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On The Sets of $f-$Strongly Cesàro Summable Sequences

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Abstract. In this paper, we establish relations between the sets of strongly Cesàro summable sequences of complex numbers for modulus functions f and g satisfying various conditions. Furthermore, for some special modulus functions, we obtain relations between the sets of strongly Cesàro summable and statistically convergent sequences of complex numbers.

1. Introduction

The principle of statistical convergence arose from the first version of the monograph of Zygmund [29] in 1935, and its definition was given in a short note by Fast [12] and later independently by Schoenberg [25] where some specific characteristics of statistical convergence were identified. In recent decades, statistical convergence has arisen in several fields under different names. It appears in such fields as measure theory, approximation theory, Banach spaces, hopfield neural networks, locally convex spaces, summability theory, ergodic theory, number theory, turnpike theory, trigonometric series, Fourier analysis, and optimization. Such authors as Connor [6], Fridy [13], Šalát [27], Rath and Tripathy [22], Et [10], Duman [9], León-Saavedra et al. [18], Weisz [28] have explored statistical convergence from the perspective of spaces of sequences; this is referred to as the theory of summability.

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In 1953, Nakano [21] presented the idea of a modulus function for the first time. In 2014, with the benefit of an unbounded modulus function, Aizpuru et al. [1] characterized the notion of f-density, and so introduced a new nonmatrix convergence principle. Using this notion, Bhardwaj et al. [4] have recently extended statistical convergence to the notion of f-statistical boundedness. It has been demonstrated that bounded sequences are definitely those sequences which are f-statistically bounded for every unbounded modulus.

By using a modulus function, Maddox [20], Connor [7], Ruckle [23], Gosh and Srivastava [14], Altin and Et [2], Sarma [24], Kamber [17] and others have constructed various sequence spaces. Further details and applications of the principles of statistical convergence and strong Cesàro summability are available in [5, 8, 11, 15, 16, 26].

2. Definitions and Preliminaries

In this study, the symbols c and ℓ_∞ denote the spaces of convergent and bounded sequences, respectively. The symbols $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$ denote the sets of all complex, real and natural numbers, respectively.

Definition 2.1. [27] Let $U \subset \mathbb{N}$. The *natural density* of U and is defined as

$$\delta\left(U\right)=\lim_{n\to\infty}\frac{1}{n}\left|U_n\right|,$$

in the case the limit exists, where $\left|U_n\right|=\left|\{u\leq n: u\in U\}\right|$ is the cardinality of the indicated set.

It is obvious that $\delta\left(\mathbb{N}\right)=1$ and $\delta\left(U\right)=\delta\left(\mathbb{N}\right)-\delta\left(\mathbb{N}\backslash U\right)=1-\delta\left(\mathbb{N}\backslash U\right)$ and also $\delta\left(U\right)=0$ if U is a finite subset of $\mathbb{N}$.

Definition 2.2. [27] A sequence (x_k) of complex numbers is *statistically convergent* (or $S-$convergent) to $l\in\mathbb{C}$ if

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k\leq n: |x_k-l|\geq\varepsilon\}\right|=0$$

for every $\varepsilon>0$. We write $S-\lim x_k=l$ or $x_k\to l\left(S\right)$ in this particular case. Throughout the paper, the class of all $S-$convergent sequences will be symbolized by S. That is, we set

$$S=\left\{x=(x_k):\forall\varepsilon>0,\lim_{n\to\infty}\frac{1}{n}\left|\{k\leq n: |x_k-l|\geq\varepsilon\}\right|=0\text{ for some } l\in\mathbb{C}\right\}.$$

Definition 2.3. [21] A function $f:\mathbb{R}^+\cup\{0\}\to\mathbb{R}^+\cup\{0\}$ is a *modulus function* (or simply a modulus) if

1. $f\left(h\right)=0\Leftrightarrow h=0,$

2. $f(h_1 + h_2) \leq f(h_1) + f(h_2)$ for all $h_1, h_2 \in \mathbb{R}^+ \cup \{0\}$,
3. f is increasing,
4. f is continuous at 0 from the right.

From the above characteristics we clearly get that a modulus f is continuous on $\mathbb{R}^+ \cup \{0\}$. There are bounded and unbounded modulus functions. As an example, $f(h) = \frac{h}{h+1}$ is a bounded modulus, but $f(h) = \log(h+1)$ is an unbounded modulus. Furthermore, for every modulus f and each positive integer n, we have $f(nh) \leq nf(h)$ from condition (2).

