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ON THE WEAKENED HYPOTHESES-BASED GENERALIZATIONS OF THE ENESTRÖM-KAKEYA THEOREM

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Abstract. According to the well-known Eneström-Kakeya Theorem, all the zeros of a polynomial $P(z) = \sum_{s=0}^{n} a_s z^s$ of degree n with real coefficients satisfying $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$ lie in the complex plane $|z| \leq 1$. We provide comparable results with hypotheses relating to the real and imaginary parts of the coefficients as well as the coefficients' moduli in response to recent findings about an Eneström-Kakeya "type" condition on real coefficients. Our findings so broadly extend the other previous findings.

1. Introduction

The classical Eneström-Kakeya Theorem addresses where the complex zeros of a real polynomial with non-negative monotone coefficients are located. Gustav Eneström [3] and Sòichi Kakeya [8] separately demonstrated it in 1893 and 1912, respectively.

Theorem 1.1. ***(Eneström-Kakeya)*** *If $P(z) = \sum_{s=0}^{n} a_s z^s$ is a complex polynomial of degree n and its real coefficients meet the conditions $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$, then all the zeros of $P(z)$ are located in the circle $|z| \leq 1$.*

There are several different generalizations of the Eneström-Kakeya Theorem. The majority of these entail loosening the restriction on the coefficients. See [5] for a summary of these outcomes until 2014. Govil and Rahman [6], for instance, demonstrated the following in 1968.

Theorem 1.2. *If $P(z) = \sum_{s=0}^{n} a_s z^s$ is a polynomial of degree n with complex coefficients such that $|arg(a_s) - \mu| \leq \theta \leq \pi/2$ for $0 \leq s \leq n$ for some real μ, and $|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_0|$, then all the zeros of P lie in*

$$|z| \leq \cos\theta + \sin\theta + \frac{2\sin\theta}{|a_n|}\sum_{s=0}^{n-1}|a_s|.$$

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Theorem 1.2 reduces to the Eneström-Kakeya Theorem 1.1 when $\mu = \theta = 0$.
The following result regarding a monotonicity constraint on the real and imaginary parts of the coefficients was stated by Gardner and Govil $\big($ [4], Corollary 1$\big)$ as a corollary to a more comprehensive result.

Theorem 1.3. *Let $P(z) = \sum_{s=0}^{n} a_s z^s$ be a polynomial of degree n with complex coefficients where $Re(a_s) = \alpha_s$ and $Im(a_s) = \beta_s$ for $s = 0, 1, \cdots, n$. Suppose that $\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_0$ and $\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_0$. Then all the zeros of P lie in*

$$\frac{|a_0|}{|a_n| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)} \leq |z| \leq \frac{|a_0| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)}{|a_n|}.$$

According to Theorem 1.3, a result of Joyal, Labelle, and Rahman [[7], Theorem 3] is implied when $\beta_s = 0$ for $s = 0, 1, \cdots, n$.
In 2012, Aziz and Zargar [1] published the following result, which involves a slight generalization of the Eneström-Kakeya monotonicity condition on the real coefficients of a polynomial.

Theorem 1.4. *Let $P(z) = \sum_{s=0}^{n} a_s z^s$ be a polynomial of degree n with real coefficients such that, for some positive numbers k and ρ with $k \geq 1$ and $0 < \rho \leq 1$, the coefficients satisfy $ka_n \geq a_{n-1} \geq \cdots \geq a_1 \geq \rho a_0 \geq 0$. Then all the zeros of P lie in the closed disk*

$$|z + k - 1| \leq k + 2a_0(1 - \rho)/a_n.$$

The Eneström-Kakeya Theorem 1.1 is implied by Theorem 1.4 with $k = \rho = 1$.
Shah et al. [9] recently established the following result, which keeps the monotonicity criterion on the "central" coefficients but does not impose one on the "tail" coefficients.

Theorem 1.5. *Let $P(z) = \sum_{s=0}^{n} a_s z^s$ be a polynomial of degree n with real coefficients such that, for some positive numbers p and q with $0 \leq q \leq p \leq n$, the coefficients satisfy $a_p \geq a_{p-1} \geq \cdots \geq a_{q+1} \geq a_q$. Then all the zeros of P lie in the closed annulus*

$$\min\left\{1, \frac{|a_0|}{N_q - a_q + a_p + M_p + |a_n|}\right\} \leq |z| \leq \frac{|a_0| + N_q - a_q + a_p + M_p}{|a_n|},$$

where $N_q = \sum\limits_{s=1}^{q} |a_s - a_{s-1}|$ and $M_p = \sum\limits_{s=p+1}^{n} |a_s - a_{s-1}|$.

A result from Joyal, Labelle, and Rahman [8] is implied by Theorem 1.5 for $p = n$ and $q = 0$.
Furthermore, when $a_0 \geq 0$, it entails the Eneström-Kakeya Theorem 1.1.
This study combines Theorems 1.4 and 1.5's hypotheses and applies them to polynomials with complex coefficients while allowing the possibility of $\alpha_s \leq \alpha_{s-1}$ and $\beta_s \leq \beta_{s-1}$ for $1 \leq s \leq n$. We apply the hypotheses to the real and imaginary parts of the coefficients, and to the moduli of the coefficients.

