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GENERALIZED α-KÖTHE TOEPLITZ DUALS OF CERTAIN DIFFERENCE SEQUENCE SPACES

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Abstract. In this paper, we compute the generalized α-Köthe Toeplitz duals of the X-valued (Banach space) difference sequence spaces $E(X, \Delta)$, $E(X, \Delta_v)$ and obtain a generalization of the existing results for α-duals of the classical difference sequence spaces $E(\Delta)$ and $E(\Delta_v)$ of scalars, $E \in \{\ell_\infty, c, c_0\}$. Apart from this, we compute the generalized α-Köthe Toeplitz duals for $E(X, \Delta^r)$ $r \geq 0$ integer and observe that the results agree with corresponding results for scalar cases.

1. Introduction

Kizmaz [12] in 1981, added to the field of sequence spaces a new idea of difference sequence spaces by introducing $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ (termed as difference sequence spaces) as follows:

$$\ell_\infty(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in \ell_\infty\}$$
$$c(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in c\}$$
$$c_0(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in c_0\}$$

where c_0, c, ℓ_∞ are Banach spaces of null, convergent and bounded sequences of scalars, normed by
$\|x\|_\infty = \sup_k |x_k|$ and ω is the space of scalar sequences.

In other words, $E(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in E\}$ for $E \in \{\ell_\infty, c, c_0\}$. It is observed that $E(\Delta)$ are Banach spaces with the norm
$\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty$ for $x = (x_k) \in E(\Delta)$, $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$.

In 1995, Et and Çolak [7] generalized the above concept as follows:

$$E(\Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in E\} \text{ for } E \in \{\ell_\infty, c, c_0\}, \text{where}$$
$$\Delta^n x = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}) \text{ for all } k \in \mathbb{N} \text{ and } \Delta^0 x_k = x_k.$$

These spaces turn out to be complete when equipped with the norm $\|x\|_\Delta = \sum_{i=1}^n |x_i| + \|\Delta^n x\|_\infty$. Obviously, for $n = 1$ the work of Et and Colak [7], reduces to that of Kizmaz [12].

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Using a multiplier sequence, Gnanaseelan and Srivastva [9] introduced the following sequence spaces

$$\ell_\infty(\Delta_v) = \{x = (x_k) \in \omega : (v_k(x_k - x_{k+1})) \in \ell_\infty\}$$
$$c(\Delta_v) = \{x = (x_k) \in \omega : (v_k(x_k - x_{k+1})) \in c\}$$
$$c_0(\Delta_v) = \{x = (x_k) \in \omega : (v_k(x_k - x_{k+1})) \in c_0\}$$

where $v = (v_k)$ is a sequence such that complex numbers $v_k \neq 0$ and

(1) $$\frac{|v_k|}{|v_{k+1}|} = 1 + O\left(\frac{1}{k}\right), \text{for each } k$$

(2) $$k^{-1}\,|v_k| \sum_{i=1}^{k} |v_i^{-1}| = O(1)$$

(3) $(k|v_k^{-1}|)$ is a monotonically $\uparrow$ sequence of positive numbers tending to infinity.

The spaces $E(\Delta_v)$ for $v = (1, 1, 1, ...)$ are noting but the spaces $E(\Delta)$ of Kizmaz and have Banach space structure when equipped with norm

$$\|x\|_{\Delta_v} = |v_1 x_1| + \sup_k |v_k(x_k - x_{k+1})|.$$

For more insight into difference sequence spaces and its various generalizations one may refer to [1–4, 8, 14, 16–21].

The theory of sequence spaces is considered to be incomplete without a touch to the concept of dual spaces. Credit of introducing dual spaces goes to G. Köthe and O. Toeplitz [13].

For a real or complex sequence space E,

$$E^\alpha = \left\{a = (a_k) \in \omega : \sum_{k=1}^{\infty} |a_k x_k| < \infty \text{ for each } x = (x_k) \in E\right\}$$

$$E^\beta = \left\{a = (a_k) \in \omega : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent for each } x = (x_k) \in E\right\}$$

are called $\alpha-$, $\beta-$ duals spaces of E, respectively.

