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Generalized Fourier–Feynman Transform of Bounded Cylinder Functions on the Function Space $C_{a,b}[0,T]$

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Abstract. In this paper, we study the generalized Fourier–Feynman transform (GFFT) for functions on the general Wiener space $C_{a,b}[0,T]$. We establish an explicit evaluation formula for the analytic GFFT of bounded cylinder functions on $C_{a,b}[0,T]$. We start by examining certain cylinder functions which belong in a Banach algebra of bounded functions on $C_{a,b}[0,T]$. We then obtain an explicit formula for the analytic GFFT of the bounded cylinder functions.

1. Introduction

Let $C_0[0,T]$ be the classical Wiener space. In [4], Cameron and Storvick introduced a Banach algebra $\mathcal{S}(L_2[0,T])$ of analytic Feynman integrable functions on $C_0[0,T]$. Each function in $\mathcal{S}(L_2[0,T])$ is defined as a stochastic Fourier transform of a complex measure on $L_2[0,T]$. Cameron and Storvick showed that certain functions which arise naturally in quantum mechanics are elements of the Banach algebra $\mathcal{S}(L_2[0,T])$. Under strengthened measurability assumptions, Cameron and Storvick showed in [3] that the analytic Feynman integral of functions F having the form

$$F(x) = \exp\left\{ \int_0^T \theta(s, x(s))ds \right\} \tag{1.1}$$

gives a solution of an integral equation formally equivalent to Schrödiner equation. In (1.1), $\{\theta(s,\cdot),\, s \in [0,T]\}$ is a family of the Fourier transforms of bounded measures on $\mathbb{R}$. The functions given by equation (1.1) also are elements of the Banach algebra $\mathcal{S}(L_2[0,T])$, see [3, 4, 17].

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A study of the analytic Fourier–Feynman transform is an interesting topic concerning with the analytic Feynman integral theory. The theory of the analytic Fourier–Feynman transform suggested by Brue [1] now plays a noteworthy role in infinite dimensional analysis.

In [9, 11], the authors used a generalized Brownian motion process (GBMP) to define a generalized analytic Feynman integral and an $L_p(1 \le p \le 2)$ analytic GFFT for functions on a function space $C_{a,b}[0,T]$. The general Wiener space $C_{a,b}[0,T]$ can be understood as a space of continuous sample functions of the GBMP. We refer to the references [9, 11, 19, 20] for more detailed informations about the definition of the GBMP associated with continuous functions $a(\cdot)$ and $b(\cdot)$ on the time interval $[0,T]$, and the construction of the function space $C_{a,b}[0,T]$. Standard Brownian motion is centered and stationary in time, while in general, a GBMP is neither centered nor stationary in time.

In [9], the authors studied the L_p analytic GFFT of cylinder functions on $C_{a,b}[0,T]$. However, they provided the existences of only L_1 and L_2 GFFTs for cylinder functions on $C_{a,b}[0,T]$ because the drift term $a(t)$ of the GBMP makes establishing the existences of the GFFTs very difficult. The purpose of this paper is to study the cylinder functions on $C_{a,b}[0,T]$ whose L_p analytic GFFT exists for all $p \in [1,2]$. For our purpose, we first examine certain cylinder functions which belong in a Banach algebra $\mathcal{F}(C_{a,b}[0,T])$ of functions on the function space $C_{a,b}[0,T]$. The class $\mathcal{F}(C_{a,b}[0,T])$ used in this paper is homeomorhic to the Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ studied in [11]. We then provide an explicit formula for the GFFT of the cylinder function under our consideration.

2. Definitions and Preliminaries

In this section we first provide a brief background about the general Wiener space $C_{a,b}[0,T]$ induced by the GBMP.

Let $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ denote the function space induced by a GBMP Y determined by continuous functions $a(t)$ and $b(t)$ where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra induced by sup-norm, see [19] and [20, Chapters 3 and 4]. We assume in this paper that $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L_2[0,T]$, and $b(t)$ is an increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$. Then we can consider the coordinate process $X : [0,T] \times C_{a,b}[0,T] \to \mathbb{R}$ given by $X(t,x) = x(t)$ which is the continuous realization of Y [20, Theorem 14.2]. For any $t \in [0,T]$ and $x \in C_{a,b}[0,T]$, we have $X(t,x) = x(t) \sim N(a(t), b(t))$. We then complete this function space to obtain the measure space $(C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T]), \mu)$ where $\mathcal{W}(C_{a,b}[0,T])$ is the set of all μ-Carathéodory measurable subsets of $C_{a,b}[0,T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable (s.i.m.) provided ρB is $\mathcal{W}(C_{a,b}[0,T])$-measurable for all $\rho > 0$, and a s.i.m. set N is said to be a scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A function F is said to be s.i.m. provided F is defined on a s.i.m. set and $F(\rho\,\cdot)$

is $\mathcal{W}(C_{a,b}[0,T])$-measurable for every $\rho > 0$. If two functions F and G defined on $C_{a,b}[0,T]$ are equal s-a.e., then we write $F \approx G$.

