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A Note on Theta Pairs for BCI-algebras

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Abstract. In this paper, first we show that all finite BCI-algebras are solvable. We then introduce the notion of a θ-pair for a maximal ideal of a BCI-algebra. Proving various properties of maximal θ-pairs, we use them to characterize solvable and nilpotent BCI-algebras.

1. Introduction

The concepts of BCK-algebras, and the more general BCI-algebras, were introduced by Y. Imai and K. Iséki [5, 6] in 1966. Since this time, various authors have studied and developed many concepts related to these algebraic structures; see for example [2, 8, 10, 11, 12]. In [4], Huang used the notion of nilpotency in ring theory to introduce the notion of nilpotency in BCI-algebras. See also [8], where this and a new definition of commutators and solvability in a BCI-algebra was given, and then used to prove that every finite nilpotent BCI-algebra is solvable. In this paper, we first improve this result (see [8, Theorem 6.3]) and show that every finite BCI-algebra is solvable. Also we introduce the notion of θ-pair for a maximal ideal of a BCI-algebra and give some results for nilpotency and solvability of a BCI-algebra. This is similar to the concept of θ-pair for any maximal subgroup of a group as introduced by Mukherjee and Bhattacharya [9], and Beidleman and Smith [1]. This concept has since been further studied by a number of authors, including Guo [3] and Li [7]. We also look at other useful properties of solvable and nilpotent BCI-algebras.

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2. Preliminaries and Basic Results

In this section we give some basic results which will be used in the rest of the paper.

A BCI-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$, if for any $x, y, z \in X$, it satisfies the following axioms:
$(BCI1)$ $((x * y) * (x * z)) * (z * y) = 0$,
$(BCI2)$ $(x * (x * y)) * y = 0$,
$(BCI3)$ $x * x = 0$,
$(BCI4)$ $x * y = y * x = 0$ implies $x = y$. [5, 6]

In any BCI-algebra X, one can define a partial order by putting $x \leq y$ if and only if $x * y = 0$.

Theorem 2.1.([13]) In any BCI-algebra X the following properties are satisfied for any $x, y, z \in X$:
$(BCI5)$ $x * 0 = x$.
$(BCI6)$ $(x * y) * z = (x * z) * y$.
$(BCI7)$ $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.
$(BCI8)$ $(x * z) * (y * z) \leq x * y$.
$(BCI9)$ $x * (x * (x * y)) = x * y$.
$(BCI10)$ $0 * (x * y) = (0 * x) * (0 * y)$.

Theorem 2.2.([13]) Let X be a nonempty set. Then X is a BCI-algebra if and only if there is a partial ordering $\leq$ on X such that for any $x, y, z \in X$, the following conditions hold:

(i) $(x * y) * (x * z) \leq (z * y)$,

(ii) $x * (x * y) \leq y$,

(iii) $x * y = 0$ if and only if $x \leq y$.

A nonempty subset S of a BCI-algebra X is said a subalgebra of X if $x * y \in S$ for any $x, y \in S$. Also a nonempty subset I of a BCI-algebra X is said an ideal of X if $0 \in I$ and if $x * y \in I, y \in I$, imply $x \in I$. Obviously, X and $\{0\}$ (we write 0 is an ideal of X, for convenience) are ideals of X, which called the trivial ideals of X. An ideal I is called proper, if $I \neq X$ and is called closed, if it is also a subalgebra of X.

A BCI-algebra X is called commutative if $x \leq y$ implies $x \wedge y = x$, where $x \wedge y = y * (y * x)$, for all $x, y \in X$. An ideal I of a BCI-algebra X is called commutative if $x * y \in I$ implies $x * ((y * (y * x)) * (0 * (y * x))) \in I$ for all $x, y \in X$.

In a BCI-algebra X we denote by $BCK(X)$ the BCK-part of X and set

$$BCK(X) = \{x \in X : 0 * x = 0\}.$$

If $X = BCK(X)$, then X is called a BCK-algebra. One can easily check that the BCK-part of X is a closed ideal of X. A BCI-algebra X is said to be p-semisimple if $0*(0*x)=x$, for all $x\in X$. The set $\{x\in X: 0*(0*x)=x\}$ is called the center of X and is denoted by $C(X)$ (see [12]).

Let $S\subseteq X$ be a non-empty set. The least ideal of X containing S is said the generated ideal of X by S and is denoted by $\langle S\rangle$. A proper ideal M of X is called a maximal ideal if $\langle M\cup\{x\}\rangle = X$, for any $x\in X\setminus M$. We note that M is a maximal ideal of X if and only if $M\subseteq I\subseteq X$ implies that $M=I$ or $I=X$, for any ideal I of X. We call a maximal ideal of X that is closed, as a closed maximal ideal of X. If A and B are two ideals of X, then the symbol $A+B$ will be used for $\langle A\cup B\rangle$. Moreover, If A and B are closed, then $A+B$ is a closed ideal of X (see [13], Proposition 1.4.15).