Lemma 2.4. *[19] For any modulus f, $\lim_{h\to\infty} \frac{f(h)}{h}$ exists and $\lim_{h\to\infty} \frac{f(h)}{h} = \inf_{h\in(0,\infty)} \frac{f(h)}{h}$.*

Definition 2.5. [1] Suppose f is an unbounded modulus. The $f-$*density* of a subset U of $\mathbb{N}$ is defined by

$$\delta_f(U) = \lim_{n\to\infty} \frac{1}{f(n)} f(|\{u \leq n : u \in U\}|),$$

if the limit exists.

The $f-$density becomes the natural density if we take $f(h) = h$. In the case of the natural density, it is obvious that for any $U \subset \mathbb{N}$, we have $\delta(U) + \delta(\mathbb{N}\backslash U) = 1$. But this conclusion is different for $f-$density, i.e., $\delta_f(U) + \delta_f(\mathbb{N}\backslash U) = 1$ does not have to be true, in general. To verify this situation, we may take $U = \{2, 4, 6, ...\}$ and the modulus $f(h) = \log(h+1)$, then we have $\delta_f(\mathbb{N}\backslash U) = 1 = \delta_f(U)$. But this situation happens for any unbounded modulus function when $\delta_f(U) = 0$ (for details see Remark 1.2 of [3]). For any finite $U \subset \mathbb{N}$, $f-$density and natural density have similar concepts, that is, $\delta_f(U) = 0$ and so that $\delta_f(U) + \delta_f(\mathbb{N}\backslash U) = 1$.

We know that if $U \subset \mathbb{N}$, $\delta_f(U) = 0$ implies $\delta(U) = 0$ for any unbounded modulus function f (see [1]). The converse need not hold. Indeed, take $f(h) = \log(h+1)$ and set $U = \{u^2 : u \in \mathbb{N}\}$. One gets that $\delta(U) = 0$ but $\delta_f(U) = \frac{1}{2}$. Moreover, if $U \subset \mathbb{N}$ is finite and $\delta(U) = 0$, then $\delta_f(U) = 0$.

Definition 2.6. Suppose f is an unbounded modulus. Then, the sequence (x_k) of complex numbers is $f-$*statistically convergent* (or S_f-convergent) to $l \in \mathbb{C}$ if

$$\lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : |x_k - l| \geq \varepsilon\}|) = 0$$

for every $\varepsilon > 0$. We write this as $S_f - \lim x_k = l$ or $x_k \to l\,(S_f)$. The class of all S_f-convergent sequences will be symbolized by S_f throughout the paper, that is,

$$S_f = \left\{x = (x_k) : \forall\varepsilon > 0, \lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : |x_k - l| \geq \varepsilon\}|) = 0 \text{ for some } l \in \mathbb{C}\right\}.$$

Note that S_f-convergence reduces to $S-$convergence in the case $f(h)=h$.

Lemma 2.7. *Suppose (x_k) is any sequence of complex numbers. If $(x_k)\in S_f$, then its S_f-limit is unique.*

Definition 2.8. Suppose f is a modulus. Then, the sequence (x_k) of complex numbers is $f-$*strongly Cesàro summable* to $l\in\mathbb{C}$ if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f(|x_k-l|)=0.$$

The symbol w_f denotes the class of all $f-$strongly Cesàro summable sequences, that is,

$$w_f=\left\{x=(x_k):\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f(|x_k-l|)=0\text{ for some }l\in\mathbb{C}\right\}.$$

Note that this definition does not require the modulus function f to be unbounded.

The concepts of $f-$strong Cesàro summability and strong Cesàro summability are the same in the case $f(h)=h$ and the set of all strongly Cesàro summable sequences will be denoted by w, that is, w_f will reduce to w if $f(h)=h$.

3. Main Results

In this section, we give the main results of the study.