Let $L^2_{a,b}[0,T]$ be the space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue–Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$: i.e.,

$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(t)db(t) < \infty \ \text{ and } \ \int_0^T v^2(t)d|a|(t) < \infty \right\}$$

where $|a|(\cdot)$ is the total variation function of $a(\cdot)$. Then $L^2_{a,b}[0,T]$ is a separable Hilbert space with inner product defined by $(u,v)_{a,b} = \int_0^T u(t)v(t)d[b(t) + |a|(t)]$. For more details, see [9, 11].

Consider the function space

$$C'_{a,b}[0,T] = \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)db(s) \ \text{ for some } \ z \in L^2_{a,b}[0,T] \right\}.$$

For $w \in C'_{a,b}[0,T]$, let the operator $D : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ be defined by the formula

$$Dw(t) = \frac{w'(t)}{b'(t)}. \tag{2.1}$$

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product $(w_1, w_2)_{C'_{a,b}} = \int_0^T Dw_1(t)Dw_2(t)db(t)$ is a separable Hilbert space.

Note that the two separable Hilbert spaces $L^2_{a,b}[0,T]$ and $C'_{a,b}[0,T]$ are (topologically) homeomorphic under the linear operator given by (2.1). The inverse operator of D is given by $(D^{-1}z)(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$. In the case that $a(t) \equiv 0$, then the operator $D : C'_{0,b}[0,T] \to L^2_{0,b}[0,T]$ is an isometry.

In this paper, in addition to the conditions put on $a(t)$ above, we now add the condition

$$\int_0^T |a'(t)|^2 d|a|(t) < +\infty \tag{2.2}$$

from which it follows that

$$\begin{aligned}\int_0^T |Da(t)|^2 d[b(t) + |a|(t)] &= \int_0^T \left| \frac{a'(t)}{b'(t)} \right|^2 d[b(t) + |a|(t)] \\ &< M\|a'\|_{L_2[0,T]} + M^2 \int_0^T |a'(t)|^2 d|a|(t) < +\infty,\end{aligned}$$

where $M = \sup_{t\in[0,T]}(1/b'(t))$. Thus, the function $a : [0,T] \to \mathbb{R}$ satisfies the condition (2.2) if and only if $a(\cdot)$ is an element of $C'_{a,b}[0,T]$.

Let $\{e_n\}_{n=1}^\infty$ be a complete orthonormal set of functions in $(C'_{a,b}[0,T], \|\cdot\|_{C'_{a,b}})$ such that the De_n's are of bounded variation on $[0,T]$. For $w \in C'_{a,b}[0,T]$ and

$x \in C_{a,b}[0,T]$, we define the Paley–Wiener–Zygmund stochastic integral $(w,x)^{\sim}$ as follows:

$$(w,x)^{\sim} = \lim_{n\to\infty} \int_0^T \sum_{j=1}^{n} (w,e_j)_{C'_{a,b}} De_j(t)dx(t)$$

if the limit exists. We will emphasize the following fundamental facts. For each $w \in C'_{a,b}[0,T]$, the Paley–Wiener–Zygmund stochastic integral $(w,x)^{\sim}$ exists for μ-a.e. $x \in C_{a,b}[0,T]$. If $Dw = z \in L^2_{a,b}[0,T]$ is of bounded variation on $[0,T]$, then the Paley–Wiener–Zygmund stochastic integral $(w,x)^{\sim}$ equals the Riemann–Stieltjes integral $\int_0^T Dw(t)dx(t) = \int_0^T z(t)dx(t)$. Also we note that for $w, x \in C'_{a,b}[0,T]$, $(w,x)^{\sim} = (w,x)_{C'_{a,b}}$. Furthermore for each $w \in C'_{a,b}[0,T]$, the Paley–Wiener–Zygmund stochastic integral $(w,x)^{\sim}$ is a Gaussian random variable on $C_{a,b}[0,T]$ with mean $(w,a)_{C'_{a,b}} = \int_0^T Dw(t)da(t)$ and variance $\|w\|^2_{C'_{a,b}} = \int_0^T \{Dw(t)\}^2 db(t)$.