2. Statement of Results

Theorem 2.1. *Let $P(z) = \sum\limits_{s=0}^{n} a_s z^s$ be a polynomial of degree n with complex coefficients. Let $\alpha_s = Re(a_s)$ and $\beta_s = Im(a_s)$ for $0 \leq s \leq n$. Suppose that, for some* *positive numbers* k_s, k'_s, ρ, ρ', p *and* q *with* $k_s \geq 1$, $k'_s \geq 1$, $0 < \rho \leq 1$, $0 < \rho' \leq 1$, $1 \leq s \leq p-q$ *and* $0 \leq q \leq p \leq n$, *the coefficients satisfy*

$$\rho\alpha_q \leq k_1\alpha_{q+1} \leq k_2\alpha_{q+2} \leq \cdots \leq k_{p-q-1}\alpha_{p-1} \leq k_{p-q}\alpha_p,$$

and

$$\rho'\beta_q \leq k'_1\beta_{q+1} \leq k'_2\beta_{q+2} \leq \cdots \leq k'_{p-q-1}\beta_{p-1} \leq k'_{p-q}\beta_p.$$

Then all the zeros of P *lie in the closed annulus*

$$\min\Bigg\{1 \quad , \quad |a_0|\Bigg/\Bigg(N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| + 2\sum_{s=1}^{p-q}(k_s-1)|\alpha_{q+s}|$$

$$+(k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| + 2\sum_{s=1}^{p-q}(k'_s-1)|\beta_{q+s}| + M_p + |a_n|\Bigg)\Bigg\} \leq |z|$$

$$\leq \Bigg(|a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| + 2\sum_{s=1}^{p-q}(k_s-1)|\alpha_{q+s}|$$

$$+(k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| + 2\sum_{s=1}^{p-q}(k'_s-1)|\beta_{q+s}| + M_p\Bigg)\Bigg/|a_n|,$$

where $N_q = \sum\limits_{s=1}^{q} |a_s - a_{s-1}|$ and $M_p = \sum\limits_{s=p+1}^{n} |a_s - a_{s-1}|$.

It is important to note that Theorem 2.1 becomes Theorem 1.5 when $\beta_s = 0$ for $0 \leq s \leq n$ and $k_s = \rho = 1$ for $1 \leq s \leq p-q$ and also Theorem 2.1 reduces to a result of Joyal, Labelle, and Rahman [8] when $\beta_s = 0$ for $0 \leq s \leq n$, $k_s = \rho = 1$ for $1 \leq s \leq p-q$, $q = 0$, and $p = n$. It further simplifies to the Enestrom-Kakeya Theorem 1.1 if, moreover, $a_0 \geq 0$.

The following example illustrates Theroem 2.1.

Example: Consider the polynomial

$$100z^6 + (-1-i)z^5 + (5+4i)z^4 + (1+i)z^3 + (2+i)z^2 + (2-2i)z + 5.$$

Take, $q = 1,\ p = 4$, so that $N_q = \sqrt{13}$ and $M_p = \sqrt{61} + \sqrt{10202}$.

Choose,

$$\rho = 1/3,\ k_1 = 1,\ k_2 = 2,\ k_3 = 1,$$

$$\rho' = 1/2,\ k'_1 = 1,\ k'_2 = 1,\ k'_3 = 1.$$

Clearly, the coefficients satisfy the hypotheses

$$\rho\alpha_q \leq\ k_1\alpha_{q+1} \leq k_2\alpha_{q+2} \leq k_3\alpha_{q+3},$$

$$\rho'\beta_q \leq\ k'_1\beta_{q+1} \leq k'_2\beta_{q+2} \leq k'_3\beta_{q+3}.$$

Applying Theorem 2.1, it is easy to verify that all the zeros of p lie in $\alpha \leq |z| \leq \beta$, where $\alpha \approx 0.02246$ and $\beta \approx 1.3108$.

Next, we prove the following result by imposing a similar hypothesis as that of Theorem 2.1 on the moduli of the coefficients of a polynomial.

Theorem 2.2. *Let* $P(z) = \sum\limits_{s=0}^{n} a_s z^s$ *be a polynomial of degree* n *with complex coefficients. Suppose that, for some positive numbers* k_s, ρ, p *and* q *with* $k_s \geq 1$,

$0 < \rho \leq 1$, $1 \leq s \leq p-q$ *and* $0 \leq q \leq p \leq n$, *the coefficients satisfy* $|\arg(k_s a_{q+s}) - \mu| \leq \theta \leq \pi/2$ *for some real* μ, *and*

$$\rho|a_q| \leq k_1|a_{q+1}| \leq k_2|a_{q+2}| \leq \cdots \leq k_{p-q-1}|a_{p-1}| \leq k_{p-q}|a_p|.$$

Then all the zeros of P *lie in the closed annulus*

$$\min\Bigg\{1 \quad , \quad |a_0| \Bigg/ \Bigg(N_q + |a_q| + 2\sum_{s=1}^{p-q-1}(k_s - 1)|a_{q+s}| + \rho|a_q|\big(\sin\theta - \cos\theta - 1\big)$$
$$+2\sin\theta\sum_{s=1}^{p-q-1} k_s|a_{q+s}| + k_{p-q}|a_p|\big(\sin\theta + \cos\theta + 1\big) - |a_p| + M_p$$
$$+|a_n|\Bigg)\Bigg\} \leq |z| \leq \frac{1}{|a_n|}\Bigg(|a_0| + N_q + |a_q| + 2\sum_{s=1}^{p-q-1}(k_s - 1)|a_{q+s}| + \rho|a_q|\big(\sin\theta$$
$$-\cos\theta - 1\big) + 2\sin\theta\sum_{s=1}^{p-q-1} k_s|a_{q+s}| + k_{p-q}|a_p|\big(\sin\theta + \cos\theta + 1\big) - |a_p| + M_p\Bigg),$$

where N_q and M_p are same as defined in Theorem 2.1.

Remark 2.3. For $q = 0$, $p = n$ and $\rho = k_s = 1$, $1 \leq s \leq n$, Theorem 2.2 implies that all the zeros of P lie in

$$|z| \leq \frac{1}{|a_n|}\Bigg(\big(\cos\theta + \sin\theta\big)|a_n| + |a_0|\big(1 - \cos\theta - \sin\theta\big) + 2\sin\theta\sum_{s=0}^{n-1}|a_s|\Bigg).$$

This is an improvement of Theorem 1.2. More over choosing $\theta = \mu = 0$, it results in the Eneström-Kakeya Theorem 1.1.