Kizmaz [12] observed that

$$[\ell_\infty(\Delta)]^\alpha = [c_0(\Delta)]^\alpha = [c(\Delta)]^\alpha = \left\{a = (a_k) \in \omega : \sum_k k|a_k| < \infty\right\}.$$

Also we have in view of [7, 9]

$$[\ell_\infty(\Delta^r)]^\alpha = [c_0(\Delta^r)]^\alpha = [c(\Delta^r)]^\alpha = \left\{a = (a_k) \in \omega : \sum_k k^r|a_k| < \infty\right\}$$

and

$$[\ell_\infty(\Delta_v)]^\alpha = [c_0(\Delta_v)]^\alpha = [c(\Delta_v)]^\alpha = \left\{a = (a_k) \in \omega : \sum_k k|v_k^{-1}||a_k| < \infty\right\}.$$

The above introduced notion of Köthe Toeplitz duals [13] was further generalized by Maddox [15] and termed as generalized Köthe Toeplitz duals (or operator duals). To have a view of this, we first have the following:

Let us consider Banach spaces $(X, \|.\|)$ and $(Y, \|.\|)$ with θ as zero element. By $B(X,Y)$, we notate the class of bounded linear operator from X to Y which turn out to be Banach space with usual operator norm and $\omega(X)$ as the space of X-valued sequences. Then for any nonempty subset $E(X)$ of $\omega(X)$

$$[E(X)]^{\alpha} = \left\{ (A_k) : \sum_k \|A_k x_k\| < \infty \text{ for } x = (x_k) \in E(X) \right\}$$

and

$$[E(X)]^{\beta} = \left\{ (A_k) : \sum_k A_k x_k \text{ converges in } Y \text{ for } x = (x_k) \in E(X) \right\}$$

are termed as generalized $\alpha-$, $\beta-$ dual spaces of $E(X)$ respectively. Here $\langle A_k \rangle$ is a sequence of linear (not necessarily bounded) operators from X to Y. Due to the completeness of Y, $[E(X)]^{\alpha} \subset [E(X)]^{\beta}$.

It is to be noted that, the generalized dual spaces $[E(X)]^{\alpha}$ and $[E(X)]^{\beta}$ reduce to classical dual spaces E^{α} and E^{β} for the case $X = Y = \mathbb{C}$, because in this case the operator A_k may be identified with scalar a_k.

Maddox [15], Duyar [6], Haryadi et al. [10], Khan [11] and many more investigated generalized Köthe Toeplitz duals, for sequence spaces $c_0(X)$, $c(X)$ and $\ell_{\infty}(X)$ (the Banach spaces of null, convergent and bounded X-valued sequences respectively) normed by $\|x\|_{\infty} = \sup_k \|x_k\|$. It was shown that $[\ell_{\infty}(X)]^{\alpha} = [c(X)]^{\alpha} = [c_0(X)]^{\alpha}$ which is natural generalization of the scalar case $c_0^{\alpha} = c^{\alpha} = \ell_{\infty}^{\alpha} = \ell_1$.

Bhardwaj and Gupta [5] introduced and studied the following difference sequence spaces $E(X, \Delta)$, $E(X, \Delta_v)$ and $E(X, \Delta^r)$ as follows:

$$\begin{aligned} E(X, \Delta) &= \{x = (x_k) \in \omega(X) : (\Delta x_k) \in E(X)\} \\ E(X, \Delta_v) &= \{x = (x_k) \in \omega(X) : (v_k(x_k - x_{k+1})) \in E(X)\} \\ E(X, \Delta^r) &= \{x = (x_k) \in \omega(X) : (\Delta^r x_k) \in E(X)\} \end{aligned}$$

for $E \in \{\ell_{\infty}, c, c_0\}$ and compute their generalized β-Köthe Toeplitz duals.

In the present paper, we compute the generalized α-Köthe Toeplitz duals of difference sequence spaces $E(X, \Delta)$, $E(X, \Delta^r)$ and $E(X, \Delta_v)$ for $E \in \{\ell_{\infty}, c, c_0\}$. The results agree with those of the classical spaces $E(\Delta)$, $E(\Delta^r)$ and $E(\Delta_v)$ for $X = Y = \mathbb{C}$.