Let I be an ideal of a BCI-algebra X and $x,y\in X$. Following [8], we call the element

$$[x,y]=((x\wedge y)*(y\wedge x))*((0*(y*x))),$$

is the commutator of x_1 and x_2 of weight 2.
In general, the element $[x_1,x_2,...,x_n]=[[x_1,...,x_{n-1}],x_n]$ is a commutator of weight $n\geq 2$, where $[x_1]=x_1$.

Let I be an ideal of a BCI-algebra X. Then the relation $\sim$ defined by $x\sim y$ if and only if $x*y, y*x\in I$ is a congruence relation on X. Let I_x denote the class of $x\in X$ and X/I denote the set of all classes I_x, where $x\in X$. Then $(X/I,*,I_0)$ is a BCI-algebra, where $I_x*I_y=I_{x*y}$ and $I_x=I_y$ if and only if $x\leqslant y$. The BCI-algebra X/I is called the quotient BCI-algebra of X determined by I. Obviously, for any $x\in I$, $I_x=I$ if I is a closed ideal of X. Throughout the paper, X means a BCI-algebra $(X,*,0)$.

Lemma 2.3.([8]) For any $x,y\in X$, the following hold:

(i) $[x,y]*x\leq(0*x)$,

(ii) $0*[x,y]=0$,

(iii) $[0*x,y]=0$.

Theorem 2.4.([13]) An ideal I of a BCI-algebra X is closed if and only if $0*x\in I$, for any $x\in I$. Moreover, if X is of finite order, then any ideal of X is closed.

Theorem 2.5.([13]) Let S be a nonempty subset of a BCI-algebra X and let $A=\{x\in X: (\cdots((x*a_1)*a_2)*\cdots)*a_n=0; \textit{for some } a_1,a_2,...,a_n\in S\}$. Then $\langle S\rangle = A\cup\{0\}$. Moreover, if I is an ideal of X, then

$$\langle A\cup S\rangle=\{x\in X: (\cdots((x*a_1)*a_2)*\cdots)*a_n\in A; \textit{for some } a_1,...,a_n\in S\}.$$

Theorem 2.6.([13]) A closed ideal I of a BCI-algebra X is commutative if and only if the quotient algebra X/I is a commutative BCI-algebra.

Definition 2.7.([8]) Let $X_1,X_2,...,X_n$ be a non-empty subsets of X. A commutator of X_1 and X_2 is defined as $[X_1,X_2]=\langle\{[x_1,x_2]: x_1\in X_1, x_2\in X_2\}\rangle$. Moreover, for $n \geq 2$, $[X_1, ..., X_n] = [[X_1, ..., X_{n-1}], X_n]$. Hence $[X, X] = \langle\{[x, y] : x, y \in X\}\rangle$ and is called the derived ideal of X, which is denoted by $X' = X^{(1)}$.

Theorem 2.8.([8]) Let I be an ideal of X. Then X/I is commutative if and only if $X^{(1)} \subseteq I$.

By Lemma 2.3(ii), for any $[x, y] \in X^{(1)}$, $0 * [x, y] = 0 \in X^{(1)}$, which implies that $X^{(1)}$ is a closed ideal of X.

Corollary 2.9. Let X be a BCI-algebra. Then X is commutative if and only if $X^{(1)} = 0$.

By using Theorems 2.6 and 2.8, we conclude the following result.

Corollary 2.10. Let I be a closed ideal of BCI-algebra X. Then I is commutative if and only if $[x, y] \in I$, for all $x, y \in X$.

Lemma 2.11. Let X be a BCI-algebra and I be an ideal of X. Then for any $x, y \in X$, $[I_x, I_y] = I_{[x,y]}$.

Proof. Let $x, y \in X$. Then

$$\begin{aligned}[I_x, I_y] &= ((I_y * (I_y * I_x)) * (I_x * (I_x * I_y))) * (I_0 * (I_y * I_x)) \\ &= (I_{(y*(y*x))} * I_{(x*(x*y))}) * I_{(0*(y*x))} \\ &= I_{((y*(y*x))*(x*(x*y)))*(0*(y*x))} = I_{[x,y]}.\end{aligned}$$

□

Theorem 2.12.([13]) If I is a commutative ideal of a BCI-algebra X, then every closed ideal A of X with $I \subseteq A$, is commutative.

Definition 2.13. Let I be a subalgebra of a BCI-algebra X. The set

$$R_X(I) = \{x \in X : [x, a] \in I, \textit{for any } a \in I\},$$

is called the normalizer of I in X. Since I is a subalgebra of X, it follows that $I \subseteq R_X(I)$. Obviously, if X is commutative, then $R_X(I) = X$.

Lemma 2.14. Let X be a BCI-algebra and I be a subalgebra of X. Then $C(X) \subseteq R_X(I)$.

Proof. Suppose that $x \in C(X)$. Then $0 * (0 * x) = x$. Now for any $y \in X$,

$$\begin{aligned}[x, y] &= ((y * (y * x)) * (x * (x * y))) * (0 * (y * x)) \\ &\leq (x * (x * (x * y))) * (0 * (y * x)) \qquad \textit{by } (BCI7) \textit{ and Theorem } 2.2(ii)\end{aligned}$$

$$= (x * y) * (0 * (y * x)) \qquad \textit{by } (BCI9)$$

$$= (x * y) * ((0 * y) * (0 * x)) \qquad \textit{by } (BCI10)$$

$$= (x * y) * ((0 * (0 * x)) * y) \qquad \textit{by } (BCI6)$$

$$= (x * y) * (x * y) = 0 \qquad \textit{by assumption and } (BCI3)$$

Hence for all $y \in X, [x, y] = 0 \in I$ and so $C(X) \subseteq R_X(I)$, as required. □

Corollary 2.15. Let X be a p-semisimple BCI-algebra and I be a subalgebra of X. Then $R_X(I) = X$.