The class of polynomials of the form $P(z) = a_0 + \sum_{s=m}^{n} a_s z^s$ was taken into consideration in relation to Bernstein type inequalities by Chan and Malik [2]. By enforcing the inequality hypotheses of Theorems 2.1 and 2.2 to this class of polynomials, we get the following.

Corollary 2.4. *Let* $P(z) = a_0 + \sum_{s=m}^{n} a_s z^s$ *be a polynomial of degree* n *with complex coefficients. Let* $\alpha_s = Re(a_s)$ *and* $\beta_s = Im(a_s)$ *for* $0 \leq s \leq n$. *Suppose that, for some positive numbers* k_s, k'_s, ρ, ρ', p *and* q *with* $k_s \geq 1$, $k'_s \geq 1$, $0 < \rho \leq 1$, $0 < \rho' \leq 1$, $1 \leq s \leq p-q$ *and* $0 \leq q \leq p \leq n$, *the coefficients satisfy*

$$\rho\alpha_q \leq k_1\alpha_{q+1} \leq k_2\alpha_{q+2} \leq \cdots \leq k_{p-q-1}\alpha_{p-1} \leq k_{p-q}\alpha_p,$$

and

$$\rho'\alpha_q \leq k'_1\alpha_{q+1} \leq k'_2\alpha_{q+2} \leq \cdots \leq k'_{p-q-1}\alpha_{p-1} \leq k'_{p-q}\alpha_p.$$

Then all the zeros of P lie in the closed annulus defined in Theorem 2.1 where N_q is to be replaced as $N_q = \sum_{s=m}^{q}|a_s - a_{s-1}|$.

Corollary 2.5. *Let* $P(z) = a_0 + \sum_{s=m}^{n} a_s z^s$ *be a polynomial of degree* n *with complex coefficients. Suppose that, for some positive numbers* k_s, ρ, p *and* q *with* $k_s \geq 1$, $0 < \rho \leq 1$, $1 \leq s \leq p-q$ *and* $0 \leq q \leq p \leq n$, *the coefficients satisfy*

$|\arg(k_s a_{q+s}) - \mu| \leq \theta \leq \pi/2$ *for some real* μ, *and*

$$\rho|a_q| \leq k_1|a_{q+1}| \leq k_2|a_{q+2}| \leq \cdots \leq k_{p-q-1}|a_{p-1}| \leq k_{p-q}|a_p|.$$

Then all the zeros of P lie in the closed annulus defined in Theorem 2.1 where N_q is to be replaced as $N_q = \sum\limits_{s=m}^{q} |a_s - a_{s-1}|$.

3. Lemma

In proving Theorem 2.2, we require the following lemma by Govil and Rahman $\big($ [6], equation (6)$\big)$.

Lemma 3.1. *Let* $\{a_s\}_{s=1}^{n}$ *be a set of complex numbers which satisfy* $|\arg(a_s) - \mu| \leq \theta \leq \pi/2$ *for* $0 \leq s \leq n$ *and for some real* μ. *Suppose that* $|a_0| \leq |a_1| \leq \cdots \leq |a_n|$, *then*

$$|a_k - a_{k-1}| \leq (|a_k| - |a_{k-1}|)\cos\theta + (|a_k| + |a_{k-1}|)\sin\theta, \ \ 1 \leq k \leq n.$$

4. Proof of the Results

We first give a proof of Theorem 2.1.

Proof. Let $P(z) = \sum_{s=0}^{n} a_s z^s,\ a_n \neq 0$ be the given polynomial satisfying the stated hypotheses. Define f by the equation

$$P(z)(1-z) \ = \ a_0 + \sum_{s=1}^{n}(a_s - a_{s-1})z^s = f(z) - a_n z^{n+1}.$$

For $|z| = 1$, we have

$$\begin{aligned} |f(z)| &= \Big|a_0 + \sum_{s=1}^{n}(a_s - a_{s-1})z^s\Big| \\ &\leq |a_0| + \sum_{s=1}^{n}|a_s - a_{s-1}| \\ &= |a_0| + \sum_{s=1}^{q}|a_s - a_{s-1}| + |\alpha_{q+1} + i\beta_{q+1} - \alpha_q - i\beta_q| + |\alpha_{q+2} + i\beta_{q+2} - \alpha_{q+1} - i\beta_{q+1}| \\ &\quad + \cdots + |\alpha_{p-1} + i\beta_{p-1} - \alpha_{p-2} - i\beta_{p-2}| + |\alpha_p + i\beta_p - \alpha_{p-1} - i\beta_{p-1}| + \sum_{s=p+1}^{n}|a_s - a_{s-1}|. \end{aligned}$$

Let $N_q = \sum\limits_{s=1}^{q}|a_s - a_{s-1}|$ and $M_p = \sum\limits_{s=p+1}^{n}|a_s - a_{s-1}|$. Therefore for $|z| = 1$, we get

$$\begin{aligned} |f(z)| \ \leq \ & |a_0| + N_q + \big(|\alpha_{q+1} - \alpha_q| + |\alpha_{q+2} - \alpha_{q+1}| + \cdots + |\alpha_{p-1} - \alpha_{p-2}| + |\alpha_p - \alpha_{p-1}|\big) \\ & \big(|\beta_{q+1} - \beta_q| + |\beta_{q+2} - \beta_{q+1}| + \cdots + |\beta_{p-1} - \beta_{p-2}| + |\beta_p - \beta_{p-1}|\big) + M_p. \end{aligned}$$