2. Generalized α-Köthe Toeplitz duals of difference sequence spaces $E(X, \Delta)$ and further generalizations

In the present snippet, we compute the generalized α-Köthe Toeplitz duals of difference sequence spaces $E(X, \Delta)$, $E \in \{\ell_{\infty}, c, c_0\}$. It is observed that results obtained agree with that of Kizmaz [12] for $X = Y = \mathbb{C}$ and hence a generalization from scalar valued theory to Banach space valued theory. Apart from this, we extend these results in the setting of generalized difference sequence spaces $E(X, \Delta^r)$, $E \in \{\ell_{\infty}, c, c_0\}$, $r \geq 0$ integer.

Proposition 2.1. $(A_k) \in c_0^{\alpha}(X, \Delta)$ *iff there exists integer* $m > 0$ *such that* *(i)* $A_k \in B(X,Y)$ *for all* $k \geq m$ *and*

(ii) $\sum_{k\geq m} k\|A_k\| < \infty$.

Proof. Sufficiency: Let (i) and (ii) hold and $x = (x_k) \in c_0(X,\Delta)$. Then $x_k - x_{k+1} \to \theta$ as $k \to \infty$ and so $\sup_k \|x_k - x_{k+1}\| < \infty$. Now

$$\|x_k - x_{k+1}\| \leq \left\|\sum_{v=1}^{k}(x_v - x_{v+1})\right\| \leq \sum_{v=1}^{k} \|x_v - x_{v+1}\| = O(k).$$

Also

$$\|x_k\| \leq \|x_k - x_{k+1}\| + \|x_{k+1} - x_1\| + \|x_1\| \text{ for every } k,$$

which implies $k^{-1}\|x_k\| \leq k^{-1}O(1) + O(1) + k^{-1}\|x_1\|$. Thus $\sup_k k^{-1}\|x_k\| < \infty$. Also by (ii) for given $\varepsilon > 0$, there exists an integer $k_1 \geq m$ such that $\sum_{k\geq k_1} k\|A_k\| < \frac{\varepsilon}{M}$ where $M = \sup_k k^{-1}\|x_k\|$. Now

$$\sum_{k\geq k_1} \|A_k x_k\| \leq \sum_{k\geq k_1} (k\|A_k\|)(k^{-1}\|x_k\|) < M.\frac{\varepsilon}{M} = \varepsilon.$$

Thus $\sum_{k\geq k_1} \|A_k x_k\|$ converges and $(A_k) \in c_0^{\alpha}(X,\Delta)$.

Necessity: suppose $(A_k) \in c_0^{\alpha}(X,\Delta)$ but no $m \in \mathbb{N}$ exists for which $A_k \in B(X,Y)$ for all $k \geq m$. Then there exists a sequence (k_i) of natural numbers $m \leq k_1 < k_2 < \ldots$ with $A_{k_i} \notin B(X,Y)$ for each $i \geq 1$. Thus for each $i \geq 1$, we can find $z_i \in S$ such that $\|A_{k_i} z_i\| > i$. Define

$$x_k = \begin{cases} \dfrac{z_i}{i} & \text{for } k = k_i\,, i \geq 1 \\ \theta & \text{otherwise.} \end{cases}$$

Then $x = (x_k) \in c_0(X,\Delta)$ and $\|A_{k_i} x_{k_i}\| > 1$ for each $i \geq 1$. This implies $\sum_k \|A_k x_k\|$ diverges, which contradicts that $(A_k) \in c_0^{\alpha}(X,\Delta)$. Hence the $A_k{}'s$ are ultimately bounded.