3. Various Functions in the Banach Algebra $\mathcal{F}(C_{a,b}[0,T])$

The Banach algebra $\mathcal{F}(C_{a,b}[0,T])$ is defined as the space of all functions F on $C_{a,b}[0,T]$ having the form

$$F(x) = \int_{C'_{a,b}[0,T]} \exp\{i(w,x)^{\sim}\} d\sigma(w) \tag{3.1}$$

for s-a.e. $x \in C_{a,b}[0,T]$, where σ is in $M(C'_{a,b}[0,T])$, the space of complex-valued Borel measures on $\mathcal{B}(C'_{a,b}[0,T])$, the Borel σ-algebra of subsets of the Cameron–Martin space $C'_{a,b}[0,T]$. Note that every function given by (3.1) is s.i.m..

A function F on $C_{a,b}[0,T]$ is called a cylinder function if

$$F(x) = f((h_1,x)^{\sim}, \ldots, (h_n,x)^{\sim}), \quad x \in C_{a,b}[0,T] \tag{3.2}$$

for μ-a.e. $x \in C_{a,b}[0,T]$, where f is a complex-valued Lebesgue measurable function on $\mathbb{R}^n$ and $\{h_1, \ldots, h_n\}$ is a finite set of functions in $C'_{a,b}[0,T]$.

Example 3.1. Let $F_1 : C_{a,b}[0,T] \to \mathbb{C}$ be given by

$$F_1(x) = f((w_1,x)^{\sim}, \ldots, (w_n,x)^{\sim}), \tag{3.3}$$

where $\{w_1, \ldots, w_n\}$ is a lineally independent set of functions in $C'_{a,b}[0,T]$. The GFFT of functions given by the right-hand side of (3.3) are studied in [9]. Let $0 = t_0 < t_1 < \cdots < t_n \le T$ be a subdivision of $[0,T]$.

(i) For each $l \in \{1, \ldots, n\}$, let $w_l(t) = \int_0^t \chi_{[0,t_l]}(s)db(s)$ on $[0,T]$. Then we can rewrite equation (3.3) as

$$F_2(x) = f(x(t_1), \ldots, x(t_n)). \tag{3.4}$$

(ii) For each $l \in \{1, \ldots, n\}$, let $w_l(t) = \int_0^t \chi_{[t_{l-1},t_l]}(s)db(s)$ on $[0,T]$. Then we can rewrite equation (3.3) as

$$F_3(x) = f(x(t_1), x(t_2) - x(t_1), \ldots, x(t_n) - x(t_{n-1})). \tag{3.5}$$

Letting $a(t) = 0$ and $b(t) = t$ on $[0,T]$, the general Wiener space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$. In [2, 5, 6, 14], the authors studied certain classes of functions of the forms (3.4) and (3.5) on $C_0[0,T]$ and they used those classes to complete their researches concerning the analytic Feynman integral and the analytic Fourier–Feynman transform on $C_0[0,T]$.

Let $S : C'_{a,b}[0,T] \to C'_{a,b}[0,T]$ be the linear operator given by

$$Sw(t) = \int_0^t w(s)db(s). \tag{3.6}$$

Then the adjoint operator S^* of S is given by

$$S^*w(t) = w(T)b(t) - \int_0^t w(s)db(s) = \int_0^t [w(T) - w(s)]db(s).$$

It is easily shown that S^* is injective. For a more detailed study of the operator S and S^*, see [10].

Example 3.2. Let $F_4 : C_{a,b}[0,T] \to \mathbb{C}$ be given by

$$F_4(x) = f\bigg(\int_0^T z_1(t)x(t)db(t), \dots, \int_0^T z_n(t)x(t)db(t)\bigg), \tag{3.7}$$

where $\{z_1, \dots, z_n\}$ is a lineally independent subset of $L^2_{a,b}[0,T]$. Then

$$\{w_1, \dots, w_n\} = \bigg\{\int_0^{\cdot} z_1(s)db(s), \dots, \int_0^{\cdot} z_n(s)db(s)\bigg\}$$

is a lineally independent subset of $C'_{a,b}[0,T]$, see [10]. Since S^* is linear and injective, $\{S^*w_1, \dots, S^*w_n\}$ also is an independent subset of $C'_{a,b}[0,T]$. Furthermore, by an integration by parts formula, it follows that

$$(S^*w_l, x)^\sim = \int_0^T x(t)Dw_l(t)db(t) = \int_0^T x(t)z_l(t)db(t) \tag{3.8}$$

for each $l \in \{1, \dots, n\}$. Hence

$$F_4(x) = f((S^*w_1, x)^\sim, \dots, (S^*w_n, x)^\sim)$$

is a cylinder function on $C_{a,b}[0,T]$.