Proof. Since X is p-semisimple, $C(X) = X$. Now the result holds by Lemma 2.14. □

Example 2.16. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a BCI-algebra with the Cayley table as follows:

*	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	1	5
2	2	2	0	2	0	5
3	3	3	3	0	0	5
4	4	3	4	1	0	5
5	5	5	5	5	5	0

By simple calculations we obtain $I = \{0, 2, 4\}$ is a subalgebra of X such that $1 \notin R_X(I)$, because $[1, 4] = 1$, and $R_X(I) = \{0, 2, 3, 4, 5\}$.

Lemma 2.17. Suppose that A is a subalgebra and B is a closed ideal of a BCI-algebra X. Then $[A, B] \subseteq B$ if and only if $A \subseteq R_X(B)$.

Proof. Let $a \in A$. Then for any $b \in B$, $[a, b] \in [A, B] \subseteq B$. Hence $[a, b] \in B$ and so $A \subseteq R_X(B)$.

Conversely, if $A \subseteq R_X(B)$, then for any $a \in A, b \in B$, $[a, b] \in B$. Now let $u \in [A, B] = \langle [a, b] : a \in A, b \in B \rangle$. By Theorem 2.5, we get

$$(\cdots((u * x_1) * x_2) * \cdots) * x_n = 0 \in B,$$

for some $n \in \mathbb{N}$, where $x_i = [a_i, b_i], a_i \in A, b_i \in B$ and $i = 1, ..., n$. Since B is an ideal of X, it follows that $u \in B$, which shows that $[A, B] \subseteq B$, as required. □

Lemma 2.18. Let I be a subalgebra of a BCI-algebra X. Then $N_X(I) = \langle R_X(I) \rangle = \bigcap_{R_X(I) \subseteq J} J$ where J is any ideal of X such that $R_X(I) \subseteq J$, is the closed ideal of X contains I. Moreover, if I is a commutative closed ideal of X, then $N_X(I)$ is the largest commutative closed ideal of X containing I.

Proof. First by Lemma 2.3(iii), for any $x \in N_X(I)$ and $a \in I, [0 * x, a] = 0 \in I$ and thus $0 * x \in R_X(I)$, which shows that $N_X(I)$ is the closed ideal of X. Also if I is a commutative closed ideal of X, then by Theorem 2.12, $N_X(I)$ is commutative. Next, let K be a commutative closed ideal of X such that $I \subseteq K$. By Corollary 2.10, for all $k \in K$ and $a \in I$, $[k, a] \in I$ and so $K \subseteq R_X(I) \subseteq N_X(I)$, which shows that $N_X(I)$ is the largest commutative closed ideal of X containing I, as required. □

We observe that if I is a closed ideal of a BCI-algebra X, then $C_X(I) \subseteq N_X(I)$, where $C_X(I) = \langle \{x \in X : [x, a] = 0, [a, x] = 0, \forall a \in I\} \rangle$ is said the centralizer of I in X (see [8]).

Definition 2.19. For an ideal U of X, let U_X, the core (with respect to X) of U, be the largest closed ideal of X contained in U. Obviously if U is a closed ideal of X, then $U_X = U$.

Corollary 2.20. If X is a finite BCI-algebra and I is an ideal of X, then $I_X = I$.

Proof. Since by Theorem 2.4, all ideals of X are closed, we deduced that $I_X = I$, for any ideal I of X. □

Theorem 2.21.([13]) Suppose that A and B are ideals of a BCI-algebra X and let $AB = \cup_{a \in A} B_a$, where B_a is an equivalence class in X/B. If B is closed, then $AB = A + B$, where $A + B = \langle A \cup B \rangle$.

Theorem 2.22.([13]) If H is a subalgebra of X and K is a closed ideal of X, then $HK/K \cong H/(H \cap K)$.

3. Nilpotent and Solvable BCI-algebras

In this section, we provide some results concerning nilpotent and solvable BCI-algebras. In [8], Mohammadzadeh and Borzooei introduced the concept of nilpotent BCI-algebra, according to nilpotency in group theory, as follows:

Definition 3.1.([8]) Let $Z_0(X) = 0$, $Z_n(X) = \langle \{x : [x, y_1, ..., y_n] = 0, \text{ for any } y_1, ..., y_n \in X\} \rangle$ for any $n \geq 1$. By Lemma 2.3(iii), $Z_n(X)$ is a closed ideal of X. The sequence of ideals

$$0 = Z_0(X) \subseteq Z_1(X) \subseteq ... \subseteq Z_n(X),$$

is called the upper central series of X. Its i-th term $Z_i(X)$ is called the i-th center of X. Now, X is called nilpotent, if there exists $n \in \mathbb{N}$ such that $Z_n(X) = X$. The smallest such integer is called the class of X. We note that $Z_1(X) = Z(X) = \langle \{x : [x, y] = 0, \text{ for any } y \in X\} \rangle$.