Now suppose (ii) does not holds, i.e., $\sum_{k\geq m} k\|A_k\| = \infty$. Following Maddox [15], there exists natural numbers $n(1) < n(2) < \ldots$ with $n(1) \geq m$ such that for each $i \geq 1$, $\sum_{1+n(i)}^{n(i+1)} k\|A_k\| > 2^{n(i+1)}$. Moreover, for each $k \geq m$, there exists $z_k \in S$ such that $\|A_k\| \leq 2\|A_k z_k\|$. Define

$$x_k = \begin{cases} \frac{k}{2^k} z_k & \text{for } n(i) < k \leq n(i+1),\ i \geq 1 \\ \theta & \text{otherwise.} \end{cases}$$

Then $x = (x_k) \in c_0(X,\Delta)$ but

$$\begin{aligned} \sum_{1+n(i)}^{n(i+1)} \|A_k x_k\| &= \sum_{1+n(i)}^{n(i+1)} \frac{k}{2^k}\|A_k z_k\| \\ &> \frac{1}{2}\sum_{1+n(i)}^{n(i+1)} \frac{k}{2^k}\|A_k\| \\ &> \frac{1}{2}\sum_{1+n(i)}^{n(i+1)} \frac{k\|A_k\|}{2^{n(i+1)}} > \frac{1}{2} \text{ for each } i \geq 1, \end{aligned}$$

shows that $\sum_k \|A_k x_k\| = \infty$, which is a contradiction to $\sum_k \|A_k x_k\| < \infty$. Hence (ii) holds. □

Proposition 2.2. $(A_k) \in c^\alpha(X,\Delta)$ *iff there exists integer* $m > 0$ *with*

(i) $A_k \in B(X,Y)$ *for all* $k \geq m$ *and*

(ii) $\sum_{k\geq m} k\|A_k\| < \infty$.

Proof. Sufficiency: Let (i) and (ii) hold and $x = (x_k) \in c(X,\Delta)$. Then $(x_k - x_{k+1}) \in c(X)$ and so $\sup_k \|x_k - x_{k+1}\| < \infty$. Arguing in the same way, as in sufficiency portion of Proposition 2.1, we get $(A_k) \in c^\alpha(X,\Delta)$.

Necessity: Since $c^\alpha(X,\Delta) \subset c_0^\alpha(X,\Delta)$, so the necessary part follows from the necessary part of Proposition 2.1. □

Proposition 2.3. $(A_k) \in \ell_\infty^\alpha(X,\Delta)$ *iff there exists integer* $m > 0$ *such that*

(i) $A_k \in B(X,Y)$ *for all* $k \geq m$ *and*

(ii) $\sum_{k\geq m} k\|A_k\| < \infty$.

Proof. Sufficiency: Let (i) and (ii) hold and $x = (x_k) \in \ell_\infty(X,\Delta)$. Then $(x_k - x_{k+1}) \in \ell_\infty(X)$ and so $\sup_k \|x_k - x_{k+1}\| < \infty$. Arguing in the same way, as in Proposition 2.1, we get $(A_k) \in \ell_\infty^\alpha(X,\Delta)$.

Necessity: Since $\ell_\infty^\alpha(X,\Delta) \subset c_0^\alpha(X,\Delta)$, so the result follows in view of Proposition 2.1. □

Corollary 2.4. $[c_0(X,\Delta)]^\alpha = [c(X,\Delta)]^\alpha = [\ell_\infty(X,\Delta)]^\alpha$.

Corollary 2.5. $[c_0(\Delta)]^\alpha = [c(\Delta)]^\alpha = [\ell_\infty(\Delta)]^\alpha = \{(a_k) : \sum_k k|a_k| < \infty\}$.

Proof. As in case $X = Y = \mathbb{C}$, the operator A_k may be replace by scalar a_k, hence the result follows from Proposition 2.1, Proposition 2.2 and Proposition 2.3. □

Before proceeding further in this section, we recall the following

$$\begin{aligned} c_0(X,\Delta^r) &= \{(x_k) : (\Delta^r x_k) \in c_0\} \\ c(X,\Delta^r) &= \{(x_k) : (\Delta^r x_k) \in c\} \\ \ell_\infty(X,\Delta^r) &= \{(x_k) : (\Delta^r x_k) \in \ell_\infty\}. \end{aligned}$$

Obviously, taking $X = \mathbb{C}$, the above spaces reduce to $\ell_\infty(\Delta^r)$, $c(\Delta^r)$ and $c_0(\Delta^r)$ respectively of [7].

Lemma 2.6. *If* $\sup_k \|\Delta^r x_k\| < \infty$ *then* $\sup_k k^{-1}\|\Delta^{r-1}x_k\| < \infty$, $r \in \mathbb{N}$.