Let $0 = t_0 < t_1 < \cdots < t_n \le T$ be a subdivision of $[0,T]$ and for each $l \in \{1, \dots, n\}$, let $z_l(s) = \chi_{[0,t_l]}(s)$ on $[0,T]$. Then we can rewrite equation (3.7) as

$$F_5(x) = f\bigg(\int_0^{t_1} x(s)db(s), \int_0^{t_2} x(s)db(s), \dots, \int_0^{t_n} x(s)db(s)\bigg).$$

In view of the fact that $L_1(\mathbb{R}^n)\setminus L_\infty(\mathbb{R}^n)\neq\emptyset$, one can see that every cylinder function on $C_{a,b}[0,T]$ is not necessarily in the Banach algebra $\mathcal{F}(C_{a,b}[0,T])$. Thus the rest of this section, we consider a class of cylinder functions on $C_{a,b}[0,T]$ and provide necessary and sufficient conditions for the cylinder functions given by (3.2) to be in the Banach algebra $\mathcal{F}(C_{a,b}[0,T])$.

Let $M(\mathbb{R}^n)$ denote the space of complex-valued Borel measures on $\mathcal{B}(\mathbb{R}^n)$, the Borel σ-algebra of $\mathbb{R}^n$. Let ν be in $M(\mathbb{R}^n)$. Then the Fourier transform $\widehat{\nu}$ of ν given by the formula

$$\widehat{\nu}(\vec{u})=\int_{\mathbb{R}^n}\exp\bigg\{i\sum_{l=1}^n u_l v_l\bigg\}d\sigma(\vec{v}), \tag{3.9}$$

is a complex-valued function on $\mathbb{R}^n$.

Next theorem provide necessary and sufficient conditions for the cylinder functions on $C_{a,b}[0,T]$ to be in $\mathcal{F}(C_{a,b}[0,T])$. This result subsumes similar known results given in [5, 6, 7, 13].

Theorem 3.3. *Let $\{w_1,\dots,w_n\}$ be a linearly independent subset of $C'_{a,b}[0,T]$. Let $F: C_{a,b}[0,T]\to\mathbb{C}$ be a cylinder function on $C_{a,b}[0,T]$ given by the right-hand side of (3.3). Then F is in $\mathcal{F}(C_{a,b}[0,T])$ if and only if there exists a measure $\sigma\in M(\mathbb{R}^n)$ such that $\widehat{\sigma}=f$ almost everywhere on $\mathbb{R}^n$.*

We will provide a more basic theorem ensuring that various functions are in $\mathcal{F}(C_{a,b}[0,T])$.

Theorem 3.4. *Let (Q,Σ,γ) be a σ-finite measure space and let $\varphi_l: Q\to C'_{a,b}[0,T]$ be Σ–$\mathcal{B}(C'_{a,b}[0,T])$ measurable for each $l\in\{1,\dots,n\}$. Let $\theta: Q\times\mathbb{R}^n\to\mathbb{C}$ be given by $\theta(\eta;\cdot)=\widehat{\nu}_\eta(\cdot)$ where ν_η is in $M(\mathbb{R}^n)$ for every $\eta\in Q$ and where the family $\{\nu_\eta:\eta\in Q\}$ satisfies:*

(i) $\nu_\eta(B)$ is a Σ-measurable function of η for every $B\in\mathcal{B}(\mathbb{R}^n)$,

(ii) $\|\nu_\eta\|\in L_1(Q,\Sigma,\gamma)$.

Under these conditions, the function $F: C_{a,b}[0,T]\to\mathbb{C}$ given by

$$F(x)=\int_Q\theta\big(\eta;(\varphi_1(\eta),x)^\sim,\dots,(\varphi_n(\eta),x)^\sim\big)d\gamma(\eta) \tag{3.10}$$

is in the class $\mathcal{F}(C_{a,b}[0,T])$ and satisfies the inequality $\|F\|\le\int_Q\|\nu_\eta\|d\gamma(\eta)$.

Proof. Using the techniques similar to those used in [7], we can show that $\|\nu_\eta\|$ is measurable as a function of η, that θ is $\Sigma\times\mathcal{B}(\mathbb{R}^n)$-measurable, and that the integrand in equation (3.10) is a measurable function of η for every $x\in C_{a,b}[0,T]$.

We define a measure τ on $\Sigma\times\mathcal{B}(\mathbb{R}^n)$ by

$$\tau(B)=\int_Q\nu_\eta(B^{(\eta)})d\gamma(\eta)\ \text{for}\ B\in\Sigma\times\mathcal{B}(\mathbb{R}^n). \tag{3.11}$$

Then by the first assertion of [17, Theorem 3.1] with the current condition (ii), τ satisfies $\|\tau\| \le \int_Q \|\nu_\eta\| d\gamma(\eta)$. Now let $\Phi : Q \times \mathbb{R}^n \to C'_{a,b}[0,T]$ be defined by

$$\Phi(\eta; v_1, \ldots, v_n) = \sum_{l=1}^{n} v_l \varphi_l(\eta). \tag{3.12}$$

Then Φ is $\Sigma \times \mathcal{B}(\mathbb{R}^n)$–$\mathcal{B}(C'_{a,b}[0,T])$-measurable using the hypothesis for φ_l, $l \in \{1, \ldots, n\}$. Let $\sigma = \tau \circ \Phi^{-1}$. Then clearly $\sigma \in M(C'_{a,b}[0,T])$ and satisfies $\|\sigma\| \le \|\tau\|$.

