. Using (13), we have

$$d(u, v) = d(Su, Tv) \geq \lambda\alpha_1 \ d(u, v),$$

$$i.e., \ (\lambda\alpha_1 - 1)d(u, v) \leq 0,$$

which shows that $u = v$.

Case 2: If $\tau(x, y) = \beta_1 \ \frac{d(Sx,x)d(Ty,y)}{d(x,y)} + \beta_2 \ d(x, y)$, then

$$d(Sx, Ty) \geq \lambda \ \left(\beta_1 \ \frac{d(Sx, x)d(Ty, y)}{d(x, y)} + \beta_2 \ d(x, y)\right) \quad \forall \ x, y \in X. \tag{14}$$

Using (12) and (14), we get

$$d(x_{2n+1}, x_{2n+2}) \leq h \ d(x_{2n}, x_{2n+1}), \quad \text{where } h = \frac{1}{\lambda\beta_2} \in [0, 1).$$

Similar to the previous case, it can be shown that $\{x_n\}$ is a Cauchy sequence in X, which converges to some $u \in X$. Clearly u is the unique common fixed point of S and T.

Case 3: If $\tau(x, y) = \gamma_1 \ d(Sx, x) + \gamma_2 \ d(Ty, y) + \gamma_3 \ d(x, y)$, then

$$d(Sx, Ty) \geq \lambda \ [\gamma_1 \ d(Sx, x) + \gamma_2 \ d(Ty, y) + \gamma_3 \ d(x, y)] \quad \forall \ x, y \in X. \tag{15}$$

Using (12) and (15), we get

$$d(x_{2n+1}, x_{2n+2}) \leq h \ d(x_{2n}, x_{2n+1}), \quad \text{where } h = \frac{1}{\lambda(\gamma_2 + \gamma_3)} \in [0, 1).$$

So $\{x_n\}$ is a Cauchy sequence in X, converging to some $u \in X$. It can be easily shown that u is the unique common fixed point of S and T.

Case 4: If $\tau(x, y) = \delta_1 \ d(Sx, y) + \delta_2 \ d(Ty, x) + \delta_3 \ d(x, y)$, then

$$d(Sx, Ty) \geq \lambda \ [\delta_1 \ d(Sx, y) + \delta_2 \ d(Ty, x) + \delta_3 \ d(x, y)] \quad \forall \ x, y \in X. \tag{16}$$

Using (12) and (16), we get

$$d(x_{2n+1}, x_{2n+2}) \leq h \ d(x_{2n}, x_{2n+1}), \quad \text{where } h = \frac{1}{\lambda\delta_3} \in [0, 1).$$

So $\{x_n\}$ is a Cauchy sequence in X, which converges to some $u \in X$. Then u is the unique common fixed point of S and T. □

Putting $\beta_1 = \gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 0$ and $\alpha_1 = \beta_2 = \gamma_3 = \delta_3 = 1$ in Theorem 3.7, we get the following result.

Corollary 3.8. *Let (X, d) be a complete dq-metric space and S, T be two onto self-mappings on X such that*

$$d(Sx, Ty) \geq \lambda \, d(x, y),$$

for all $x, y \in X$ with $\lambda > 1$. Then S and T have a unique common fixed point in X.

The following example exhibits Theorem 3.7

Example 3.9. Consider $X = \mathbb{R}$ and define a complete dq-metric $d : X \times X \longrightarrow [0, \infty)$ such that

$$d(x, y) = |y| \quad \forall \; x, y \in X.$$

We define two onto self-mappings S and T on X by

$$Sx = \begin{cases} 6x, & x \in X - \{2, 3\} \\ 18, & x = 2 \\ 12, & x = 3 \end{cases} , \quad Tx = \begin{cases} 5x, & x \leq 2 \\ 4x, & x > 2 \end{cases} . \tag{17}$$

Consider $\lambda = \frac{3}{2}, \alpha_1 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \gamma_3 = \delta_1 = \delta_2 = \delta_3 = 1.$

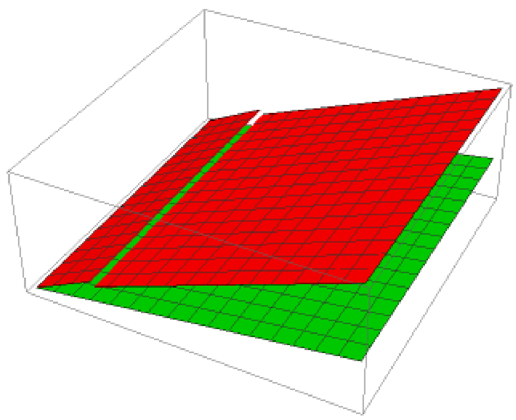

Figure 2.