Proof. Let $\sup_k \|\Delta^r x_k\| < \infty$, i.e., $\sup_k \|\Delta^{r-1}x_k - \Delta^{r-1}x_{k+1}\| < \infty$. Now

$$\begin{aligned} \|\Delta^{r-1}x_1 - \Delta^{r-1}x_{k+1}\| &= \left\| \sum_{v=1}^{k} \left(\Delta^{r-1}x_v - \Delta^{r-1}x_{v+1}\right) \right\| \\ &\leq \sum_{v=1}^{k} \|\Delta^{r-1}x_v - \Delta^{r-1}x_{v+1}\| = O(k) \end{aligned}$$

and this holds for each $k \in \mathbb{N}$. Also

$$\|\Delta^{r-1}x_k\| \leq \|\Delta^{r-1}x_1\| + \|\Delta^{r-1}x_{k+1} - \Delta^{r-1}x_1\| + \|\Delta^{r-1}x_k - \Delta^{r-1}x_{k+1}\|$$

for each $k \in \mathbb{N}$, which implies $\sup_k k^{-1}\|\Delta^{r-1}x_k\| < \infty$. □

Lemma 2.7. *If* $\sup_k k^{-i}\|\Delta^{r-i}x_k\| < \infty$ *then* $\sup_k k^{-(i+1)}\left\|\Delta^{r-(i+1)}x_k\right\| < \infty$, *for all* $i, r \in \mathbb{N}$ *and* $1 \leq i \leq r$.

Proof. Let $\sup_k k^{-i}\|\Delta^{r-i}x_k\| < \infty$. Then

$$\left\|\Delta^{r-(i+1)}x_1 - \Delta^{r-(i+1)}x_{k+1}\right\| = \left\|\sum_{v=1}^{k}\left(\Delta^{r-(i+1)}x_v - \Delta^{r-(i+1)}x_{v+1}\right)\right\|$$
$$\leq \sum_{v=1}^{k}\left\|\Delta^{r-(i+1)}x_v - \Delta^{r-(i+1)}x_{v+1}\right\|$$
$$= \sum_{v=1}^{k}\left\|\Delta^{r-i}x_v\right\| = O\left(k^{i+1}\right).$$

Also

$$\left\|\Delta^{r-(i+1)}x_k\right\| \leq \left\|\Delta^{r-(i+1)}x_1\right\| + \left\|\Delta^{r-(i+1)}x_{k+1} - \Delta^{r-(i+1)}x_1\right\| + \left\|\Delta^{r-(i+1)}x_k - \Delta^{r-(i+1)}x_{k+1}\right\|$$

which implies $\sup_k k^{-(i+1)}\left\|\Delta^{r-(i+1)}x_k\right\| < \infty$. □

Corollary 2.8. *If* $\sup_k k^{-1}\left\|\Delta^{r-1}x_k\right\| < \infty$ *then* $\sup_k k^{-r}\left\|x_k\right\| < \infty$.

Proof. Repeated application of Lemma 2.7, yields the result. □

Proposition 2.9. $(A_k) \in c_0^{\alpha}(X, \Delta^r)$ *iff there exists integer* $m > 0$ *such that*

(i) $A_k \in B(X, Y)$ *for all* $k \geq m$ *and*
(ii) $\sum_{k\geq m} k^r\|A_k\| < \infty$.

Proof. Sufficiency: Let (i) and (ii) hold and $x = (x_k) \in c_0^{\alpha}(X, \Delta^r)$. Then $(\Delta^r x_k) \in c_0(X)$ and so $\sup_k \|\Delta^r x_k\| < \infty$. Now Lemma 2.6 and Corollary 2.8, yields $\sup_k k^{-r}\|x_k\| < \infty$. Also for given $\varepsilon > 0$, there exists an integer $k_1 \geq m$ such that $\sum_{k\geq k_1} k^r\|A_k\| < \frac{\varepsilon}{M}$ where $M = \sup_k k^{-r}\|x_k\|$. Now

$$\sum_{k\geq k_1}\|A_k x_k\| \leq \sum_{k\geq k_1}\|A_k\|\|x_k\|$$
$$= \sum_{k\geq k_1}(k^r\|A_k\|)(k^{-r}\|x_k\|) \leq M\frac{\varepsilon}{M} = \varepsilon.$$

Thus $\sum_k \|A_k x_k\|$ converges and $(A_k) \in c_0^{\alpha}(X, \Delta^r)$.

