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RESEARCH ARTICLE

High school students’ evaluation of mathematical arguments as proof: Exploring relationships between understanding, convincingness, and evaluation

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Abstract

Researchers continue to emphasize the centrality of proof in the context of school mathematics and the importance of proof to student learning of mathematics is well articulated in nationwide curricula. However, researchers reported that students’ performance in proving tasks is not promising and students are not likely to see the need to prove a proposition even if they learned mathematical proof previously. Research attributes this issue to students’ tendencies to accept an empirical argument as proof for a mathematical proposition, thus not being able to recognize the limitation of an empirical argument as proof for a mathematical proposition. In Korea, there is little research that investigated high school students’ views about the need for proof in mathematics and their understanding of the limitation of an empirical argument as proof for a mathematical generalization. Sixty-two 11th graders were invited to participate in an online survey and the responses were recorded in writing and on either a four- or five-point Likert scale. The students were asked to express their agreement with the need of proof in school mathematics and to evaluate a set of mathematical arguments as to whether the given arguments were proofs. Results indicate that a slight majority of students were able to identify a proof amongst the given arguments with the vast majority of students acknowledging the need for proof in mathematics.

Keywords: high school students’ perceptions, mathematical argument, proof validation, survey

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I. INTRODUCTION

Researchers continue to emphasize the centrality of proof in the context of school mathematics and the importance to student learning of mathematics (Hanna, 2020; Knuth, 2002a, 2002b; Schoenfeld, 1994; Stylianides, Bieda, & Morselli, 2016; Stylianides, Stylianides, & Weber, 2016; Zaslavsky et al., 2012). Similarly, the importance and prominence of proof is well reflected in nation-wide curricula and recommendations (e.g., Council of Chief State School Officers [CCSSO], 2010; Department of Education [DoE], 2014; Ministry of Education [MoE], 2015, 2022; National Council of Teachers of Mathematics [NCTM], 2000). In the revised mathematics curriculum of the Korean Ministry of Education (2015, 2022), proof and reasoning are considered as one of the core practices that students should engage in throughout their K-12 education. However, researchers reported that students' performance in proving tasks was not robust enough to distinguish invalid mathematical arguments from proofs (Kim, 2022; Harel & Sowder, 2007; Knuth et al., 2009a; Reid & Knipping, 2010; Sowder & Harel, 1998; Stylianides et al., 2016) and that students were not likely to see the need to prove a mathematical proposition even if they learned about mathematical proof previously (Roh & Kang, 2015).

Researchers tend to attribute this issue to students' tendencies to accept empirical arguments as proofs. By *empirical arguments* here and thereafter in this paper, I refer to mathematical arguments that provide the reason why a mathematical proposition holds true based on testing a proper subset of examples in the domain of the mathematical proposition. Research shows that students tend to misunderstand an empirical argument for a proof (Chazan, 1993; Herbst & Brach, 2006; Knuth et al., 2009a, 2009b), thus not failing to recognize the need of a proof in the presence of an empirical argument (Roh & Kang, 2015). In Korea, though there have been a few studies that examined students' understandings about proof at elementary level (e.g., Kim et al., 2014; Song et al., 2006) and at middle level (Hong & Son, 2021; Kang & Shin, 2022; Kim, 2012; Roh & Kang, 2015), there is little research that investigated high school students' understandings about proof in general, their understanding about the limitation of an empirical argument as a proof in particular. The results of the study would provide insights into high school students' understandings of proof and evaluations of mathematical arguments (i.e. proof validation) in relation to the convincingness and understanding the mathematical arguments. Implications of the results may call mathematics teachers' and teacher educators' attention to issues related with the teaching and learning of mathematical proof.

This study was designed to investigate the nature of high school students' understandings about proof in mathematics and to examine their understanding of the limitation of an empirical argument as a proof with the given mathematical proposition "For a natural number n, $1 + 2 + \cdots + (n-1) + n + (n-1) + (n-2) + \cdots 2 + 1 = n^2$"(Kim, 2022, p. 72). The research questions which this study aimed to address were:

a) What are students' views about the need of proof in mathematics?

b) How likely are students to reject an empirical argument as a proof? What characteristics of an empirical argument do students consider when evaluating a mathematical argument with respect to convincingness and validity as a proof?