3.1. Modulus functions and strong Cesàro summability

Theorem 3.1. *Suppose f and g are any modulus functions. If $\sup_{h\in(0,\infty)}\frac{f(h)}{g(h)}<\infty$, then $w_g\subset w_f$.*

Proof. Assume that $p=\sup_{h\in(0,\infty)}\frac{f(h)}{g(h)}<\infty$. Then, we have $\frac{f(h)}{g(h)}\leq p$ and so that $f(h)\leq pg(h)$ for every $h\in[0,\infty)$. Now, it is apparent that $p>0$ and if $x=(x_k)$ is $g-$strongly Cesàro summable to l, we may write

$$\frac{1}{n}\sum_{k=1}^{n}f(|x_k-l|)\leq\frac{1}{n}\sum_{k=1}^{n}pg(|x_k-l|).$$

Taking the limits on both sides as $n\to\infty$, we obtain that $x\in w_g$ implies $x\in w_f$. □

Remark 3.2. The converse of Theorem 3.1 does not have to be correct for every modulus functions f and g such that $\sup\limits_{h\in(0,\infty)} \frac{f(h)}{g(h)} < \infty$, in general. The example below demonstrates that at least for certain specific modulus functions, the inclusion $w_g \subset w_f$ can be strict.

Example 3.3. Define the sequence $x = (x_k)$ as

$$x_k = \begin{cases} k & \text{if } k = n^3 \\ & \\ 0 & \text{if } k \neq n^3 \end{cases} \quad n \in \mathbb{N},$$

and take the modulus functions $f(h) = \frac{h}{h+1}$ and $g(h) = h$. Then, $\sup\limits_{h\in(0,\infty)} \frac{f(h)}{g(h)} = 1 < \infty$ and so that $w_g \subset w_f$. By using the equality $f(0) = 0$, we have

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} f(|x_k|) &= \frac{1}{n}\sum_{\substack{k=1\\k=m^3}}^{n} f(k) + \frac{1}{n}\sum_{\substack{k=1\\k\neq m^3}}^{n} f(0) \\ &= \frac{1}{n}\sum_{\substack{k=1\\k=m^3}}^{n} \frac{k}{1+k} \\ &< \frac{1}{n}\sum_{\substack{k=1\\k=m^3}}^{n} 1 \leq \frac{\sqrt[3]{n}}{n}. \end{aligned}$$

Since $\frac{\sqrt[3]{n}}{n} \to 0$ as $n \to \infty$, we get $x \in w_f$. However,

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} g(|x_k|) &= \frac{1}{n}\sum_{k=1}^{n} g(x_k) \\ &= \frac{1}{n}\sum_{\substack{k=1\\k=m^3}}^{n} k + \frac{1}{n}\sum_{\substack{k=1\\k\neq m^3}}^{n} g(0) \\ &= \frac{1}{n}\left(1^3 + 2^3 + 3^3 + ... + i^3\right), \ \max_{i\in\mathbb{N}} i^3 \leq n \\ &= \frac{1}{n}\left[\frac{i(i+1)}{2}\right]^2 \\ &\geq \frac{1}{n}\left[\frac{\left(\left[\sqrt[3]{n}\right]-1\right)\left(\left[\sqrt[3]{n}\right]\right)}{2}\right]^2. \end{aligned}$$

Since $\frac{1}{n}\left[\frac{([\sqrt[3]{n}]-1)([\sqrt[3]{n}])}{2}\right]^2 \to \infty$ as $n \to \infty$ so that $x \notin w_g$, where $[r]$ denotes an integral part of the real number r. Hence, $x \in w_f - w_g$ and the inclusion $w_g \subset w_f$ is strict.

Theorem 3.4. *Suppose f and g are any modulus functions. If $\inf\limits_{h\in(0,\infty)} \frac{f(h)}{g(h)} > 0$, then $w_f \subset w_g$.*

Proof. Suppose that $q = \inf\limits_{h\in(0,\infty)} \frac{f(h)}{g(h)} > 0$. Then, we have $\frac{f(h)}{g(h)} \geq q$ and so that $qg(h) \leq f(h)$ for every $h \in [0,\infty)$. Now, if $x = (x_k)$ is $f-$strongly Cesàro summable to l, we may write

$$\frac{1}{n}\sum_{k=1}^{n} g(|x_k - l|) \leq \frac{1}{q}\frac{1}{n}\sum_{k=1}^{n} f(|x_k - l|).$$

Taking the limits on both sides as $n \to \infty$, we obtain that $x \in w_f$ implies $x \in w_g$ and this fulfills the proof. □

Remark 3.5. The converse of Theorem 3.4 does not have to be correct for every modulus functions f and g if $\inf\limits_{h\in(0,\infty)} \frac{f(h)}{g(h)} > 0$, in general. For this, recall the sequence $x = (x_k)$ in Example 3.3 and take the modulus functions $f(h) = h$ and $g(h) = \frac{h}{h+1}$. Then, $\inf\limits_{h\in(0,\infty)} \frac{f(h)}{g(h)} > 0$ and $x \in w_g$ but $x \notin w_f$. This shows that at least for certain specific modulus functions, the inclusion $w_f \subset w_g$ can be strict.