That is

$$
\begin{aligned}
|f(z)| \leq\ & |a_0| + N_q + |k_1\alpha_{q+1} - \rho\alpha_q + \alpha_q(\rho-1) + \alpha_{q+1}(1-k_1)| + |k_2\alpha_{q+2} - k_1\alpha_{q+1} \\
& +\alpha_{q+1}(k_1-1) + \alpha_{q+2}(1-k_2)| + \cdots + |k_{p-q}\alpha_p - k_{p-q-1}\alpha_{p-1} + \alpha_{p-1}(k_{p-q-1}-1) \\
& +\alpha_p(1-k_{p-q})| + |k'_1\beta_{q+1} - \rho'\beta_q + \beta_q(\rho'-1) + \beta_{q+1}(1-k'_1)| + |k'_2\beta_{q+2} - k'_1\beta_{q+1} \\
& +\beta_{q+1}(k'_1-1) + \beta_{q+2}(1-k'_2)| + \cdots + |k'_{p-q}\beta_p - k'_{p-q-1}\beta_{p-1} + \beta_{p-1}(k'_{p-q-1}-1) \\
& +\beta_p(1-k'_{p-q})| + M_p \\
\leq\ & |a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| + 2\sum_{s=1}^{p-q}(k_s-1)|\alpha_{q+s}| \\
& +(k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| + 2\sum_{s=1}^{p-q}(k'_s-1)|\beta_{q+s}| + M_p.
\end{aligned}
$$

Notice that, we have

$$\max_{|z|=1}\left|z^n f\left(\frac{1}{z}\right)\right| = \max_{|z|=1}\left|f\left(\frac{1}{z}\right)\right| = \max_{|z|=1}|f(z)|,$$

it is clear that $z^n f\left(\frac{1}{z}\right)$ has the same bound on $|z|=1$ as f, that is

$$
\begin{aligned}
\left|z^n f\left(\frac{1}{z}\right)\right| \leq\ & |a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| + 2\sum_{s=1}^{p-q}(k_s-1)|\alpha_{q+s}| \\
& +(k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| + 2\sum_{s=1}^{p-q}(k'_s-1)|\beta_{q+s}| + M_p \quad \text{for } |z|=1.
\end{aligned}
$$

Since $z^n f(\frac{1}{z})$ is a polynomial and hence is analytic in $|z| \leq 1$, it follows by Maximum Modulus Theorem, that

$$
\begin{aligned}
\left|z^n f\left(\frac{1}{z}\right)\right| \leq\ & |a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| + 2\sum_{s=1}^{p-q}(k_s-1)|\alpha_{q+s}| \\
& +(k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| + 2\sum_{s=1}^{p-q}(k'_s-1)|\beta_{q+s}| + M_p \text{ for } |z| \leq 1.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left|f\left(\frac{1}{z}\right)\right| \leq\ & \frac{1}{|z|^n}\Bigg(|a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| + 2\sum_{s=1}^{p-q}(k_s-1)|\alpha_{q+s}| \\
& +(k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| + 2\sum_{s=1}^{p-q}(k'_s-1)|\beta_{q+s}| + M_p\Bigg) \text{ for } |z| \leq 1.
\end{aligned}
$$

Replacing z by $1/z$, we see that

$$
\begin{aligned}
|f(z)| \leq\ & \Bigg(|a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| + 2\sum_{s=1}^{p-q}(k_s-1)|\alpha_{q+s}| \\
& +(k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| + 2\sum_{s=1}^{p-q}(k'_s-1)|\beta_{q+s}| + M_p\Bigg)|z|^n \text{ for } |z| \geq 1.
\end{aligned}
$$

Therefore, for $|z| \geq 1$, we have

$$\begin{aligned} |(1-z)P(z)| &= |f(z) - a_n z^{n+1}| \geq |a_n||z|^{n+1} - |f(z)| \\ &\geq |z|^n \Bigg[|a_n||z| - \Bigg(|a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| \\ &\quad + 2\sum_{s=1}^{p-q}(k_s - 1)|\alpha_{q+s}| + (k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| \\ &\quad + 2\sum_{s=1}^{p-q}(k'_s - 1)|\beta_{q+s}| + M_p \Bigg) \Bigg]. \end{aligned}$$

Hence, if

$$\begin{aligned} |z| &> \frac{1}{|a_n|}\Bigg(|a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| + 2\sum_{s=1}^{p-q}(k_s - 1)|\alpha_{q+s}| \\ &\quad + (k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| + 2\sum_{s=1}^{p-q}(k'_s - 1)|\beta_{q+s}| + M_p \Bigg), \end{aligned}$$

then $|(1-z)P(z)| > 0$, that is $(1-z)P(z) \neq 0$ and so $P(z) \neq 0$. Therefore all the zeros of P lie in

$$\begin{aligned} |z| &\leq \frac{1}{|a_n|}\Bigg(|a_0| + N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| + 2\sum_{s=1}^{p-q}(k_s - 1)|\alpha_{q+s}| \\ &\quad + (k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| + 2\sum_{s=1}^{p-q}(k'_s - 1)|\beta_{q+s}| + M_p \Bigg). \end{aligned}$$

Next, consider the polynomial

$$T(z) = z^n P\left(\frac{1}{z}\right) = \sum_{s=0}^{n} a_s z^{n-s}.$$

Let

$$\begin{aligned} R(z) &= (1-z)T(z) \\ &= -a_0 z^{n+1} + \sum_{s=1}^{q}(a_{s-1} - a_s)z^{n+1-s} + \sum_{s=q+1}^{p}(a_{s-1} - a_s)z^{n+1-s} + \sum_{s=p+1}^{n}(a_{s-1} - a_s)z^{n+1-s} + a_n. \end{aligned}$$