Necessity: $(A_k) \in c_0^{\alpha}(X, \Delta^r)$ but no m exists such that $A_k \in B(X, Y)$ for all $k \geq m$. Then there exists natural numbers $k_1 < k_2 < \ldots$ and $z_i \in S$ such that for each $i \geq 1$, $\|A_{k_i} z_i\| > 2^i$. Define

$$x_k = \begin{cases} \dfrac{z_i}{2^i} & \text{for } k = k_i, \text{ for each } i \geq 1 \\ \theta & \text{otherwise.} \end{cases}$$

Then $x = (x_k) \in c_0(X, \Delta^r)$ but $\|A_k x_k\| > 1$ for $k = k_i$, $i \geq 1$, which is a contradiction as $\sum_k \|A_k x_k\|$ converges. Hence condition (i) holds.

Next, suppose if possible, that $\sum_{k\geq m} k^r\|A_k\| = \infty$. Then there exists an increasing(strictly) sequence $\langle n(i)\rangle$ with $n(1) \geq m$ and sequence $\langle z_k\rangle$ in S such that $2\|A_k z_k\| \geq \|A_k\|$ and $\sum_{1+n(i)}^{n(i+1)} k^r\|A_k\| > 2^{n(i+1)}$. Define

$$x_k = \begin{cases} \dfrac{k^r}{2^k} z_k & \text{for } n(i) < k \leq n(i+1),\ i \geq 1 \\ \theta & \text{otherwise.} \end{cases}$$

Then $x = (x_k) \in c_0(X, \Delta^r)$ but

$$\sum_{1+n(i)}^{n(i+1)} \|A_k x_k\| = \sum_{1+n(i)}^{n(i+1)} \frac{k^r}{2^k} \|A_k z_k\|$$
$$> \frac{1}{2} \sum_{1+n(i)}^{n(i+1)} \frac{k^r \|A_k\|}{2^{n(i+1)}} > \frac{1}{2} \quad \text{for each } i \geq 1,$$

show that $\sum_k \|A_k x_k\|$ diverges, contrary to the fact that $\sum_k \|A_k x_k\| < \infty$. Hence (ii) holds and proposition is proved. □

The proofs of the following runs on the similar lines as that of the above propositions and corollaries and hence omitted.

Proposition 2.10. $(A_k) \in c^\alpha(X, \Delta^r)$ *iff there exists integer* $m > 0$ *with*

(i) $A_k \in B(X, Y)$ *for all* $k \geq m$ *and*
(ii) $\sum_{k \geq m} k^r \|A_k\| < \infty$.

Proof. The proof runs on similar lines as that of Proposition 2.2 and hence omitted. □

Proposition 2.11. *The conditions (i) and (ii) of Proposition 2.10 are also necessary as well as sufficient for* $(A_k) \in \ell_\infty^\alpha(X, \Delta^r)$.

Corollary 2.12. $c_0^\alpha(X, \Delta^r) = c^\alpha(X, \Delta^r) = \ell_\infty^\alpha(X, \Delta^r)$.

Remark 2.13. From Corollary 2.12,

1. For $r = 1$, we obtained $[c_0(X, \Delta)]^\alpha = [c(X, \Delta)]^\alpha = [\ell_\infty(X, \Delta)]^\alpha$, i.e. Corollary 2.4.
2. For $r = 0$, we get results obtained by Maddox [15], i.e. $[c_0(X)]^\alpha = [c(X)]^\alpha = [\ell_\infty(X)]^\alpha$.
3. For $r = 1$, and $X = Y = \mathbb{C}$, we get the corresponding results of Kizmaz [12].
4. For $X = Y = \mathbb{C}$, we deduce the corresponding results of Et and Çolak [7].

3. Generalized α-Köthe Toeplitz Dual of sequence space $E(X, \Delta_v)$

In this snippet, we investigate the generalized α-Köthe Toeplitz duals of the following sequence spaces $E(X, \Delta_v)$ for $E \in \{\ell_\infty, c, c_0\}$ and give a generalization to the existing results (for scalar valued) on duality theory.