The first research question was formulated based on the assumption that students who viewed mathematical proof as indispensable in mathematics would have developed a better understanding about the limitation of empirical arguments as proofs, so that they were more likely to reject empirical arguments as proofs than those not. The second research question was the focal research question to this study. To address the second research question, four mathematical arguments were developed to explore relationships between one's understanding of a mathematical argument, the convincingness of the argument to him[1], and his evaluation of the mathematical argument as a proof.

II. LITERATURE REVIEW

Proof is considered to be an important and essential practice which students are expected to master and to develop their understanding of throughout K-12 education (MoE, 2022; NCTM, 2000). In curricula and recommendations, authors argue that proof should play a central role in school mathematics across grade levels (ACARA, 2022; CCSSO, 2010; Kim, 2022; NCTM, 2009) and some researchers go on to argue that proof should become a routine as part of the daily instruction (e.g., Bieda, 2010; Kim, 2021; MoE, 2022). Bieda (2010) states, "greater emphasis is needed for middle school teacher preparation, professional development, and curricular support to make justifying and proving a routine part of middle school students' opportunities to learn" (p. 380).

However, in the literature, it has been documented that students struggle to develop a robust understanding of proof that is related to rejecting empirical arguments as proofs (Harel & Sowder, 2007; Knuth et al., 2009a; Reid & Knipping, 2010; Stylianides et al., 2016). One of the difficulties that students tend to have with mathematical proof is the misunderstanding of an empirical argument for a proof (Kim et al., 2014; Roh & Kang, 2015; Weber, 2010). This tendency was observed by Chazan (1993): students fail to recognize the limitation of an empirical argument as a proof so they are not likely to reject an empirical argument as a proof. Also, it was also confirmed that the same tendency existed among secondary students in Knuth et al. (2009a) and among many of the secondary mathematics teachers in Kim (2022). Additionally, the teachers might present empirical arguments as proof in their classrooms. Though researchers (Alcock & Inglis, 2008; Buchbinder & Zaslavsky, 2018; Ellis et al., 2019; Iannone et al., 2011; Sandefur et al., 2013) documented that examples lay foundations for students to progress from empirical arguments to proofs (Bills & Rowland, 1999) or to provide support for proofs through the use of examples (Epstein & Levy, 1995). In particular, examples help one recognize patterns that may be generalizable (Ellis et al., 2019) and provide insights into what proofs for such patterns may look like (Epstein & Levy, 1995; Iannone et al., 2011; Sandefur et al., 2013). What is problematic about the tendency of not rejecting empirical arguments as proofs is that students would likely lose opportunities to develop a proper

[1] This use of a male pronoun is made due to the fact that this study was conducted with a sample of male students without any intention to imply gender-dominance throughout the report.

understanding about mathematical proof and eventually reinforce their misconceptions about proof. The latter case was reported by Coe & Ruthven (1994), Knuth (2002b), and Kim (2022).

In Korea, students are introduced explicitly to formal ideas of mathematical proof in Grade 10. In the grade, as prescribed in the national mathematics curriculum (MoE, 2015, 2022), students learn that there must be a proof to validate a mathematical proposition and about several proof methods including modus ponens, reductio ad absurdum, and proof by mathematical induction. In the context of Korea, previous studies attempted to investigate students' understandings about the relationship between proof and generalization with 9th graders (Kim, 2012), students' recognitions of the need for a proof for a given proposition with 9th graders (Roh & Kang, 2015), students' preferences of deductive, experiential, and formal justification with 7th and 8th graders (Hong & Son, 2021), and students' recognitions of the need of justification and levels of understanding about justification with 6th graders (Kim et al., 2014). Given that there has been little research that examined high school students' understandings of the limitation of an empirical argument as a proof, there is a need for a study that explicitly examines high school students' understandings of the limitation of an empirical argument as a proof.