Example 3.2. Let $-$ be the subtraction of integers. Then $X = (\mathbb{Z}, -, 0)$ is a p-semisimple BCI-algebra and so $C(X) = X$ (see [13], Example 5.3.2). It follows that $[x, y] = 0$ for all $x, y \in X$, by the proof of Lemma 2.14 and hence $Z_1(X) = X$, which shows that X is nilpotent of class at most 1.

Lemma 3.3.([8]) Let $i > 0$. Then $[Z_i(X), X] \subseteq Z_{i-1}(X)$.

Theorem 3.4.([8]) Let X be a BCI-nilpotent algebra and I be a nontrivial proper closed ideal of X. Then $I \neq N_X(I)$.

Proof. Assume that X is nilpotent of class r with the upper central series:

$$0 = Z_0(X) \subseteq Z_1(X) \subseteq ... \subseteq Z_r(X) = X.$$

Let $A = \{m : \ Z_m(X) \nsubseteq I, 1 \leq m \leq r\}$. It is obvious that $r \in A$. Let $k = \min \ A$. Hence $Z_k(X) \nsubseteq I$ and $Z_{k-1}(X) \subseteq I$. Now, we observe that by Lemma 3.3,

$$[Z_k(X), I] \subseteq [Z_k(X), X] \subseteq Z_{k-1}(X) \subseteq I.$$

Hence by Lemma 2.17, $Z_k(X) \subseteq R_X(I)$, it follows that $I \subset R_X(I) \subseteq N_X(I)$ and the result holds. □

The following immediate corollary of the above theorem is straightforward.

Corollary 3.5. If X is a BCI-nilpotent algebra and M is a closed maximal ideal of X, then $N_X(M) = X$.

Theorem 3.6.([8]) A BCI-algebra X is commutative if and only if it is nilpotent of class at most 1.

Corollary 3.7. For any BCI-algebra X, the following properties are equivalent:

(i) X is nilpotent of class at most 1,

(ii) the zero ideal $\{0\}$ is commutative,

(iii) every closed ideal of X is commutative.

Proof. The proof is trivial from Theorem 2.5.19 and Corollary 2.5.20 of [13], and Theorem 3.6. □

Lemma 3.8.([8]) If X is nilpotent BCI-algebra, then any subalgebra of X is nilpotent. Also if I is a BCI-ideal of X, then X/I is nilpotent.

Theorem 3.9.([8]) Let I be an ideal of BCI-algebra X and $n, m \in \mathbb{N}$. If I is a nilpotent BCI-ideal of class m and X/I is nilpotent of class n, then X is nilpotent of class $n+m$.

Theorem 3.10.([8]) Let X be a nilpotent BCI-algebra of class $n \geq 1$ and N be a nontrivial closed ideal of X. Then $N \cap Z(X) \neq 0$.

Corrollary 3.11.([8]) Let X be a nilpotent BCI-algebra of class $n \geq 1$. If N is a minimal (closed) ideal of X, then $N \subseteq Z(X)$.

The following concept was introduced by Mohammadzadeh and Borzooei [8], by applying a new definition of derived ideal (Definition 2.7).

Definition 3.12. Let X be a BCI-algebra, $X^{(1)} = [X, X]$ and for any $n \in \mathbb{N}, X^{(n)} = [X^{(n-1)}, X^{(n-1)}]$. Then X is called solvable if there exists $n \in \mathbb{N}$ such that $X^{(n)} = 0$. The smallest such n is called derived length of X. Clearly by Lemma 2.3(iii), $X^{(n)}$ for any $n \in \mathbb{N}$ is a closed ideal of X.

Lemma 3.13.([8]) If X is a solvable BCI-algebra, then any subalgebra of X is solvable. Also if I is a BCI-ideal of X, then X/I is solvable.

Theorem 3.14.([8]) Let I be an ideal of X. If I and X/I are solvable BCI-algebras, then X is a solvable BCI-algebra.

Theorem 3.15.([8]) Let X be a finite BCI-algebra and $[x, y] \leq x$, for any $x, y \in X$. Then X is solvable.

In the following theorem, we give a necessary and sufficient condition on a BCI-algebra X, such that X to be solvable.

Theorem 3.16. The BCI-algebra X is solvable if and only if there exists a chain of closed ideals $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_k = 0$ such that each quotient X_l/X_{l+1}, $0 \leq l \leq k-1$ is commutative.

Proof. Since X is solvable, there exists $i \in \mathbb{N}$ such that $X^{(i)} = 0$. Then we have a sequence of closed ideals:

$$X \supseteq X^{(1)} \supseteq X^{(2)} \supseteq \cdots \supseteq X^{(i)} = 0.$$

We show that each quotient $X^{(l)}/X^{(l+1)}(1 \leq l \leq i-1)$, is commutative. Let $a, b \in X^{(l)}$. Since $X^{(l+1)} = [X^{(l)}, X^{(l)}]$, it follows that $[a, b] \in X^{(l+1)}$ and then by Lemma 2.11,

$$[X_a^{(l+1)}, X_b^{(l+1)}] = X_{[a,b]}^{(l+1)} = X^{(l+1)} = X_0^{(l+1)}.$$

Therefore $X^{(l)}/X^{(l+1)}$ is commutative, by Corollary 2.9.