From the change of variables theorem and the second assertion of [17, Theorem 3.1], it follows that for a.e. $x \in C_{a,b}[0,T]$ and for every $\rho > 0$,

$$\begin{aligned} F(\rho x) &= \int_Q \widehat{\nu}_\eta\big((\varphi_1(\eta), \rho x)^\sim, \ldots, (\varphi_n(\eta), \rho x)^\sim\big) d\gamma(\eta) \\ &= \int_Q \bigg[\int_{\mathbb{R}^n} \exp\bigg\{ i \sum_{l=1}^{n} v_l (\varphi_l(\eta), \rho x)^\sim \bigg\} d\nu_\eta(v_1, \ldots, v_n) \bigg] d\gamma(\eta) \\ &= \int_{Q \times \mathbb{R}^n} \exp\bigg\{ i \sum_{l=1}^{n} v_l (\varphi_l(\eta), \rho x)^\sim \bigg\} d\tau(\eta; v_1, \ldots, v_n) \\ &= \int_{Q \times \mathbb{R}^n} \exp\big\{ i(\Phi(\eta; v_1, \ldots, v_n), \rho x)^\sim \big\} d\tau(\eta; v_1, \ldots, v_n) \\ &= \int_{C'_{a,b}[0,T]} \exp\big\{ i(w, \rho x)^\sim \big\} d\tau \circ \Phi^{-1}(w) \\ &= \int_{C'_{a,b}[0,T]} \exp\big\{ i(w, \rho x)^\sim \big\} d\sigma(w). \end{aligned} \tag{3.13}$$

Clearly, σ is a complex measure in $M(C'_{a,b}[0,T])$. Thus the function F given by equation (3.10) belongs to $\mathcal{F}(C_{a,b}[0,T])$ and satisfies the inequality

$$\|F\| = \|\sigma\| \le \|\tau\| \le \int_Q \|\nu_\eta\| d\gamma(\eta)$$

as desired. □

The following corollaries are relevant to Feynman integration theories and quantum mechanics where exponential functions play an important role. Our next corollary comes from the fact that $\mathcal{F}(C_{a,b}[0,T])$ is a Banach algebra

Corollary 3.5. *Let F be given by equation (3.10), and let $\Xi : \mathbb{C} \to \mathbb{C}$ be an entire function. Then $(\Xi \circ F)(x)$ is in $\mathcal{F}(C_{a,b}[0,T])$. In particular,* $\exp\{F(x)\} \in \mathcal{F}(C_{a,b}[0,T])$.

Corollary 3.6 (Necessary condition of Theorem 3.3 with weaker condition). *Let $\{g_1, \ldots, g_n\}$ be a finite (not necessarily linearly independent) subset of $C'_{a,b}[0,T]$. Given $\Theta = \widehat{\nu}$ with $\nu \in M(\mathbb{R}^n)$, define a function $F : C_{a,b}[0,T] \to \mathbb{C}$ by*

$$F(x) = \Theta((g_1, x)^\sim, \ldots, (g_n, x)^\sim).$$

Then F is in the class $\mathcal{F}(C_{a,b}[0,T])$.

Proof. Let (Q,Σ,γ) be a probability space and for $l\in\{1,\dots,n\}$, let $\varphi_l(\eta)\equiv g_l$. Take $\theta(\eta;\cdot)=\Theta(\cdot)=\widehat{\nu}(\cdot)$. Then for all $\rho>0$ and for a.e. $x\in C_{a,b}[0,T]$,

$$\begin{aligned}&\int_Q \theta\big(\eta;(\varphi_1(\eta),\rho x)^\sim,\dots,(\varphi_n(\eta),\rho x)^\sim\big)d\gamma(\eta)\\ &=\int_Q \Theta\big((g_1,\rho x)^\sim,\dots,(g_n,\rho x)^\sim\big)d\gamma(\eta)\\ &=\Theta\big((g_1,\rho x)^\sim,\dots,(g_n,\rho x)^\sim\big)\\ &=F(\rho x).\end{aligned}\tag{3.14}$$