An empirical argument for a mathematical proposition has been distinguished in various ways by researchers. With the terms coined by Balacheff (1988), naïve empiricism and generic examples are the cases of point for this study. Though both of these two terms are concerned with empirical arguments, the distinction between arguments with naïve empiricism and generic example is the nature of the operation involved in an argument:

> *The generic example* involves making explicit the reasons for the truth of an assertion using operations or transformations on an object that is not there in its own right, but as *a characteristic representative of its class*. The account involves *the characteristic properties and structures of a class*, while doing so in terms of the names and illustrations of its representative. (p. 219, italics added)

While operations underlying arguments concerned with naïve empiricism are rooted in the particularity of examples involved in the arguments, operations involved in arguments concerned with generic example rest on the generality and representativeness of a class. In this study, two confirming examples for a mathematical proposition were provided to investigate how likely high school students are to accept one with relatively higher generality over the other with relatively less generality. Given that visual representation is a tool with which students conduct deductive reasoning in proving a theorem (Kang & Shin, 2022), it is worth investigating whether the presence of a visual representation in a mathematical argument has an effect on students' tendencies to accept an empirical argument as a proof. For instance, Healy & Hoyles (2000) reported that students find mathematical arguments with symbolic representations more valid as proofs. Similarly, Harel & Sowder (1998) observed similar tendencies among preservice mathematics teachers. Knuth (2002b) reported a similar result that secondary teachers tend to rely on surface features of mathematical arguments when asked to evaluate mathematical arguments. In addition to that, the use of pictorial representation tends to influence one's evaluation of mathematical arguments (Roh & Kang, 2015). In this study, four mathematical arguments were developed to investigate how likely students are to accept empirical arguments with varying degrees of generality and the presence (or absence) of visual or symbolic representations.

III. METHODOLOGY

Data Collection

In a metropolitan city of Korea, a sample of grade 11 students from a boys high school were invited to participate in an online survey (see Table 1 for an overview of the questions included in the survey). This sampling was a convenient sample for the author due to the fact that the author was the mathematics teacher for all of the participants at the time of their participation to the survey. In addition to that, per the national curriculum (MoE, 2015), the students learned about mathematical proof and proof methods including direct proof, proof by contradiction, and mathematical induction during their studies of mathematics in grade 10. In this regard, 11th graders are expected to have developed a robust understanding about mathematical proof, particularly the understanding about the limitation of an empirical argument as proof. Prior to the time of their participation into this study, in grade 11, there were often times when the concept of mathematical proof was explicitly discussed with respect to three aspects outlined by Stylianides (2009): that is, *set of accepted statements*, *valid mode of reasoning*, and *appropriate mode of representation*. The participating students were all who declared to major in STEM-related fields upon graduation and took the same course of high school mathematics at grade 10 through which they were first introduced to formal ideas of proof and proof methods per the Korean national curriculum (MoE, 2015).

The average time of their participation in the online survey was about eighteen minutes. The number of participants who opened the survey was sixty-two and fifty of the participants answered in full, so the number of the responses for each question ranged from fifty to sixty-two.

The online survey was developed when the previous study (Kim, 2022) was conducted. Some of the questions from the online survey were selected for this study and the language was modified appropriately for students. The selection of questions was made based on the analysis of the previous study considering the cognitive load for students, the comprehension of the language used in the questionnaire, and the expected duration of participation. From the previous study, the average time of answering each of the questions was recorded through the platform and the selection of the questions for this study was made to keep the expected time of participation less than 20 minutes. Additionally, a pilot study with the initial set of questions was conducted with other 11th graders in the same school who did not participate in this study. Then, the initial set of the questions were modified and the modified set of the questions was used for this study. Given that the survey items used in the previous study were developed for secondary mathematics teachers, it was inevitable to modify the language and to reduce the number of the survey items. Among the survey items of the previous study, nine questions were selected for this study and the resulting set of questions is shown below (see Table 1).

Table 1. The survey questions (Adapted from Kim, 2022)

1*. Please type in your student ID.
2*. Please explain what a proof means in mathematics.
3. Do you think proof is essential in mathematics?
4*. Please explain why you think so.
5*. Please explain what examples can do and can't in mathematics.
6. Which of approaches do you find most convincing to you?
7*. Please explain why you ranked order the approaches in question 6 in that order.
8. Which of approaches do you think are close to a mathematical proof?
9*. Please explain why you ranked order the approaches in question 8 in that order.

Note: Question numbers with asterisks denote that the corresponding responses for the questions were recorded in writing. Otherwise, the responses were recorded on a four- or five-point Likert scale.