Conversely, let $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_k = 0$ be the closed ideals of X such that each quotient X_l/X_{l+1}, $0 \leq l \leq k-1$ is commutative. Therefore X_k

and X_{k-1}/X_k are solvable, which show that X_{k-1} is solvable by Theorem 3.14. Similarly, X_{k-2}/X_{k-1} is solvable, and thus X_{k-2} is solvable. Continuing this way, X_1 is solvable. Next, since X/X_1 is commutative, it follows that $X_0 = X$ is solvable, as desired. □

By using the concept of solvable BCI-algebras, we have the following main theorem.

Theorem 3.17. Let X be a finite BCI-algebra. Then X is solvable.

Proof. Suppose that X is a finite BCI-algebra and $B = BCK(X)$ is the BCK-part of X. Since by Lemma 2.3(ii), $0 * [x, y] = 0$ for all $x, y \in X$, it follows that $X^{(1)} \subseteq B$. We observe that if $|B| \leq 2$, then $X^{(1)} = 0$ and so by Corollary 2.9, X is commutative, which shows that X is solvable. Next, we assume that $|B| > 2$. Then by Lemma 2.3(i), $[x, y] * x \leq 0 * x = 0$ for all $x, y \in B$. Thus $[x, y] * x = 0$ and so $[x, y] \leq x$. Hence by Theorem 3.15, B is solvable. Now X/B is solvable, since X/B is commutative by Theorem 2.8 which, together with Theorem 3.14, implies that X is solvable. □

Remark 3.18. Note that, the above theorem was proved in [8, Theorem 6.3] with extra condition that G must be a nilpotent BCI-algebra.

As an application of Theorem 3.17, we give the following main result.

Corollary 3.19. A finite BCI-algebra is nilpotent.

Proof. Assume that X is a finite BCI-algebra. By Theorem 3.17, X is solvable. Suppose on the contrary, that X is non-nilpotent of the smallest order. If $X^{(1)} = 0$ then X is commutative by Corollary 2.9 and so X is nilpotent, which is impossible. Moreover, if $X^{(1)} = X$, then $X = X^{(i)}$ for any $i \in \mathbb{N}$, which shows that X is non-solvable, a contradiction. Now $X/X^{(1)}$ and $X^{(1)}$ are nilpotent and so by Theorem 3.9, X is nilpotent, again a contradiction. This completes the proof. □

4. θ-pairs in BCI-algebras

In this section, we determine the concept of θ-pair for a maximal ideal in a BCI-algebra. Moreover, we obtain some results on the maximal θ-pairs when the BCI-algebra is solvable or nilpotent. For convenience, we denote $M \lessdot X$ to indicate that M is a maximal ideal of a BCI-algebra X. The following definition is essential in our investigation.

Definition 4.1. Let M be a maximal ideal of BCI-algebra X. A pair (C, D) of subalgebras of X is said to be a θ-pair for M if it satisfies the following conditions:

(a) D is a closed ideal of X, contained in C;

(b) $D \subseteq M$ and $C \nsubseteq M$;

(c) C/D includes properly no nonzero closed ideal of X/D.

Furthermore, if C is an (a closed) ideal of X, then the pair (C, D) is called an (a closed) ideal θ-pair for M. This concept will be use to investigate the influence of the maximal ideals on the structure of certain BCI-algebras.

If M is a maximal ideal of X, then we denote by $\theta(M)$, the set of all θ-pairs of M, and define a partial order on it by means of $(C_1, D_1) \leq (C_2, D_2)$ if and only if C_1 is a subalgebra of C_2, whence $\theta(M)$ will contain maximal elements with respect to this ordering, which called maximal θ-pairs. We denote the set of all maximal θ-pairs for M, by $\theta_{max}(M)$. Also we call a closed ideal θ-pair $(A, B) \in \theta(M)$ is maximal closed ideal θ-pair, if there is no closed ideal θ-pair $(C, D) \in \theta(M)$ such that A is a proper subalgebra of C.

This is similar to the concept of θ-pair for any maximal subgroup of a finite group as suggested by Mukherjee and Bhattacharya [9], which has since been further investigated by a number of authors (see [3, 7]). Also Beidleman and Smith applied this concept for infinite group (see [1]).

Example 4.2. Let $X = \{0, a, b, c, d\}$ and the binary operation $*$ be defined as follows:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	a	0	b	a
c	c	c	c	0	c
d	d	d	d	d	0

Then X is a BCI-algebra. Let $M = \{0, a, b, c\}$, $A = \{0, c, d\}$ and $B = \{0, c\}$. Hence M is a closed maximal ideal of X, A is a subalgebra of X which is not an ideal, $B \subset M = M_X$ and $(A, B) \in \theta(M)$.