With $\alpha_s = Re(a_s)$ and $\beta_s = Im(a_s)$ for $q \leq s \leq p$, we have

$$\begin{aligned}
|R(z)| &\geq |a_0||z|^{n+1} - \Bigg(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_q - a_{q+1}||z|^{n-q} + \cdots \\
&\quad + |a_p - a_{p+1}||z|^{n-p} + \cdots + |a_{n-1} - a_n||z| + |a_n|\Bigg) \\
&= |a_0||z|^{n+1} - \Bigg(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} \\
&\quad + |\alpha_q + i\beta_q - \alpha_{q+1} - i\beta_{q+1}| + |a_{q-1} - a_q||z|^{n-q} + |\alpha_{q+1} + i\beta_{q+1} - \alpha_{q+2} \\
&\quad - i\beta_{q+2}||z|^{n-q-1} + \cdots + |\alpha_{p-1} + i\beta_{p-1} - \alpha_p - i\beta_p||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} + \\
&\quad \cdots + |a_{n-1} - a_n||z| + |a_n|\Bigg) \\
&\geq |a_0||z|^{n+1} - \Bigg(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} \\
&\quad + |\rho\alpha_q - k_1\alpha_{q+1} + \alpha_q(1-\rho) + \alpha_{q+1}(k_1 - 1)||z|^{n-q} + |\rho'\beta_q - k_1'\beta_{q+1} \\
&\quad + \beta_q(1-\rho') + \beta_{q+1}(k_1' - 1)||z|^{n-q} + |k_1\alpha_{q+1} - k_2\alpha_{q+2} + \alpha_{q+1}(1 - k_1) \\
&\quad + \alpha_{q+2}(k_2 - 1)||z|^{n-q-1} + |k_1'\beta_{q+1} - k_2'\beta_{q+2} + \beta_{q+1}(1 - k_1') \\
&\quad + \beta_{q+2}(k_2' - 1)||z|^{n-q-1} + \cdots + |k_{p-q-1}\alpha_{p-1} - k_{p-q}\alpha_p + \alpha_{p-1}(1 - k_{p-q-1}) \\
&\quad + \alpha_p(k_{p-q} - 1)||z|^{n-p+1} + |k'_{p-q-1}\beta_{p-1} - k'_{p-q}\beta_p + \beta_{p-1}(1 - k'_{p-q-1}) \\
&\quad + \beta_p(k'_{p-q} - 1)||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} + \cdots + |a_{n-1} - a_n||z| + |a_n|\Bigg) \\
&\geq |z|^n\Bigg[|a_0||z| - \Bigg(|a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \ldots + \frac{|a_{q-1} - a_q|}{|z|^{q-1}} + \frac{k_1\alpha_{q+1} - \rho\alpha_q}{|z|^q} \\
&\quad + \frac{|\alpha_q|(1-\rho)}{|z|^q} + \frac{|\alpha_{q+1}|(k_1 - 1)}{|z|^q} + \frac{k_1'\beta_{q+1} - \rho'\beta_q}{|z|^q} + \frac{|\beta_q|(1-\rho')}{|z|^q} \\
&\quad + \frac{|\beta_{q+1}|(k_1' - 1)}{|z|^q} + \frac{k_2\alpha_{q+2} - k_1\alpha_{q+1}}{|z|^{q+1}} + \frac{|\alpha_{q+1}|(k_1 - 1)}{|z|^{q+1}} + \frac{|\alpha_{q+2}|(k_2 - 1)}{|z|^{q+1}} \\
&\quad + \frac{k_2'\beta_{q+2} - k_1'\beta_{q+1}}{|z|^{q+1}} + \frac{|\beta_{q+1}|(k_1' - 1)}{|z|^{q+1}} + \frac{|\beta_{q+2}|(k_2' - 1)}{|z|^{q+1}} + \cdots + \\
&\quad \frac{k_{p-q}\alpha_p - k_{p-q-1}\alpha_{p-1}}{|z|^{p-1}} + \frac{|\alpha_{p-1}|(k_{p-q-1} - 1)}{|z|^{p-1}} + \frac{|\alpha_p|(k_{p-q} - 1)}{|z|^{p-1}} + \\
&\quad \frac{k'_{p-q}\beta_p - k'_{p-q-1}\beta_{p-1}}{|z|^{p-1}} + \frac{|\beta_{p-1}|(k'_{p-q-1} - 1)}{|z|^{p-1}} + \frac{|\beta_p|(k'_{p-q} - 1)}{|z|^{p-1}} + \frac{|a_p - a_{p+1}|}{|z|^p} \\
&\quad + \cdots + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\Bigg)\Bigg].
\end{aligned}$$

Now for $|z|>1$, implies $\dfrac{1}{|z|^{n-s}}<1,\ 0\le s\le n-1$, so that we have

$$\begin{aligned}
|R(z)| &\ge |z|^n\Bigg[|a_0||z|-\Bigg(|a_0-a_1|+|a_1-a_2|+\cdots+|a_{q-1}-a_q|+(k_{p-q}\alpha_p-\rho\alpha_q)\\
&\quad+(1-\rho)|\alpha_q|+2\sum_{s=1}^{p-q}(k_s-1)|\alpha_{q+s}|+(k'_{p-q}\beta_p-\rho'\beta_q)+(1-\rho')|\beta_q|\\
&\quad+2\sum_{s=1}^{p-q}(k'_s-1)|\beta_{q+s}|+|a_p-a_{p+1}|+\cdots+|a_{n-1}-a_n|+|a_n|\Bigg)\Bigg]\\
&= |z|^n\Bigg[|a_0||z|-\Bigg(\sum_{s=1}^{q}|a_{s-1}-a_s|+(k_{p-q}\alpha_p-\rho\alpha_q)+(1-\rho)|\alpha_q|\\
&\quad+2\sum_{s=1}^{p-q}(k_s-1)|\alpha_{q+s}|+(k'_{p-q}\beta_p-\rho'\beta_q)+(1-\rho')|\beta_q|\\
&\quad+2\sum_{s=1}^{p-q}(k'_s-1)|\beta_{q+s}|+\sum_{s=p+1}^{n}|a_{s-1}-a_s|+|a_n|\Bigg)\Bigg]\\
&= |z|^n\Bigg[|a_0||z|-\Bigg(N_q+(k_{p-q}\alpha_p-\rho\alpha_q)+(1-\rho)|\alpha_q|+2\sum_{s=1}^{p-q}(k_s-1)|\alpha_{q+s}|\\
&\quad+(k'_{p-q}\beta_p-\rho'\beta_q)+(1-\rho')|\beta_q|+2\sum_{s=1}^{p-q}(k'_s-1)|\beta_{q+s}|+M_p+|a_n|\Bigg)\Bigg]\\
&> 0,
\end{aligned}$$

if

$$\begin{aligned}
|z| &> \frac{1}{|a_0|}\Bigg(N_q+(k_{p-q}\alpha_p-\rho\alpha_q)+(1-\rho)|\alpha_q|+2\sum_{s=1}^{p-q}(k_s-1)|\alpha_{q+s}|\\
&\quad+(k'_{p-q}\beta_p-\rho'\beta_q)+(1-\rho')|\beta_q|+2\sum_{s=1}^{p-q}(k'_s-1)|\beta_{q+s}|+M_p+|a_n|\Bigg).
\end{aligned}$$