The questions were developed to learn about students' views about proof and how likely they are to accept an empirical argument as a proof for the given proposition. Based on the taxonomy (Stylianides & Stylianides, 2009), argument 1 is concerned with näive empirical justification scheme that a mathematical proposition is validated through testing the mathematical proposition against a proper subset of cases in its domain. Argument 2 is concerned with crucial experiment justification scheme in that argument 2 involves a particular case but it also makes explicit the concern about the potential existence of counterexamples to the mathematical proposition. While argument 3 is related to crucial experiment justification scheme, the argument contains a figure that makes the argument more convincing than argument 2 (Roh & Kang, 2015). Argument 4 is a valid proof that is concerned with the nonempirical justification scheme that one can recognize the limitation of an empirical argument as a proof. The developed arguments are provided in Figure 1 which also provides an overview of the arguments used in the online survey using the taxonomy (Stylianides & Stylianides, 2009).

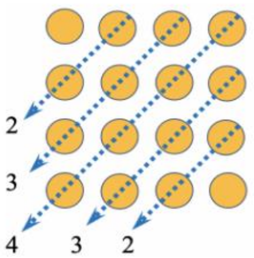
Argument 1	I substitute n with 3 and it worked out: $1+2+3+2+1=9=3^2$
	Näive empirical justification scheme ***Commentary:*** Argument 1 is purely empirical in the sense that the truth of the proposition is confirmed against the case $n=3$ without consideration of its generalizability to other natural numbers.
Argument 2	I substitute n with 4 and it worked out: $1+2+3+4+3+2+1=16=4^2$ and the same will hold true for all whole numbers.
	Crucial experiment justification scheme ***Commentary:*** argument 2 involves a particular case ($n=4$) with consideration that the argument would hold true for all other natural numbers than 4.
Argument 3	I substitute n with 4 and it worked out: $1+2+3+4+3+2+1=16=4^2$ As shown below, for other cases, squares with n circles in a diagonal line will be constructed and the squares should have n times n circles. Therefore, the given conjecture holds true for all whole numbers. 
	Crucial experiment justification scheme ***Commentary:*** Even though argument 3 involves a particular case ($n=4$), it contains a figure that demonstrates the general structure shared among all natural numbers and potentially leads to the development of a proof for the proposition.
Argument 4	$1+2+\cdots+(n-1)+n+(n-1)+(n-2)+\cdots 2+1$ $=2\{1+2+\cdots+(n-1)\}+n$ $=[\{1+(n-1)\}+\{2+(n-2)\}+\cdots+\{k+(n-k)\}+\cdots+\{(n-1)+1\}]+n$ $=(n+n+\cdots+n)+n$ $=(n-1)n+n$ $=n^2-n+n$ $=n^2$
	Nonempirical justification scheme ***Commentary:*** argument 4 is a proof that involves a general case, showing that the proposition holds true through algebraic manipulation. These four arguments were developed to learn about which characteristic of an argument is more convincing or valid as proof to students.

Figure 1. Four arguments to prove the mathematical proposition "For a natural number n, $1+2+\cdots+(n-1)+n+(n-1)+(n-2)+\cdots 2+1=n^2$" (Adapted from Kim, 2022, p. 72)

Data Analysis

The overall process of analyzing the data was two-fold. Responses were bifurcated into two types (i.e. either multiple-choice or writing) and each type of response was undergone a different analysis. The responses for questions 3, 6, and 8 that were recorded in a four- or five-point Likert scale were statistically analyzed and the written responses corresponding to questions 4, 7, and 9 provided the reasoning behind the statistical analysis of the responses recorded in a four- or five-point Likert scale. In analyzing the responses to the questions of which responses were recorded in writing, the responses for questions 2, 4, 5, 7, and 9 underwent a reiterative inductive coding (Creswell & Poth, 2016). In this process of inductive coding, I used as a priori the roles of proof (Ellis et al., 2019; Knuth, 2002a; Weber, 2010) and excluded some of the codes that have no relevant responses in the data. The resulting set of codes is as shown below (see Table 2).