Example 4.3. Let $X = \{0, 1, a, b, c\}$ and define the binary operation $*$ on X by the following table:

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Then $(X, *, 0)$ is a BCI-algebra. Let $M = \{0, 1, a\}$, $C = \{0, 1, b\}$ and $D = \{0, 1\}$. Then $M \lessdot X$ and (C, D) is a closed maximal ideal θ-pair for M.

Example 4.4. For a BCI-algebra $X = (\mathbb{Z}, -, 0)$, where $-$ be the subtraction of integers, we define two closed maximal ideals of X, as $M_1 = \{2n : n \in \mathbb{Z}\}$ and $M_2 = \{3n : n \in \mathbb{Z}\}$. We conclude that $(X, M_1) \in \theta_{max}(M_1)$ and $(X, M_2) \in \theta_{max}(M_2)$.

Lemma 4.5. Let M be a maximal ideal of a BCI-algebra X and $(C, D) \in \theta(M)$. Then $D \subset M_X$.

Proof. Since D is a closed ideal of X such that $D \subset M$, it follows that $D \subset M_X$. □

The following theorem is a useful fact in proving our next results.

Theorem 4.6. Let X be a BCI-algebra, M be a maximal ideal of X and (C, D) be an ideal θ-pair for M. Then

(i) $D = (C \cap M)_X$.

(ii) $D = C \cap M_X$ if (C, D) is a closed ideal θ-pair for M; otherwise $D = C_X$.

Proof. (i) Obviously, $D \subseteq (C \cap M)_X$. For the converse, assume by way of contradiction that $(C \cap M)_X \nsubseteq D$. Then $D + (C \cap M)_X$ is a closed ideal of X containing properly D and contained in C. It follows that $C = D + (C \cap M)_X$, because $(C, D) \in \theta(M)$. Hence $X = \langle C, M \rangle = M + D + (C \cap M)_X = M$, which is a contradiction.

(ii) First suppose that C is a closed ideal of X. Then $C \cap M_X \subseteq (C \cap M)_X$ and so by (i), $C \cap M_X \subseteq D$, whence $D = C \cap M_X$. Next, assume that C is not closed ideal of X. Then C_X is a closed ideal of X containing D. Since C_X is a proper ideal of C, it follows that $D = C_X$ by the definition of θ-pair. □

Corollary 4.7. If $(A, B), (C, D) \in \theta(M)$ and $(A, B) \leq (C, D)$, then $B \subseteq D$.

Proof. Since $A \subseteq C$, it follows that $B = (A \cap M)_X \subseteq (C \cap M)_X = D$, by Theorem 4.6. □

Lemma 4.8. Let X be a BCI-algebra, $M \lessdot X$ and I be an ideal of X such that $I \subseteq M$.

(i) If (A, B) is a (an ideal) θ-pair for M and $I \subseteq B$, then $(A/I, B/I)$ is a (an ideal) θ-pair for M/I. Conversely, if $(A/I, B/I)$ is a (an ideal) θ-pair for M/I, then (A, B) is a (an ideal) θ-pair for M. In particular, (A, B) is a maximal member in $\theta(M)$ if and only if $(A/I, B/I)$ is a maximal member in $\theta(M/I)$.

(ii) If (A, B) is a closed ideal θ-pair of M, then $\theta(M)$ contains a maximal closed ideal θ-pair (C, D) such that $(A, B) \leq (C, D)$ and $A/B \cong C/D$.

Proof. (i) This is trivial from the definition of θ-pair.

(ii) If (A, B) is not a maximal member in $\theta(M)$, then $(A, B) \leq (A_1, B_1)$, where $(A_1, B_1) \in \theta(M)$. If $B = B_1$, then $B_1 = B \subset A \subseteq A_1$, and so A_1/B_1 includes properly nonzero closed ideal A/B_1 of X/B_1, contracting the fact that $(A_1, B_1) \in \theta(M)$, whence by Corollary 4.7, B is a proper subalgebra of B_1. Also $A \cap B_1 = B$; otherwise we will have $B \subset A \cap B_1 \subset A$, which is impossible by definition of θ-pair. It is readily verified that $A + B_1 = A_1$. Next, we prove that A_1 is a closed ideal of X. Let $y * x, x \in A_1$. Since $A_1 = B_1 + A$, then by Theorem 2.5, and (BCI6) we get

$$(\cdots((y * x) * a_1) * \cdots) * a_n = (\cdots((y * a_1) * a_2) * \cdots * a_n) * x \in B_1,$$

for some $n \in \mathbb{N}$ and $a_1, ..., a_n \in A$. Since $A \subseteq A_1$, it follows that $y \in A_1 + B_1 = A + B_1 + B_1 = A + B_1 = A_1$, as required. Also since A is a closed ideal of X, then as $A + B_1 = A_1$, we deduced that A_1 is closed. Now by Theorems 2.21 and 2.22

$$A_1/B_1 = (A + B_1)/B_1 = AB_1/B_1 \cong A/A \cap B_1 \cong A/B.$$

Finally, if (A_1, B_1) is not maximal in $\theta(M)$, we may replace (A, B) by (A_1, B_1) in the above and derive a same conclusion.

□

Lemma 4.9. There exists a closed ideal θ-pair in $\theta(M)$ for every maximal ideal M of X.