The outcome below is a result of Theorem 3.1 and Theorem 3.4.

Corollary 3.6. *Suppose f and g are any modulus functions. If*

$$0 < \inf_{h\in(0,\infty)} \frac{f(h)}{g(h)} \leq \sup_{h\in(0,\infty)} \frac{f(h)}{g(h)} < \infty,$$

then $w_f = w_g$.

Corollary 3.7. *Suppose f is a modulus function. If $\inf\limits_{h\in(0,\infty)} \frac{f(h)}{h} > 0$, then $w_f = w$.*

Proof. Since $w \subset w_f$ for any modulus function by the first part of Theorem 3.4 of [3] for the case $\alpha = 1$, taking $g(h) = h$ in Theorem 3.4, we obtain $w_f \subset w$ if $\inf\limits_{h\in(0,\infty)} \frac{f(h)}{h} > 0$. Therefore, $w_f = w$ if $\inf\limits_{h\in(0,\infty)} \frac{f(h)}{h} > 0$. □

3.2. Relations between statistical convergence and strong Cesàro summability according to modulus functions

Theorem 3.8. *suppose f and g are unbounded modulus functions. If $\inf\limits_{h\in(0,\infty)} \frac{f(h)}{g(h)} > 0$ and $\lim\limits_{h\to\infty} \frac{g(h)}{h} > 0$, then every $f-$strongly Cesàro summable sequence is $g-$statistically convergent, that is, $w_f \subset S_g$.*

Proof. Suppose that $\beta = \inf\limits_{h\in(0,\infty)} \frac{f(h)}{g(h)} > 0$. Then, we have $\frac{f(h)}{g(h)} \geq \beta$ and so that $\beta g(h) \leq f(h)$ for every $h \in [0,\infty)$. Now, if $x = (x_k)$ is $f-$strongly Cesàro summable to l, we may write

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} f(|x_k - l|) &\geq \beta\frac{1}{n}\sum_{k=1}^{n} g(|x_k - l|) \geq \beta\frac{1}{n}\sum_{\substack{k=1\\|x_k-l|\geq\varepsilon}}^{n} g(|x_k - l|)\\ &\geq \beta\frac{1}{n}\left|\{k\leq n : |x_k - l| \geq \varepsilon\}\right| g(\varepsilon)\\ &\geq \beta\frac{1}{n} g\left(\left|\{k\leq n : |x_k - l| \geq \varepsilon\}\right|\right)\frac{g(\varepsilon)}{g(1)}\\ &= \frac{g\left(\left|\{k\leq n : |x_k - l| \geq \varepsilon\}\right|\right)}{g(n)}\frac{g(n)}{n}\frac{g(\varepsilon)}{g(1)}\beta.\end{aligned}$$

Taking the limits on both sides as $n\to\infty$, we obtain that $x \in w_f$ implies $x \in S_g$ since $\lim\limits_{h\to\infty} \frac{g(h)}{h} > 0$. □

Remark 3.9. The converse of Theorem 3.8 does not have to be correct for every unbounded modulus functions f and g if $\inf\limits_{h\in(0,\infty)} \frac{f(h)}{g(h)} > 0$ and $\lim\limits_{h\to\infty} \frac{g(h)}{h} > 0$, in general. The following illustration can demonstrate that at least for certain specific unbounded modulus functions, the inclusion $w_f \subset S_g$ can be strict.