Table 2. The resulting set of codes for what proof means (Adapted from Kim, 2022)

Code	Definition	Example Response
Verification	Proof is a means to verify the truth/falsity of a mathematical claim.	*"A proof enables one to determine the truth/falsity of a conjecture."*
Explanation	Proof is a means to provide explanation about why a mathematical claim holds true/false.	*"An explanation that logically provide the reason why a mathematical claim holds true based on the known facts."*
Derivation	Proof is a means to derive formulae from the mathematical facts which are known to be true.	*"A derivation from known formulae for a new fact."*
Problem solving	Proof is a result of solving a problem.	*"Through a proof, an answer is yielded."*

V. RESULTS

Students' Views about the Need of Proof

Students were asked to provide their definitions for mathematical proof. The total number of responses was 57. The responses were inductively and reiteratively coded and the responses were classified into the four codes (see Table 2) and the code other which is used for cases not fallen under any of the four codes. Though double coding was allowed, there was no instance to which two codes were given. The code with the most frequency (26 responses, 45.6%) was *explanation* that proof serves as a means to provide a reason why a mathematical proposition holds true, followed by the code *verification* with the occurrence of twelve (21.1%). There were also codes, *derivation* and *problem solving*, tied with the same frequency of 5 (8.8%). The rest was coded for others. Under the code other, there were responses including: *"Something that must be correct," "A mathematical one," "To make explicit,"* and *"An explanation about what a concept means."* These responses were ambiguous in meaning and seemed to be too general. A summary of the results is shown below in Table 3.

Table 3. A Summary of student's views about the role of mathematical proof

Code	Count (percentage)
Explanation	26 (45.6%)
Derivation	5 (8.8%)
Verification	12 (21.1%)
Problem Solving	5 (8.8%)
Other	9 (15.8%)

Students' views about the role of mathematical proof were concerned mostly with explanation and verification. This result resonates with the results of teachers (Basturk, 2010; Knuth, 2002b; Na, 2014).

Students' Tendencies to Accept Empirical Arguments as Proofs and Evaluation of the Convincingness of The Arguments to Their Eyes

Students were asked to consider the mathematical proposition: For a natural number n, $1 + 2 + \cdots + n - 1 + n + n - 1 + n - 2 + \cdots 2 + 1 = n^2$. They were then asked to evaluate the given arguments with regard to convincingness to their eyes and validity as a mathematical proof. Mathematically, argument 4 is in the best proximity to a valid proof while arguments 3, 2, and 1 are invalid as proofs due to the fact that the arguments do not involve a general case and fall short of providing an operation that is readily applicable to other natural numbers. The author assumed that this rank order would be reversed when students were asked to evaluate the same set of the arguments with respect to convincingness to their eyes.

Students generally found argument 4 as being most convincing to them and arguments 3, 2, and 1 as being somewhat, less, and the least convincing, respectively. The slight majority (52%) of the students found argument 4 as most convincing and argument 3 as somewhat convincing (43.1%) followed by arguments two (less convincing, 70.6%) and one (least convincing, 67.3%). This is in resonance with the results of the study involving secondary teachers in Korea (Kim, 2022). Table 4 provides a summary of the results.

Table 4. Students' evaluations of the given arguments with respect to convincingness to them

Argument	Least Convincing	Less Convincing	Somewhat Convincing	Most Convincing
1	35 (67.3%)	3 (5.8%)	5 (9.6%)	9 (17.3%)
2	6 (11.8%)	36 (70.6%)	9 (17.7%)	0 (0%)
3	5 (9.8%)	8 (15.7%)	22 (43.1%)	16 (31.4%)
4	5 (10%)	4 (8%)	15 (30%)	26 (52%)

Generally, the rank order observed from the data was close to the author's anticipation that most students tend to consider argument 4 as being most convincing followed by arguments 3, 2, and 1. Some of the quotes in this view were:

> *"Logical solutions seemed to be more convincing than intuitive ones."*
> *"(I) ranked order the arguments with more details."*
> *"(I) ranked order them in the order of easiness of apprehension."*
> *"Argument 1 only shows an example and does not guarantee that it works for all other natural numbers. Argument 2 uses a figure and the same works for natural numbers up to 4. But the same might not work for other numbers. Argument 3 instantiates how the same would work for all other numbers beyond the example provided and seems to be valid. Argument 4 shows how the two quantities are mathematically equal, thus indicating that the argument is no way invalid."*

The quotes shown above bring to the fore several grounds on which students considered as important when evaluating the convincingness of the given arguments to their eyes. Albeit the use of general terms such as logical solution, detail, and easiness, the terms make explicit what aspects students consider as important in evaluating the convincingness of a mathematical argument. These superficial characteristics of mathematical arguments were also evident in students' written responses when they were asked to evaluate the validity of the given arguments as mathematical proofs:

> *"Arguments with [figures] do not seem right at all."*
> *"The substitution of a number into the equation is very incorrect. And argument 3 is of most [readability]. (Thus, I rated argument 3 as most convincing)."*
> *"Without an instantiation of a confirming example, it would be very difficult to follow a purely logical solution with [formulae]."*

These responses allow for a glimpse at student's reasoning behind their evaluations of the given arguments with respect to validity as proof. The students' evaluations seem to be based on superficial characteristics (e.g., figure, readability, formulae) that are readily approachable to them. These tendencies to rely more on the superficial characteristics of a mathematical argument than on the substance of it rescore the results of previous studies (Coe & Ruthven, 1994; Healy & Hoyles, 2000; Inglis & Alcock, 2012; Knuth, 2002a) that students tend to rely heavily on the surface features of mathematical arguments when asked to evaluate mathematical arguments.

Students' evaluations of the given mathematical arguments with respect to validity as proof generally follow the same trend documented previously (Kim, 2022) that the ascending order of the numbers assigned to the arguments is reversed when students were asked to evaluate the given arguments with respect to validity as mathematical proof. In other words, by a majority vote, students tended to consider argument 4 as being most valid (56%) followed by arguments 3 (54%), 2 (56%), and 1 (60%), consecutively. In comparison to the previous study (Kim, 2022), the proportion that the majority accounts for in each argument is much less than that of the secondary teachers' evaluations of the given arguments. Twenty-eight students (56%) considered argument 4 as being most valid as a mathematical proof while twenty-seven students (54%) evaluated argument 3 as somewhat valid. Arguments 2 and 1 were given an evaluation of the mathematical arguments with respect to the validity as proof as less valid (56%) and the least valid (60%), respectively. The detailed results are provided below in Table 5.

Table 5. Students' evaluations of the mathematical arguments with respect to validity as proof

Approach	Least Valid	Less Valid	Somewhat Valid	Most Valid
1	30 (60%)	4 (8%)	3 (6%)	13 (26%)
2	4 (8%)	28 (56%)	15 (30%)	3 (6%)
3	7 (14%)	10 (20%)	27 (54%)	6 (12%)
4	9 (18%)	8 (16%)	5 (10%)	28 (56%)

The evaluations students made with respect to the validity of given arguments as proofs were generally in the same order that the author anticipated. However, the proportion of the majority vote to each of the mathematical arguments was lower than expected. In other words, only the slight majority (56%) of the students were able to distinguish the proof (i.e. argument 4) from the empirical arguments.

The Necessity of Proof in School Mathematics

The vast majority of the students considered proof as necessary in mathematics. The students were asked to show their degrees of agreement with the argument that proof is indispensable in mathematics using a five-point Likert scale: Strongly agree, somewhat agree, neither agree nor disagree, somewhat disagree, and strongly disagree. The vast majority (87.8% accumulatively) of the students agreed with the necessity of proof in mathematics to varying degrees while 12.3% of the students remained neutral or disagreed with the argument. Table 6 provides an overview of this result.

Table 6. Students' Perceptions about the Need of Proof in Mathematics

Degree of Agreement	Count (percentage)
Strongly agree	25 (43.9%)
Somewhat agree	25 (43.9%)
Neither agree nor disagree	5 (8.8%)
Somewhat disagree	2 (3.5%)
Strongly disagree	0 (0%)

The students' views about the need for proof in mathematics were well articulated in written responses. Some of the written responses provided by those who showed strong agreement with the statement are as follows:

> *"Proof is the only means that establishes everything in mathematics. Without proof, mathematics would become vague and unclear."*
> *"Proof makes properties and formulae available to use."*
> *"Because (mathematics) needs reasons and evidence (for theorems)."*
> *"Things unproved are grounded in no evidence. Those ungrounded would cause errors in mathematics."*

The written responses given above and others make explicit reasons why students thought proof is necessary in mathematics. Based on their responses, students generally considered proof as a means that establishes the truth of mathematical facts. One of the responses provided by those students who considered proof as indispensable in mathematics caught the author's attention.

> *"(I think) proof is absolutely necessary in mathematics. However, I also think proof needs not compulsory for high school students."*

The student seemed to be ambivalent about the necessity of proof in mathematics and his high school studies of mathematics given that he showed strong agreement with the statement that mathematics is indispensable in mathematics. His response fell short of providing more reason behind the backdrop, however, other students who showed neutral or somewhat disagreement with the necessity of proof articulated reasons why they considered proof as not necessary in mathematics.