Hence $F\in\mathcal{F}(C_{a,b}[0,T])$. □

4. Generalized Fourier–Feynman Transform for the Bounded Cylinder Functions

In this section, we obtain an explicit formula for the L_p analytic GFFT of the cylinder functions in $\mathcal{F}(C_{a,b}[0,T])$. Let $\mathbb{C}_+=\{\lambda\in\mathbb{C}:\mathrm{Re}(\lambda)>0\}$ and let $\widetilde{\mathbb{C}}_+=\{\lambda\in\mathbb{C}\setminus\{0\}:\mathrm{Re}(\lambda)\geq 0\}$. Throughout the rest of this paper, $\lambda^{-1/2}$(or $\lambda^{1/2}$) always is chosen to have positive real part for all $\lambda\in\widetilde{\mathbb{C}}_+$. Let F be a s.i.m. function on $C_{a,b}[0,T]$ such that $J_F(\lambda)=\int_{C_{a,b}[0,T]}F(\lambda^{-1/2}x)d\mu(x)$ exists and is finite for all $\lambda>0$. If there exists a function $J_F^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J_F^*(\lambda)=J_F(\lambda)$ for all $\lambda>0$, then $J_F^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda\in\mathbb{C}_+$ we write $E^{\mathrm{an}_\lambda}[F]\equiv E_x^{\mathrm{an}_\lambda}[F(x)]=J_F^*(\lambda)$. Let $q\in\mathbb{R}\setminus\{0\}$ and let F be a s.i.m. function whose analytic function space integral $J_F^*(\lambda)$ exists for all $\lambda\in\mathbb{C}_+$. If the following limit exists, we call it the analytic generalized Feynman integral of F with parameter q, and we write

$$E^{\mathrm{anf}_q}[F]\equiv E_x^{\mathrm{anf}_q}[F(x)]=\lim_{\lambda\to -iq}E_x^{\mathrm{an}_\lambda}[F(x)]\tag{4.1}$$

where $\lambda\to -iq$ through $\mathbb{C}_+$.

We are now ready to state the definition of the analytic GFFT of functions F on $C_{a,b}[0,T]$.

Definition 4.1. Let F be a s.i.m. function on $C_{a,b}[0,T]$. For $\lambda\in\mathbb{C}_+$ and $y\in C_{a,b}[0,T]$, let $T_\lambda(F)(y)=E_x^{\mathrm{an}_\lambda}[F(y+x)]$. For $p\in(1,2]$, we define the L_p analytic GFFT, $T_q^{(p)}(F)$ of F, by the formula

$$T_q^{(p)}(F)(y)=\underset{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}{\mathrm{l.i.m.}}\,T_\lambda(F)(y)$$

if it exists; i.e., for each $\rho>0$,

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}\int_{C_{a,b}[0,T]}\big|T_\lambda(F)(\rho y)-T_q^{(p)}(F)(\rho y)\big|^{p'}d\mu(y)=0$$

where $1/p+1/p'=1$. We define the L_1 analytic GFFT, $T_q^{(1)}(F)$ of F, by the formula

$$T_q^{(1)}(F)(y)=\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} T_\lambda(F)(y)=\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} E_x^{\mathrm{an}_\lambda}[F(y+x)], \tag{4.2}$$

for s-a.e. $y\in C_{a,b}[0,T]$, if the limit exists.

Remark 4.2. In [2, pp. 5–7], Cameron and Storvick exhibited two measurable functions F and G on the classical Wiener space $C_0[0,T]$ such that $F(x)=G(x)$ for a.e. $x\in C_0[0,T]$ and yet their Fourier–Feynman transforms are unequal a.e.. Based on this fact, Johnson and Skoug [15] defined the L_p analytic Fourier–Feynman transform for functions on $C_0[0,T]$ under the concept of the scale-invariant measurability. In fact, it was pointed out in [16] that the concept of 'scale-invariant measurability' is correct for the analytic Fourier–Feynman transform and the analytic Feynman integration theories. For more details, see [18, pp. 1155–1157].

We note that for $1\le p\le 2$, $T_q^{(p)}(F)$ is defined only s-a.e.. If $T_q^{(p)}(F)$ exists and if $F\approx G$, then $T_q^{(p)}(G)$ exists and $T_q^{(p)}(G)\approx T_q^{(p)}(F)$. For more detailed studies of the GFFT of functions on $C_{a,b}[0,T]$, see [9, 11].

In view of (4.1) and (4.2), we set

$$T_q^{(1)}(F)(0)=E_x^{\mathrm{anf}_q}[F(x)]. \tag{4.3}$$

Theorem 4.3 below is a simple modification of the result [12, Theorem 9]. The condition (4.4) below will guarantee the existence of the right-hand side of (4.5) below.