Therefore all the zeros of $R(z)$ whose modulus is greater than 1 lie in

$$\begin{aligned}
|z| &\le \frac{1}{|a_0|}\Bigg(N_q+(k_{p-q}\alpha_p-\rho\alpha_q)+(1-\rho)|\alpha_q|+2\sum_{s=1}^{p-q}(k_s-1)|\alpha_{q+s}|\\
&\quad+(k'_{p-q}\beta_p-\rho'\beta_q)+(1-\rho')|\beta_q|+2\sum_{s=1}^{p-q}(k'_s-1)|\beta_{q+s}|+M_p+|a_n|\Bigg).
\end{aligned}$$

This gives all zeros of $R(z)$ and hence of $T(z)$ lie in

$$\begin{aligned}|z| \quad \leq \quad \max\Bigg\{1, \Bigg(N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| + 2\sum_{s=1}^{p-q}(k_s - 1)|\alpha_{q+s}| \\ +(k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| + 2\sum_{s=1}^{p-q}(k'_s - 1)|\beta_{q+s}| + M_p + |a_n|\Bigg) \Bigg/ |a_0|\Bigg\}.\end{aligned}$$

Hence all the zeros of $P(z)$ lie in

$$\begin{aligned}|z| \quad \geq \quad \min\Bigg\{1, |a_0| \Bigg/ \Bigg(N_q + (k_{p-q}\alpha_p - \rho\alpha_q) + (1-\rho)|\alpha_q| + 2\sum_{s=1}^{p-q}(k_s - 1)|\alpha_{q+s}| \\ +(k'_{p-q}\beta_p - \rho'\beta_q) + (1-\rho')|\beta_q| + 2\sum_{s=1}^{p-q}(k'_s - 1)|\beta_{q+s}| + M_p + |a_n|\Bigg)\Bigg\},\end{aligned}$$

as claimed. □

We now give a proof of Theorem 2.2.

Proof. Let $P(z) = \sum_{s=0}^{n} a_s z^s,\ a_n \neq 0$ be the given polynomial satisfying the stated hypotheses of Theorem 2.2. We can assume without loss of generality that $\mu = 0$. Consider

$$P(z)(1-z) = a_0 + \sum_{s=1}^{n}(a_s - a_{s-1})z^s = f(z) - a_n z^{n+1}$$

For $|z| = 1$, we have

$$\begin{aligned}|f(z)| &= |a_0 + \sum_{s=1}^{n}(a_s - a_{s-1})z^s| \\ &\leq |a_0| + \sum_{s=1}^{n}|a_s - a_{s-1}| \\ &= |a_0| + \sum_{s=1}^{q}|a_s - a_{s-1}| + |a_{q+1} - a_q| + ... + |a_{p-1} - a_{p-2}| + |a_p - a_{p-1}| \\ &\quad + \sum_{s=p+1}^{n}|a_s - a_{s-1}| \\ &= |a_0| + N_q + |a_{q+1} - a_q| + ... + |a_{p-1} - a_{p-2}| + |a_p - a_{p-1}| + M_p,\end{aligned}$$

where $N_q=\sum_{s=1}^{q}|a_s-a_{s-1}|$ and $M_p=\sum_{s=p+1}^{n}|a_s-a_{s-1}|$. Therefore for $|z|=1$

$$\begin{aligned}
|f(z)| &\leq |a_0|+N_q+|a_{q+1}-a_q|+\cdots+|a_{p-1}-a_{p-2}|+|a_p-a_{p-1}|+M_p\\
&= |a_0|+N_q+|k_1a_{q+1}-\rho a_q+a_q(\rho-1)+a_{q+1}(1-k_1)|+\\
&\quad |k_2a_{q+2}-k_1a_{q+1}+a_{q+1}(k_1-1)+a_{q+2}(1-k_2)|+\cdots+\\
&\quad |k_{p-q}a_p-k_{p-q-1}a_{p-1}+a_{p-1}(k_{p-q-1}-1)+a_p(1-k_{p-q})|+M_p\\
&\leq |a_0|+N_q+|k_1a_{q+1}-\rho a_q|+|a_q|(1-\rho)+|a_{q+1}|(k_1-1)+\\
&\quad |k_2a_{q+2}-k_1a_{q+1}|+|a_{q+1}|(k_1-1)+|a_{q+2}|(k_2-1)+\cdots+\\
&\quad |k_{p-q}a_p-k_{p-q-1}a_{p-1}|+|a_{p-1}|(k_{p-q-1}-1)+|a_p|(k_{p-q}-1)+M_p
\end{aligned}$$

Applying Lemma 3.1, we get for $|z|=1$,

$$\begin{aligned}
|f(z)| &\leq |a_0|+N_q+\big(k_1|a_{q+1}|-|\rho a_q|\big)\cos\theta+\big(k_1|a_{q+1}|+|\rho a_q|\big)\sin\theta+\\
&\quad |a_q|(1-\rho)+2\sum_{s=1}^{p-q}\big(k_s-1\big)|a_{q+s}|+\big(k_2|a_{q+2}|-k_1|a_{q+1}|\big)\cos\theta+\\
&\quad \big(k_2|a_{q+2}|+k_1|a_{q+1}|\big)\sin\theta+\cdots+\big(k_{p-q}|a_p|-k_{p-q-1}|a_{p-1}|\big)\cos\theta+\\
&\quad \big(k_{p-q}|a_p|+k_{p-q-1}|a_{p-1}|\big)\sin\theta+M_p\\
&\leq |a_0|+N_q+|a_q|+2\sum_{s=1}^{p-q-1}(k_s-1)|a_{q+s}|+\rho|a_q|\big(\sin\theta-\cos\theta-1\big)\\
&\quad +2\sin\theta\sum_{s=1}^{p-q-1}k_s|a_{q+s}|+k_{p-q}|a_p|\big(\sin\theta+\cos\theta+1\big)-|a_p|+M_p.
\end{aligned}$$