Before stepping further, we recall the spaces $E(X, \Delta_v)$, introduced by Bhardwaj and Gupta [5] as follows:

$$\ell_\infty(X, \Delta_v) = \{x = (x_k) : (v_k(x_k - x_{k+1})) \in \ell_\infty(X)\}$$
$$c(X, \Delta_v) = \{x = (x_k) : (v_k(x_k - x_{k+1})) \in c(X)\}$$
$$c_0(X, \Delta_v) = \{x = (x_k) : (v_k(x_k - x_{k+1})) \in c_0(X)\}.$$

Clearly, for $X = \mathbb{C}$, these spaces are nothing but the spaces introduced by Gnanaseelan and Srivastva [9].

Lemma 3.1. *If* $\sup_k \|v_k(x_k - x_{k+1})\| < \infty$, *then* $\sup_k k^{-1} \|v_k x_k\| < \infty$.

Proof. Let $\sup_k \|v_k(x_k - x_{k+1})\| < \infty$. We get

$$\begin{aligned}\|x_1 - x_{k+1}\| &= \left\|\sum_{i=1}^{k}(x_i - x_{i+1})\right\| \le \sum_{i=1}^{k}\|x_i - x_{i+1}\| \\ &= \sum_{i=1}^{k}\|v_i(x_i - x_{i+1})\|\left|v_i^{-1}\right| \\ &= O(1)\sum_{i=1}^{k}\left|v_i^{-1}\right| = O(1)\left(k^{-1}|v_k|\right)\sum_{i=1}^{k}\left|v_i^{-1}\right| k\left|v_k^{-1}\right| \\ &= O(k\left|v_k^{-1}\right|) \ \text{(using 2)}.\end{aligned}$$

Also

$$\|x_k\| = \|x_k - x_{k+1} + x_{k+1} - x_1 + x_1\| \le \|x_k - x_{k+1}\| + \|x_{k+1} - x_1\| + \|x_1\|$$

for every k, which implies

$$k^{-1}\|v_k x_k\| \le k^{-1}|v_k|\,\|x_k - x_{k+1}\| + k^{-1}|v_k|\,\|x_{k+1} - x_1\| + k^{-1}|v_k|\,\|x_1\|.$$

Using (3), we get, $k^{-1}\|v_k x_k\| \le k^{-1}O(1) + O(1) + k^{-1}|v_k|\,\|x_1\|$.
Hence $\sup_k k^{-1}\|v_k x_k\| < \infty$ by (3). □

Proposition 3.2. *$(A_k) \in c_0^{\alpha}(X, \Delta_v)$ iff there exists integer $m > 0$ such that $A_k \in B(X,Y)$ for all $k \ge m$ and $\sum_{k\ge m} k\left|v_k^{-1}\right|\|A_k\| < \infty$.*

Proof. Sufficiency: Let $x = (x_k) \in c_0(X, \Delta_v)$. Then $v_k(x_k - x_{k+1}) \to \theta$ as $k \to \infty$ and so $\sup_k \|v_k(x_k - x_{k+1})\| < \infty$. By Lemma 3.1, we get $\sup_k k^{-1}\|v_k x_k\| < \infty$. Also for given $\varepsilon > 0$, there exists an integer $k_1 \ge m$ such that $\sum_{k\ge k_1} k\left|v_k^{-1}\right|\|A_k\| < \frac{\varepsilon}{M}$ where $M = \sup_k k^{-1}\|v_k x_k\|$. Now

$$\sum_{k\ge k_1}\|A_k x_k\| \le \sum_{k\ge k_1}\left(k\left|v_k^{-1}\right|\|A_k\|)(k^{-1}|v_k|\,\|x_k\|\right) < M.\frac{\varepsilon}{M} = \varepsilon.$$

Thus $\sum_k \|A_k x_k\|$ converges and $(A_k) \in c_0^{\alpha}(X, \Delta_v)$.