Theorem 4.3. *Let $q_0\in\mathbb{R}\setminus\{0\}$ and let F be given by equation (3.1). Suppose that the associated measure σ of F satisfies the condition*

$$\int_{C'_{a,b}[0,T]}\exp\bigg\{\frac{1}{\sqrt{|2q_0|}}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\bigg\}d|\sigma|(w)<+\infty. \tag{4.4}$$

Then, for all $p\in[1,2]$ and all $q\in\mathbb{R}\setminus[-q_0,q_0]$, the L_p analytic GFFT $T_q^{(p)}(F)$ exists and is given by the formula

$$\begin{aligned} &T_q^{(p)}(F)(y)\\ &=\int_{C'_{a,b}[0,T]}\exp\bigg\{i(w,y)^\sim-\frac{i}{2q}\|w\|^2_{C'_{a,b}}+i(-iq)^{-1/2}(w,a)_{C'_{a,b}}\bigg\}d\sigma(w)\end{aligned} \tag{4.5}$$

for s-a.e. $y\in C_{a,b}[0,T]$.

In view of Theorems 3.4 and 4.3, we can provide the following evaluation formula for the L_p analytic GFFT of functions F in $\mathcal{F}(C_{a,b}[0,T])$.

Theorem 4.4. *Let (Q,Σ,γ), $\{\varphi_1,\dots,\varphi_n\}$, $\{\nu_\eta:\eta\in Q\}$, θ, and F be as in Theorem 3.4. Suppose that given a positive real q_0,*

$$\tag{4.6}\begin{aligned}&\int_{Q\times\mathbb{R}^n}\exp\bigg\{\frac{\|a\|_{C'_{a,b}}}{\sqrt{2|q_0|}}\sum_{l=1}^n\|\varphi_l(\eta)\|_{C'_{a,b}}|v_l|\bigg\}d(|\nu_\eta|\times\gamma)(\eta,\vec{v})\\&=\int_Q\bigg[\int_{\mathbb{R}^n}\exp\bigg\{\frac{\|a\|_{C'_{a,b}}}{\sqrt{2|q_0|}}\sum_{l=1}^n\|\varphi_l(\eta)\|_{C'_{a,b}}|v_l|\bigg\}d|\nu_\eta|(\vec{v})\bigg]d\gamma(\eta)<+\infty.\end{aligned}$$

Then for all $p\in[1,2]$ and all $q\in\mathbb{R}\setminus[-q_0,q_0]$, the L_p analytic GFFT $T_q^{(p)}(F)$ of F exists and is given by the formula

$$\tag{4.7}\begin{aligned}T_q^{(p)}(F)(y)=\int_Q\bigg[\int_{\mathbb{R}^n}\exp\bigg\{i\sum_{l=1}^n v_l(\varphi_l(\eta),y)^\sim-\frac{i}{2q}\bigg\|\sum_{l=1}^n v_l\varphi_l(\eta)\bigg\|^2_{C'_{a,b}}\\+i(-iq)^{-1/2}\sum_{l=1}^n v_l(\varphi_l(\eta),a)_{C'_{a,b}}\bigg\}d\nu_\eta(v_1,\dots,v_n)\bigg]d\gamma(\eta)\end{aligned}$$

for s-a.e. $y\in C_{a,b}[0,T]$. In particular, if $\{\varphi_1(\eta),\dots,\varphi_n(\eta)\}$ is an orthogonal set of functions in $C'_{a,b}[0,T]$, then it follows that

$$\tag{4.8}\begin{aligned}T_q^{(p)}(F)(y)=\int_Q\bigg[\int_{\mathbb{R}^n}\exp\bigg\{i\sum_{l=1}^n v_l(\varphi_l(\eta),y)^\sim-\frac{i}{2q}\sum_{l=1}^n v_l^2\|\varphi_l(\eta)\|^2_{C'_{a,b}}\\+i(-iq)^{-1/2}\sum_{l=1}^n v_l(\varphi_l(\eta),a)_{C'_{a,b}}\bigg\}d\nu_\eta(v_1,\dots,v_n)\bigg]d\gamma(\eta)\end{aligned}$$

for s-a.e. $y\in C_{a,b}[0,T]$.

Proof. From (3.13) with $\rho=1$, we see that the function F given by (3.10) is rewritten by

$$\begin{aligned}F(x)&=\int_Q\bigg[\int_{\mathbb{R}^n}\exp\bigg\{i\sum_{l=1}^n v_l(\varphi_l(\eta),x)^\sim\bigg\}d\nu_\eta(v_1,\dots,v_n)\bigg]d\gamma(\eta)\\&=\int_{C'_{a,b}[0,T]}\exp\{i(w,x)^\sim\}d\tau\circ\Phi^{-1}(w)\end{aligned}$$

for s-a.e. $y\in C_{a,b}[0,T]$, where τ and Φ are given by (3.11) and (3.12) respectively. Thus the condition (4.6) implies the condition (4.4) with $\sigma=\tau\circ\Phi^{-1}$, and by Theorem 4.3, the L_p analytic GFFT of F given by (3.10) exists and is given by the formula