Hence also,

$$\begin{aligned}
\left|z^nf\left(\frac{1}{z}\right)\right| &\leq |a_0|+N_q+|a_q|+2\sum_{s=1}^{p-q-1}(k_s-1)|a_{q+s}|+\rho|a_q|\big(\sin\theta-\cos\theta-1\big)\\
&\quad +2\sin\theta\sum_{s=1}^{p-q-1}k_s|a_{q+s}|+k_{p-q}|a_p|\big(\sin\theta+\cos\theta+1\big)-|a_p|+M_p.
\end{aligned}$$

Therefore by Maximum Modulus Theorem

$$\begin{aligned}
\left|z^nf\left(\frac{1}{z}\right)\right| &\leq |a_0|+N_q+|a_q|+2\sum_{s=1}^{p-q-1}(k_s-1)|a_{q+s}|+\rho|a_q|\big(\sin\theta-\cos\theta-1\big)\\
&\quad +2\sin\theta\sum_{s=1}^{p-q-1}k_s|a_{q+s}|+k_{p-q}|a_p|\big(\sin\theta+\cos\theta+1\big)-|a_p|+M_p
\end{aligned}$$

holds inside $|z|\leq 1$ as well.

If $R>1$, then $\frac{1}{R}e^{-i\omega}$ lies inside the unit disc for every real ω. Hence it follows that

$$
\begin{aligned}
|f(Re^{i\omega})| \leq & \Big(|a_0| + N_q + |a_q| + 2\sum_{s=1}^{p-q-1}(k_s-1)|a_{q+s}| + \rho|a_q|\big(\sin\theta - \cos\theta - 1\big) \\
& +2\sin\theta\sum_{s=1}^{p-q-1}k_s|a_{q+s}| + k_{p-q}|a_p|\big(\sin\theta+\cos\theta+1\big) - |a_p| + M_p\Big)R^n
\end{aligned}
$$

for every $R \geq 1$ and real ω. Thus for every $|z| = R > 1$,

$$
\begin{aligned}
|P(z)(1-z)| &= |-a_n z^{n+1} + f(z)| \\
&\geq |a_n|R^{n+1} - |f(z)| \\
&\geq |a_n|R^{n+1} - \Big(|a_0| + N_q + |a_q| + 2\sum_{s=1}^{p-q-1}(k_s-1)|a_{q+s}| + \rho|a_q|\big(\sin\theta - \\
&\quad \cos\theta - 1\big) + 2\sin\theta\sum_{s=1}^{p-q-1}k_s|a_{q+s}| + k_{p-q}|a_p|\big(\sin\theta+\cos\theta+1\big) - \\
&\quad |a_p| + M_p\Big)R^n \\
&= R^n\Big[|a_n|R - \Big(|a_0| + N_q + |a_q| + 2\sum_{s=1}^{p-q-1}(k_s-1)|a_{q+s}| + \rho|a_q|\big(\sin\theta - \\
&\quad \cos\theta - 1\big) + 2\sin\theta\sum_{s=1}^{p-q-1}k_s|a_{q+s}| + k_{p-q}|a_p|\big(\sin\theta+\cos\theta+1\big) - \\
&\quad |a_p| + M_p\Big)\Big] \\
&> 0,
\end{aligned}
$$

if

$$
\begin{aligned}
R > & \frac{1}{|a_n|}\Big(|a_0| + N_q + |a_q| + 2\sum_{s=1}^{p-q-1}(k_s-1)|a_{q+s}| + \rho|a_q|\big(\sin\theta - \\
& \cos\theta - 1\big) + 2\sin\theta\sum_{s=1}^{p-q-1}k_s|a_{q+s}| + k_{p-q}|a_p|\big(\sin\theta+\cos\theta+1\big) - \\
& |a_p| + M_p\Big).
\end{aligned}
$$

Therefore all the zeros of $P(z)$ lie in

$$
\begin{aligned}
|z| \leq & \frac{1}{|a_n|}\Big(|a_0| + N_q + |a_q| + 2\sum_{s=1}^{p-q-1}(k_s-1)|a_{q+s}| + \rho|a_q|\big(\sin\theta - \\
& \cos\theta - 1\big) + 2\sin\theta\sum_{s=1}^{p-q-1}k_s|a_{q+s}| + k_{p-q}|a_p|\big(\sin\theta+\cos\theta+1\big) - \\
& |a_p| + M_p\Big)
\end{aligned}
$$

as claimed.
Next, consider the polynomial

$$T(z) = z^n P\left(\frac{1}{z}\right) = \sum_{s=0}^{n} a_s z^{n-s}.$$

Let

$$\begin{aligned} R(z) &= (1-z)T(z) \\ &= -a_0 z^{n+1} + (a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_q - a_{q+1})z^{n-q} + \cdots \\ &\quad +(a_p - a_{p+1})z^{n-p} + \cdots + (a_{n-1} - a_n)z + a_n. \end{aligned}$$