Necessity: $(A_k) \in c_0^{\alpha}(X, \Delta_v)$ but no m exists such that $A_k \in B(X,Y)$ for all $k \ge m$. Then there exists natural numbers $k_1 < k_2 < \ldots$ and $z_i \in S$ such that for each $i \ge 1$, $\|A_{k_i} z_i\| > 2^i |v_{k_i}^{-1}|$. Now the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} \dfrac{z_i}{2^i}\left|v_{k_i}^{-1}\right| & \text{for } k = k_i,\ i \ge 1, \\ \theta & \text{otherwise,}\end{cases} \qquad \text{is in } c_0(X, \Delta_v)$$

but $\|A_{k_i} x_{k_i}\| > 1$ and so $\sum_k \|A_k x_k\|$ diverges, which is a contradiction to $(A_k) \in c_0^{\alpha}(X, \Delta_v)$.
Next, suppose that $\sum_{k\ge m} k\left|v_k^{-1}\right|\|A_k\| = \infty$. Then there exists an increasing(strictly) sequence $\langle n(i)\rangle$ of positive integers such that $\sum_{1+n(i)}^{n(i+1)} k\left|v_k^{-1}\right|\|A_k\| > 2^{n(i+1)}$. Moreover for each $k \ge m$, there exists a sequence $\langle z_k\rangle$ in S such that $2\|A_k z_k\| \ge \|A_k\|$. Now define $x = (x_k)$ by

$$x_k = \begin{cases} \dfrac{k}{2^k}\left|v_k^{-1}\right| z_k & \text{for } n(i) < k \le n(i+1),\ i \ge 1 \\ \theta & \text{otherwise.}\end{cases}$$

Then $x = (x_k) \in c_0(X, \Delta_v)$ but for each i,

$$\sum_{1+n(i)}^{n(i+1)} \|A_k x_k\| = \sum_{1+n(i)}^{n(i+1)} \frac{k}{2^k} \left|v_k^{-1}\right| \|A_k z_k\| \geq \frac{1}{2} \sum_{1+n(i)}^{n(i+1)} \frac{k}{2^k} \left|v_k^{-1}\right| \|A_k\|$$

$$\geq \frac{1}{2} \sum_{1+n(i)}^{n(i+1)} \frac{k}{2^{n(i+1)}} \left|v_k^{-1}\right| \|A_k\| > \frac{1}{2}$$

which show that $\sum_{k\geq m} \|A_k x_k\|$ diverges, again contradictory. □

Proposition 3.3. *$(A_k) \in c^\alpha(X, \Delta_v)$ iff there exists integer $m > 0$ with $A_k \in B(X, Y)$ for all $k \geq m$ and $\sum_{k\geq m} k \left|v_k^{-1}\right| \|A_k\| < \infty$.*

Proof. Sufficiency: Let $A_k \in B(X, Y)$ for all $k \geq m$ and $\sum_{k\geq m} k \left|v_k^{-1}\right| \|A_k\| < \infty$. In order to prove $(A_k) \in c^\alpha(X, \Delta_v)$, let $x = (x_k) \in c(X, \Delta_v)$. Then $(v_k(x_k - x_{k+1})) \in c(X)$ and so $\sup_k \|v_k(x_k - x_{k+1})\| < \infty$. Arguing in the same way as in Proposition 3.2, we get $(A_k) \in c^\alpha(X, \Delta_v)$

Necessity: As $c^\alpha(X, \Delta_v) \subset c_0^\alpha(X, \Delta_v)$, so the proof follows from Proposition 3.2. □

Proposition 3.4. *$(A_k) \in \ell_\infty^\alpha(X, \Delta_v)$ iff there exists integer $m > 0$ with $A_k \in B(X, Y)$ for all $k \geq m$ and $\sum_{k\geq m} k \left|v_k^{-1}\right| \|A_k\| < \infty$.*

Proof. Proof runs on similar lines as in Proposition 3.3 and hence left for reader. □

Corollary 3.5. *(a) $c_0^\alpha(X, \Delta_v) = c^\alpha(X, \Delta_v) = \ell_\infty^\alpha(X, \Delta_v)$*
(b) $c_0^\alpha(X, \Delta) = c^\alpha(X, \Delta) = \ell_\infty^\alpha(X, \Delta)$

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