The Figure 2 depicts the condition (11) of Theorem 3.7, with red surface denoting the left part of the condition and green surface denoting the right part of the condition. Clearly, all the conditions of Theorem 3.7 are satisfied and 0 is the unique common fixed point of S and T here.

In [11], Hu established a proof outlining the characterization of completeness of metric spaces using contraction mapping. Cobzas [4] investigated the characterization of completeness of uniformly Lipschitz connected metric spaces. Here, we provide a result concerning expansive type mappings in the context of characterizing the completeness of dislocated quasi-metric space.

Theorem 3.10. *Let (X, d) be a dq-metric space such that for some $0 < \eta < 1$, d satisfies*

$$d(x, x) \geq \eta \; max \{d(x, y), d(y, x)\} \;\; \forall \; x, y \in X. \tag{18}$$

If for every closed subset Y of X, every onto expansive self-mapping T on Y satisfying

$$d(Tx, Ty) \geq \lambda \; d(x, y) \; \forall \; x, y \in X \text{ with } x \neq y,$$

for some $\lambda > 1$, has a fixed point, then X is complete.

Proof. Let $\{x_n\}$ be an arbitrary Cauchy sequence in (X, d). If we can establish the existence of convergent subsequence, then we are done. Let us assume that $\{x_n\}$ does not have any convergent subsequence. Following [11], we consider

$$\beta(x_n) = min\,[inf\,\{d(x_n, x_m), m > n\}\,, inf\,\{d(x_m, x_n), m > n\}] > 0, \text{ for all } n \in \mathbb{N}.$$

For a real number $c > \frac{1}{\eta^2}$, we construct inductively a subsequence $\{x_{n_k}\}$ such that $d(x_i, x_j) < \frac{1}{c}\beta(x_{n_{k-1}})\ \forall\ i, j \geq n_k, k \in \mathbb{N}$. Then $Y = \{x_{n_k} : k \in \mathbb{N} \cup \{0\}\}$ is a closed subset of X.
Define $T : Y \longrightarrow Y$ by $Tx_{n_0} = x_{n_1}, Tx_{n_k} = x_{n_{k-1}}\ \forall\ k \in \mathbb{N}$. Clearly T is fixed point free.

Let $x, y \in Y$. Then the following cases will arise:
Case 1: Let $x = x_{n_0}, y = x_{n_1}$. Then

$$\begin{aligned} d(Tx, Ty) = d(Tx_{n_0}, Tx_{n_1}) &= d(x_{n_1}, x_{n_0}) \geq \beta(x_{n_0}) > c\ d(x_{n_1}, x_{n_1}) \\ &\geq c\eta\ d(x_{n_0}, x_{n_1}) \ \ \text{(by condition (18))} \\ &> c\eta^2\ d(x_{n_0}, x_{n_1}) = \lambda\ d(x, y), \text{ where } \lambda = c\eta^2. \end{aligned}$$

Case 2: For $x = x_{n_0}, y = x_{n_2}$,

$$\begin{aligned} d(Tx, Ty) = d(Tx_{n_0}, Tx_{n_2}) &= d(x_{n_1}, x_{n_1}) \geq \eta\ d(x_{n_0}, x_{n_1}) \geq \eta\ \beta(x_{n_0}) \\ &> \eta c\ d(x_{n_2}, x_{n_2}) \geq c\eta^2\ d(x_{n_0}, x_{n_2}) \ \ \text{(by condition (18))} \\ &= \lambda\ d(x, y). \end{aligned}$$

Case 3: For $x = x_{n_0}, y = x_{n_k}, k > 2$,

$$\begin{aligned} d(Tx, Ty) = d(Tx_{n_0}, Tx_{n_k}) &= d(x_{n_1}, x_{n_{k-1}}) \geq \beta(x_{n_1}) > c\ d(x_{n_k}, x_{n_k}) \\ &\geq c\eta\ d(x_{n_0}, x_{n_k}) \ \ \text{(by condition (18))} \\ &> c\eta^2\ d(x_{n_0}, x_{n_k}) = \lambda\ d(x, y). \end{aligned}$$