Proof. Suppose that C is a closed ideal of X such that $C \nsubseteq M$ and D denotes the sum of all closed ideals I in X such that $I \subseteq M \cap C$. We say that if C/D has no contains any nonzero closed ideal of X/D, then (C, D) is a closed ideal θ-pair; otherwise, assume that E/D is a minimal closed ideal of X/D which is contained in C/D. Now, it is easy to see that (E, D) is a closed ideal θ-pair for M. □

Theorem 4.10. Let X be a BCI-algebra, M be a maximal ideal of X and (C, D) be a maximal closed ideal θ-pair for M. Then $D = M_X$.

Proof. It is sufficient to show that if I is a closed ideal of X contained in M, then $I \subseteq D$. Suppose on the contrary that $I \nsubseteq D$. We note that if $I \subseteq C$, it follows that $D + I = C$, because $(C, D) \in \theta(M)$ and then $X = M + C = M + D + I = M$, which is impossible. Hence $I \nsubseteq C$ and C is a proper ideal of $C + I$. Next, we claim that $(C + I, D + I) \in \theta(M)$. It is easy to see that the pair $(C + I, D + I)$ satisfies both conditions (a) and (b) in the definition of θ-pair. Now, let A be a closed ideal of X such that $D + I \subseteq A \subseteq C + I$. Then $(C \cap A)/D$ is a closed ideal of X/D contained in C/D. Hence either $C \cap A = D$ or $C \cap A = C$. To continue the proof, we consider two cases:

Case I. $C \cap A = D$. In this case we show that $D + I = A$. Let $a \in A \subseteq I + C$. Since I and C are ideals of X, then by Theorem 2.5, we get $(\cdots((a * x_1) * x_2) \cdots) * x_n \in I$, for some $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n \in C$. It follows that there exists $i \in I$ such that $(\cdots((a * x_1) * x_2) \cdots) * x_n = i$. Hence by $(BCI3)$, and $(BCI6)$, we get

$$((\cdots((a * i) * x_2) \cdots) * x_n) * x_1 = ((\cdots((a * x_1) * x_2) \cdots) * x_n) * i = i * i = 0.$$

Since C is an ideal of X and $x_1, ..., x_n \in C$, then $a * i \in C$. Moreover $a * i \in A$ and thus $a * i \in A \cap C = D$. Therefore $a \in D + I$ and hence $A \subseteq D + I$. Therefore $A = D + I$.

Case II. $C \cap A = C$. Then $C \subseteq A$ and so $I + C \subseteq A$, because $I \subseteq A$. Hence $A = C + I$.
Therefore $(C + I, D + I) \in \theta(M)$, which contradicts the maximality of (C, D) in $\theta(M)$. □

Corollary 4.11. Let M be a closed maximal ideal of a BCI-algebra X. Then $\theta_{max}(M) = \{(X, M)\}$.

Proof. Since M is a closed ideal of X, $(X, M) \in \theta(M)$. Furthermore, if (A, B) is another maximal θ-pair for M in X, then $A = X$ and so (X, B) is a maximal closed ideal θ-pair for M in X. Hence by Theorem 4.10 and assumption, $B = M_X = M$ and so $\theta_{max}(M) = \{(X, M)\}$, as required. □

Lemma 4.12. Let X be a BCI-algebra, $M \lessdot X$ and I be an ideal of X such that $I \subseteq M$. Then $|\theta_{max}(M/I)| \leq |\theta_{max}(M)|$.

Proof. By Lemma 4.8, the map

$$\tau : \theta_{max}(M/I) \to \theta_{max}(M)$$
$$(C/I, D/I) \mapsto (C, D)$$

is well-defined. Now, it is easy to see that the map τ is one-to-one. □

As an application of Lemma 4.12, we get the following corollary.

Corollary 4.13. Let M be a closed maximal ideal of a BCI-algebra X and I be an ideal of X such that $I \subseteq M$. Then $|\theta_{max}(M/I)| = 1$.

Proof. It is sufficient to observe that $|\theta_{max}(M/I)| \leq |\theta_{max}(M)| = 1$ by Corollary 4.11 and Lemma 4.12. Thus $|\theta_{max}(M/I)| = 1$, proving the result. □

In the following corollary, we assume that $\theta_{max}(X) = \bigcup_{M \lessdot X} \theta_{max}(M)$.

Corollary 4.14. Let X be a finite BCI-algebra with exactly n maximal ideals $M_i (1 \leq i \leq n)$. Then $\theta_{max}(X) = \{(X, M_i) \mid 1 \leq i \leq n\}$.

Proof. Since X is a finite BCI-algebra, by Corollary 2.20, $(M_i)_X = M_i$ for all maximal ideal M_i of X and so (X, M_i) is the unique maximal θ-pair of M_i, by Corollary 4.11. Hence $\theta_{max}(X) = \{(X, M_i) \mid 1 \leq i \leq n\}$, as required. □

As an application of Theorem 3.17, the following result states the useful properties of finite BCI-algebras.

Theorem 4.15. Let X be a finite BCI-algebra. Then following statements are holds:

(i) For each $M \lessdot X$ and all maximal ideal θ-pair (A, B) for M, $C_{X/B}(A/B) \neq 0$.