> *"In Korea, problems are readily solved without understanding any proofs."*
> *"Using the results from proofs is far more important than proving."*
> *"Not knowing what proof is makes me incognizant of neither why proof is necessary in mathematics nor why it is not."*

These responses are concerned with the view that some students considered proof as something that is a game for mathematicians. These students seemed to view themselves as only spectators of the game rather than participants. This view coincides with the results of the extant literature that proof is not intended for all students or something that only a select few can do (Basturk, 2010; Knuth, 2002a), thus considering proof as something they can't do their own and what others do.

V. DISCUSSION

This study reports on high school students' views about the role and necessity of proof in mathematics, and their evaluations of mathematical argument with respect to convincingness and validity as proof. The students tended to consider proof as a means to provide an explanation about why a mathematical proposition holds true or as a means to verify the truth or falsity of the mathematical proposition. While acknowledging the explanatory function of proof, some students found empirical arguments more convincing than proofs and even considered empirical arguments valid as proof. This may be attributable to the lack of explanatory power of algebraic representation to students' eyes. Healy and Hoyles (2000) put "it offered them little in the way of explanation ... and [they] found them hard to follow" (p. 415). The vast majority (about 88%) of the students shared the view that proof is necessary in mathematics to varying degrees. However, it is not the author's intention that students' understandings of proof and evaluations about mathematical arguments are limited. Rather, the author intends to call for researchers' and teachers' attention to these issues to change the narrative of the instruction of proof as to where attention and efforts must be made to facilitate students to develop a robust understanding about proof and evaluation of mathematical arguments, thus being able to better distinguish empirical arguments from proofs and appreciating the indispensable role of proof in mathematics.

This study suggests various future research. Given the small sample size of the study, the results must be interpreted in a qualified manner that the results may only provide a glimpse at the views about the role and necessity of proof among the high school student population in Korea, suggesting that similar studies with varying sample sizes would allow for reconsideration of the results of this study. As the authority of the teacher may certainly play a role in students' evaluations of empirical argument as proof (Bell, 1976; Chazan, 1993; Fischbein, 1982; Harel & Sowder, 1998), teacher's knowledge package for teaching proof may help students to gradually develop their understanding about the limitation of empirical argument as proof (Stylianides, 2011). Another variation may be made through involving different samples of students such as female students or students at a different level of education (e.g., elementary, tertiary, different grades). Furthermore, another research would be possible to examine differences in students' responses when they are asked to evaluate empirical arguments as proofs for different mathematical propositions in a different setting (e.g., clinical interview, whole-class discussion).

This study contributes to teacher knowledge about students and proof and the literature on several grounds. Research indicates that there is a persistent problem in the teaching of proof: students have difficulty distinguishing empirical arguments from proofs (e.g., Bell, 1976; Chazan, 1993; Fischbein, 1982; Harel & Sowder, 1998). This study contributes to the literature in that the persistent issue in the teaching and learning of proof (i.e. tendencies to misunderstand empirical arguments for proofs) is still incumbent among students. As documented in Stylianides and Stylianides (2009), it is unlikely that students appreciate the necessity and importance of proof in mathematics if they do not realize the insecurity of validating a mathematical proposition with an empirical argument. It is teachers who help students make the progression of understanding the distinction between empirical argument and proof through cognitive conflicts in sequence (Stylianides, 2011). Some researchers attribute the reason for the limited understanding that teachers tend to have for proof to limited experiences of engaging in proving-related activities throughout their K-12 education and teacher preparation programs (e.g., Coe & Ruthven, 1994; Harel & Sowder, 1998; Mingus & Grassl, 1999). This may also be the case for students. Harel and Sowder (1998) argued that limited experiences seem to be manifested as tendencies to only accept certain forms of mathematical arguments (e.g., two-column proof, use of algebraic representations) as proofs and see the need for proof appropriate only in particular content areas of mathematics (Kim, 2022; Knuth, 2002b). In this regard, the results of this study contribute to teachers' knowledge about students and proof and call for teacher educators' attention to issues related to students' misconceptions that empirical arguments are proofs and for efforts to incorporate this knowledge into teacher education programs and professional developments.

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