Case 4: For $x = x_{n_k}, y = x_{n_{k+i}}$ for $k, i \in \mathbb{N}$,

$$\begin{aligned} d(Tx, Ty) = d(Tx_{n_k}, Tx_{n_{k+i}}) &= d(x_{n_{k-1}}, x_{n_{k+i-1}}) \geq \beta(x_{n_{k-1}}) > c\ d(x_{n_{k+i}}, x_{n_{k+i}}) \\ &\geq c\eta\ d(x_{n_k}, x_{n_{k+i}}) \ \ \text{(by condition (18))} \\ &> c\eta^2\ d(x_{n_k}, x_{n_{k+i}}) = \lambda\ d(x, y). \end{aligned}$$

The cases when $y = x_{n_0}, x = x_{n_k}, k \in \mathbb{N}$ and $y = x_{n_k}, x = x_{n_{k+i}}, k, i \in \mathbb{N}$ can be shown similarly.
Thus T is an onto expansive mapping satisfying the given condition having no fixed point on the closed subset Y of X. This is a contradiction.
Hence (X, d) is complete. □

4. Application to Fredholm integral equation

We solve a system of Fredholm integral equations as an application of our derived result. Consider the following system of integral equations:

$$\begin{aligned} x(t) &= \int_a^b k_1(t,r,x(r))dr + f(t), \\ y(t) &= \int_a^b k_2(t,r,y(r))dr + g(t), \end{aligned} \tag{19}$$

where $x, y, f, g \in C[a,b]$, which is the space of all continuous real valued functions on $[a,b] \subset \mathbb{R}$, $t, r \in [a,b]$ and $k_1, k_2 : [a,b] \times [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions. We consider x, y as onto mappings.
We take $X = C[a,b]$ with $d : X \times X \longrightarrow \mathbb{R}$ defined by

$$d(x,y) = \sup_{t\in[a,b]} \{|x(t) - y(t)| + |y(t)|\} \ \forall \ x, y \in X.$$

Then (X, d) is a complete dq-metric space.

THEOREM 4.1. *If there exists $\lambda > 1$ such that for all $x, y \in X$*

$$\Big|\int_a^b k_1(t,r,x(r))dr\Big| \geq \lambda \ |x(t) - y(t)| + \lambda \ |y(t)| \ \forall \ t \in [a,b], \tag{20}$$

then the system of integral equations (19) has a unique solution.

Proof. For $x, y \in X$, we take

$$\begin{aligned} F_x(t) &= \int_a^b k_1(t,r,x(r))dr, \\ G_y(t) &= \int_a^b k_2(t,r,y(r))dr, \end{aligned}$$

and define

$$\begin{aligned} Sx(t) &= F_x(t) + f(t) \\ Ty(t) &= G_y(t) + g(t) \ \forall \ t \in [a,b]. \end{aligned}$$

Then S and T are two onto self-mappings on X. Then the existence of solution to (19) is equivalent to the existence of common fixed point of S and T. Now,

$$\begin{aligned} d(Sx, Ty) &= \sup_{t\in[a,b]} \{|Sx(t) - Ty(t)| + |Ty(t)|\} \\ &= \sup_{t\in[a,b]} \{|F_x(t) + f(t) - G_y(t) - g(t)| + |G_y(t) + g(t)|\} \\ &\geq \sup_{t\in[a,b]} \{|F_x(t) + f(t) - G_y(t) - g(t) + G_y(t) + g(t)|\} \\ &= \sup_{t\in[a,b]} \{|F_x(t) + f(t)|\} \\ &\geq \sup_{t\in[a,b]} |F_x(t)| \\ &\geq \sup_{t\in[a,b]} \lambda \ \{|x(t) - y(t)| + |y(t)|\} \quad \text{(by condition (20))} \\ &= \lambda \ d(x,y). \end{aligned}$$

Therefore, all the conditions of Corollary 3.8 are satisfied. Hence the system of integral equations (19) has a unique solution. □

5. Conclusion

In this paper, we have extended and refined some fixed point results of [15] for single expansive type mapping in the framework of dq-metric space. Also, we established a common fixed point theorem considering such mappings. A characterization of completeness of dq-metric space is derived using expansive mappings. Furthermore, we have demonstrated the practical implications of our findings through their application in solving system of Fredholm integral equations. The study of the concept of coupled fixed point and proximity point for expansive type mappings in dq-metric space is a scope for further discussion. Additionally, the convergence of iteration schemes may be explored considering such mappings in convex dq-metric space.

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