(ii) For each $M \lessdot X$ there exists a maximal ideal θ-pair (A, B) for M such that $X = A + M$, $A \cap M_X = B$ and A/B is commutative.

(iii) For each $M \lessdot X$, there exists a maximal ideal θ-pair (A, B) for M such that $X = A + M$ and A/B is nilpotent.

Proof. (i) Let $M \lessdot X$ and (A, B) be a maximal ideal θ-pair for M. By Lemma 4.8(ii), there exists a maximal ideal θ-pair (C, D) for M such that $A/B \cong C/D$. Now, by assumption, Theorem 3.17 and Lemma 3.13, C/D is a solvable ideal of X/D and consequently $(C/D)^{(1)}$ is a closed ideal of X/D which is contained properly in C/D. So, $(C/D)^{(1)} = 0$ because $(C, D) \in \theta_{max}(M)$, which implies that C/D and then A/B are commutative algebras. It follows that $A/B \subseteq C_{X/B}(A/B)$, showing that $C_{X/B}(A/B) \neq 0$.

(ii) Since X is a solvable BCI-algebra, by Theorem 3.16, it has a series $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n = 0$ of closed ideals in X such that X_{k-1}/X_k is a commutative minimal ideal of X/X_k for $k = 1, ..., n$. Now let M be a maximal ideal of X and there exists $i \in \mathbb{N}$ such that $X_i \subseteq M$ but $X_{i-1} \nsubseteq M$. We observe that $(X_{i-1}, X_i) \in \theta(M)$. Then $X_{i-1} + M = X$ and $X_{i-1} \cap M_X = X_i$. If $(X_{i-1}, X_i) \in \theta_{max}(M)$, then as X_{i-1}/X_i is commutative, the result holds; otherwise, by Lemma 4.8(ii), we choose an ideal θ-pair $(C, D) \in \theta_{max}(M)$ such that $(X_{i-1}, X_i) \leq (C, D)$ and $C/D \cong X_{i-1}/X_i$. It follows that C/D is commutative. Obviously, $C + M = X$ and $C \cap M_X = D$, as desired.

(iii) Follows from (ii). □

Corollary 4.16. Let X be a nilpotent BCI-algebra. Then for any $M \lessdot X$ and each maximal ideal θ-pair (A, B) for M, $Z(X/B) \neq 0$.

Proof. Since X is nilpotent, the proof follows at once from Lemma 3.8 and Theorem 3.10. □

Theorem 4.17. Let X be a nilpotent BCI-algebra. Then for any $M \lessdot X$, there exists a maximal closed ideal θ-pair (A, B) for M such that $A/B \subseteq Z(X/B)$.

Proof. Let M be an arbitrary maximal ideal of X. If X/M_X has no proper closed ideal, then obviously (X, M_X) is a maximal closed ideal θ-pair for M such that $X/M_X = Z(X/M_X)$. In the contrary case, X/M_X contains a minimal closed ideal N/M_X such that (N, M_X) is a maximal closed ideal θ-pair for M. Now, using the assumption and Theorem 3.10, $(N/M_X) \cap Z(X/M_X) \neq 0$ and consequently $N/M_X \subseteq Z(X/M_X)$, as required. □

In the following theorem, we give a necessary and sufficient condition on a fixed BCI-algebra X such that X to be nilpotent algebra.

Theorem 4.18. Let X be a BCI-algebra. Then following statements are equivalent:

(i) X is nilpotent.

(ii) For any closed maximal ideal M of X, M is a commutative nilpotent ideal of X.

Proof. (i)⇒(ii) Let M be any closed maximal ideal of X. Hence M is a nilpotent ideal. Since $M_X = M$, by Corollary 4.11, $|\theta_{max}(M)| = 1$ and (X, M) is the unique maximal θ-pair of M. First, we assume that M is minimal. Then by Corollary 3.11, $M \subseteq Z(X)$. Now if $Z(X) = X$, then X is nilpotent of class 1 and so M is commutative, by Corollary 3.7. Moreover, if $Z(X) = M$, then by Theorem 4.17, $X/M \subseteq Z(X/M)$ and so

$$X/Z(X) = X/M = Z(X/M) = Z(X/Z(X)).$$

Thus $X/Z(X)$ is of class 1 and so it is commutative. It follows that $X^{(1)} \subseteq Z(X)$ by Theorem 2.8, and $M = Z(X)$ is a commutative ideal of X, as desired. Next, let M contains a minimal closed ideal N of X. Since $(X, M) \in \theta_{max}(M)$, by Lemma 4.8 and Corollary 4.13, we deduced that $(X/N, M/N)$ is the unique maximal θ-pair of M/N. Thus $(X/N)/(M/N) \subseteq Z((X/N)/(M/N))$ by Theorem 4.17, and it follows that X/M is commutative BCI-algebra by Theorem 3.6. Now M is commutative ideal, by Theorem 2.6.

(ii)⇒(i) Since M is a commutative closed ideal of X, it follows that X/M is a commutative BCI-algebra, and so X/M is nilpotent. Now the result follows from Theorem 3.9. □

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