Example 3.10. Recall the sequence $x = (x_k)$ in Example 3.3 and take the modulus functions $f(h) = g(h) = h$. Then, we have $\inf\limits_{h\in(0,\infty)} \frac{f(h)}{g(h)} > 0$ and $\lim\limits_{h\to\infty} \frac{g(h)}{h} > 0$ and also

$$\frac{1}{g(n)} g\left(\left|\{k\leq n : |x_k - 0|\}\right|\right) \leq \frac{g\left(\sqrt[3]{n}\right)}{g(n)}.$$

By taking the limits on both sides as $n\to\infty$, we get that $x \in S_g$. However, $x \notin w_f$ as shown in Example 3.3.

The outcome below is acquired by taking $g(h) = f(h)$ in Theorem 3.8.

Corollary 3.11. *Suppose f is an unbounded modulus. If $\lim\limits_{h\to\infty} \frac{f(h)}{h} > 0$, then every $f-$strongly Cesàro summable sequence is $f-$statistically convergent.*

Remark 3.12. Corollary 3.11 was given with the extra condition "$f(xy) \geq cf(x)f(y)$ for all $x \geq 0$, $y \geq 0$ and some positive number c" in [3]. It seems that this extra condition is not necessary and it should be neglected.

The outcome below is acquired by taking $g(h) = h$ in Theorem 3.8 (see also in [3]).

Corollary 3.13. *Suppose f is an unbounded modulus. If $\inf\limits_{h\in(0,\infty)} \frac{f(h)}{h} > 0$, then every $f-$strongly Cesàro summable sequence is statistically convergent.*

The outcome below is acquired by taking $f(h) = h$ in Corollary 3.13, which is the first part of Theorem 2.1 of [6], for the case $q = 1$.

Corollary 3.14. *A strongly Cesàro summable sequence is statistically convergent.*

Theorem 3.15. *Suppose f and g are unbounded modulus functions. Then, every bounded and $f-$statistically convergent sequence is $g-$strongly Cesàro summable sequence, i.e., $\ell_\infty \cap S_f \subset w_g$.*

Proof. Assuming that f and g are unbounded modulus functions. Since $S_f \subset S$ by the first part of Corollary 2.2 of [1], and since $\ell_\infty \cap S \subset w$ by the second part of Theorem 2.1 of [6], then we have $\ell_\infty \cap S_f \subset \ell_\infty \cap S \subset w$, that is, $\ell_\infty \cap S_f \subset w$. On the other hand, since $w \subset w_g$ for any modulus g by the first part of Theorem 3.4 of [3] for the case $\alpha = 1$, it follows that $\ell_\infty \cap S_f \subset w_g$. □

Remark 3.16. The converse of Theorem 3.15 does not have to be correct for every unbounded modulus functions f and g, in general. The following example demonstrates this situation.

Example 3.17. Let us consider the sequence $x = (x_k)$ as

$$x_k = \begin{cases} 1 & \text{if } k = n^2 \\ & \qquad\qquad n \in \mathbb{N}, \\ 0 & \text{if } k \neq n^2 \end{cases}$$

and take the modulus functions $g(h) = f(h) = \log(h+1)$. Then, by using the equality $g(0) = 0$, we have

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} g(|x_k - 0|) =& \frac{1}{n}\sum_{k=1}^{n} g(x_k) = \frac{1}{n}\sum_{\substack{k=1 \\ k=n^2}}^{n} g(1) + \frac{1}{n}\sum_{\substack{k=1 \\ k\neq n^2}}^{n} g(0) \\ =& \frac{1}{n}\sum_{\substack{k=1 \\ k=n^2}}^{n} \log 2 \leq \frac{\sqrt{n}}{n}\log 2 \to 0 \text{ as } n \to \infty. \end{aligned}$$

So that $x \in w_g$. Although,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : |x_k| \geq \varepsilon\}|) \\ &\geq \lim_{n\to\infty} \frac{1}{f(n)} f\left(\sqrt{n} - 1\right) = \lim_{n\to\infty} \frac{\log\left(\sqrt{n}\right)}{\log(n+1)} = \frac{1}{2} \neq 0. \end{aligned}$$

That is, $x \notin S_f$. This means that the inclusion $\ell_\infty \cap S_f \subset w_g$ is strict.

The following inclusions are a result of Theorem 3.15.

Corollary 3.18. *If f is any unbounded modulus, then we have*

1. $\ell_\infty \cap S_f \subset w_f$,
2. $\ell_\infty \cap S_f \subset w$, *and*
3. $\ell_\infty \cap S \subset w_f$.

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