This gives

$$\begin{aligned} |R(z)| &\geq |a_0||z|^{n+1} - \Big(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_q - a_{q+1}||z|^{n-q} + \cdots \\ &\quad +|a_p - a_{p+1}||z|^{n-p} + \cdots + |a_{n-1} - a_n||z| + |a_n|\Big) \\ &= |a_0||z|^{n+1} - \Big(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} + \\ &\quad |\rho a_q - k_1 a_{q+1} + a_q(1-\rho) + a_{q+1}(k_1 - 1)||z|^{n-q} + |k_1 a_{q+1} - k_2 a_{q+2} \\ &\quad +a_{q+1}(1-k_1) + a_{q+2}(k_2 - 1)||z|^{n-q-1} + \cdots + |k_{p-q-1}a_{p-1} - k_{p-q}a_p \\ &\quad +a_{p-1}(1-k_{p-q-1}) + a_p(k_{p-q} - 1)||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} + \cdots + \\ &\quad |a_{n-1} - a_n||z| + |a_n|\Big) \\ &= |z|^n\Bigg[|a_0||z| - \Bigg(|a_0 - a_1| + |a_1 - a_2|\frac{1}{|z|^{n-1}} + \cdots + |a_{q-1} - a_q|\frac{1}{|z|^{q-1}} + \\ &\quad |\rho a_q - k_1 a_{q+1} + a_q(1-\rho) + a_{q+1}(k_1 - 1)|\frac{1}{|z|^q} + |k_1 a_{q+1} - k_2 a_{q+2} \\ &\quad +a_{q+1}(1-k_1) + a_{q+2}(k_2 - 1)|\frac{1}{|z|^{q+1}} + \cdots + |k_{p-q-1}a_{p-1} - k_{p-q}a_p \\ &\quad +a_{p-1}(1-k_{p-q-1}) + a_p(k_{p-q} - 1)|\frac{1}{|z|^{p-1}} + |a_p - a_{p+1}|\frac{1}{|z|^p} + \cdots + \\ &\quad |a_{n-1} - a_n|\frac{1}{|z|^{n-1}} + |a_n|\frac{1}{|z|^n}\Bigg)\Bigg]. \end{aligned}$$

Now for $|z|>1$, implies $\dfrac{1}{|z|^{n-s}}<1,\ 0\le s<n$, so that we have

$$\begin{aligned}|R(z)| &\geq |z|^n\Bigg[|a_0||z| - \Big(|a_0-a_1| + |a_1-a_2| + \cdots + |a_{q-1}-a_q| + \\ &\quad |\rho a_q - k_1a_{q+1} + a_q(1-\rho) + a_{q+1}(k_1-1)| + |k_1a_{q+1} - k_2a_{q+2} \\ &\quad +a_{q+1}(1-k_1) + a_{q+2}(k_2-1)| + \cdots + |k_{p-q-1}a_{p-1} - k_{p-q}a_p \\ &\quad +a_{p-1}(1-k_{p-q-1}) + a_p(k_{p-q}-1)| + |a_p - a_{p+1}| + \cdots + \\ &\quad |a_{n-1}-a_n| + |a_n|\Big)\Bigg].\end{aligned}$$

Which with the help of Lemma 3.1 as done above yields

$$\begin{aligned}|R(z)| &\geq |z|^n\Bigg[|a_0||z| - \Bigg(N_q + |a_q| + 2\sum_{s=1}^{p-q-1}(k_s-1)|a_{q+s}| + \rho|a_q|\big(\sin\theta - \cos\theta - 1\big) \\ &\quad +2\sin\theta\sum_{s=1}^{p-q-1}k_s|a_{q+s}| + k_{p-q}|a_p|\big(\sin\theta + \cos\theta + 1\big) - |a_p| + M_p + |a_n|\Bigg)\Bigg] \\ &> 0,\end{aligned}$$

if

$$\begin{aligned}|z| &> \frac{1}{|a_0|}\Bigg(N_q + |a_q| + 2\sum_{s=1}^{p-q-1}(k_s-1)|a_{q+s}| + \rho|a_q|\big(\sin\theta - \cos\theta - 1\big) \\ &\quad +2\sin\theta\sum_{s=1}^{p-q-1}k_s|a_{q+s}| + k_{p-q}|a_p|\big(\sin\theta + \cos\theta + 1\big) - |a_p| + M_p + |a_n|\Bigg).\end{aligned}$$

Therefore all the zeros of $R(z)$ whose modulus is greater than 1 lie in

$$\begin{aligned}|z| &\leq \frac{1}{|a_0|}\Bigg(N_q + |a_q| + 2\sum_{s=1}^{p-q-1}(k_s-1)|a_{q+s}| + \rho|a_q|\big(\sin\theta - \cos\theta - 1\big) \\ &\quad +2\sin\theta\sum_{s=1}^{p-q-1}k_s|a_{q+s}| + k_{p-q}|a_p|\big(\sin\theta + \cos\theta + 1\big) - |a_p| + M_p + |a_n|\Bigg).\end{aligned}$$

This gives all zeros of $R(z)$ and hence of $T(z)$ lie in

$$\begin{aligned}|z| &\leq \max\Bigg\{1, \Bigg(N_q + |a_q| + 2\sum_{s=1}^{p-q-1}(k_s-1)|a_{q+s}| + \rho|a_q|\big(\sin\theta - \cos\theta - 1\big) \\ &\quad +2\sin\theta\sum_{s=1}^{p-q-1}k_s|a_{q+s}| + k_{p-q}|a_p|\big(\sin\theta + \cos\theta + 1\big) - |a_p| + M_p + |a_n|\Bigg)\Bigg/|a_0|\Bigg\}.\end{aligned}$$

Hence all the zeros of $P(z)$ lie in

$$
\begin{aligned}
|z| \;\geq\; & \min\left\{1, |a_0| \Big/ \left(N_q + |a_q| + 2\sum_{s=1}^{p-q-1}(k_s-1)|a_{q+s}| + \rho|a_q|\big(\sin\theta - \cos\theta - 1\big)\right.\right. \\
& \left.\left. +2\sin\theta \sum_{s=1}^{p-q-1} k_s|a_{q+s}| + k_{p-q}|a_p|\big(\sin\theta + \cos\theta + 1\big) - |a_p| + M_p + |a_n|\right)\right\}.
\end{aligned}
$$

This completes the proof of Theorem 2.2 □

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