$$\begin{aligned}&T_q^{(p)}(F)(y)\\&=\int_{C'_{a,b}[0,T]}\exp\bigg\{i(w,y)^\sim-\frac{i}{2q}\|w\|^2_{C'_{a,b}}+i(-iq)^{-1/2}(w,a)_{C'_{a,b}}\bigg\}d\tau\circ\Phi^{-1}(w)\end{aligned}$$

$$= \int_{Q\times\mathbb{R}^n} \exp\bigg\{ i(\Phi(\eta; v_1,\dots,v_n), y)^\sim - \frac{i}{2q}\|\Phi(\eta; v_1,\dots,v_n)\|^2_{C'_{a,b}} + i(-iq)^{-1/2}(\Phi(\eta; v_1,\dots,v_n), a)_{C'_{a,b}}\bigg\} d\tau(\eta; v_1,\dots,v_n)$$

$$= \int_Q \bigg[\int_{\mathbb{R}^n} \exp\bigg\{ i\sum_{l=1}^n v_l(\varphi_l(\eta), y)^\sim - \frac{i}{2q}\bigg\|\sum_{l=1}^n v_l\varphi_l(\eta)\bigg\|^2_{C'_{a,b}} + i(-iq)^{-1/2}\sum_{l=1}^n v_l(\varphi_l(\eta), a)_{C'_{a,b}}\bigg\} d\nu_\eta(v_1,\dots,v_n)\bigg] d\gamma(\eta)$$

for s-a.e. $y \in C_{a,b}[0,T]$. From this, we also have (4.8). □

From (4.3) and (4.7) with $p = 1$, we have the following corollary.

Corollary 4.5. *Let (Q, Σ, γ), $\{\varphi_1,\dots,\varphi_n\}$, $\{\nu_\eta : \eta \in Q\}$, θ, and F be as in Theorem 4.4. Then, for all $q \in \mathbb{R}\setminus[-q_0, q_0]$, the generalized analytic Feynman integral $E^{\mathrm{anf}_q}[F]$ of F exists and is given by the formula*

$$E_x^{\mathrm{anf}_q}[F(x)] = \int_Q \bigg[\int_{\mathbb{R}^n} \exp\bigg\{ - \frac{i}{2q}\bigg\|\sum_{l=1}^n v_l\varphi_l(\eta)\bigg\|^2_{C'_{a,b}} + i(-iq)^{-1/2}\sum_{l=1}^n v_l(\varphi_l(\eta), a)_{C'_{a,b}}\bigg\} d\nu_\eta(v_1,\dots,v_n)\bigg] d\gamma(\eta).$$

under the condition (4.6).

Given an orthonormal set $\{g_1,\dots,g_n\}$ of functions in $C_{a,b}[0,T]$, let the function $F : C_{a,b}[0,T] \to \mathbb{C}$ be given by

$$F(x) = \widehat{\nu}((g_1, x)^\sim,\dots,(g_n, x)^\sim), \quad x \in C_{a,b}[0,T], \tag{4.9}$$

where $\widehat{\nu}$ is the Fourier transform defined by equation (3.9) for a complex-valued Borel measure ν in $M(\mathbb{R}^n)$. Then F is a bounded cylinder function, since $|\widehat{\nu}(\vec{u})| \le \|\nu\| < +\infty$. In [8], Chang and Choi studied an inverse transform corresponding to the L_p analytic GFFT of the function given by (4.9). One of the main results in [8] is to establish the existence of the GFFT of the functions F given by (4.9).

Corollary 4.6. *Let $q_0 \in \mathbb{R}\setminus\{0\}$ and let F be given by equation (4.9). Suppose that the associated measure ν of F satisfies the condition*

$$\int_{\mathbb{R}^n} \exp\bigg\{ \frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q_0|}} \sum_{l=1}^n |v_l| \bigg\} d|\nu|(\vec{v}) < +\infty. \tag{4.10}$$

Then, for each $p \in [1, 2]$ and any $q \in \mathbb{R}\setminus[-q_0, q_0]$, the L_p analytic GFFT $T_q^{(p)}(F)$ exists and is given by the formula

$$T_q^{(p)}(F)(y) = \int_{\mathbb{R}^n} \exp\bigg\{ i\sum_{l=1}^n v_l(g_l, y)^\sim - \frac{i}{2q}\sum_{l=1}^n v_l^2 + i(-iq)^{-1/2}\sum_{l=1}^n v_l(g_l, a)_{C'_{a,b}}\bigg\} d\nu(\vec{v}) \tag{4.11}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

Proof. From equation (3.14), we already observe that

$$\begin{aligned} F(x) &= \widehat{\nu}\big((g_1,x)^\sim,\dots,(g_n,x)^\sim\big)\\ &= \int_Q \theta\big(\eta;(\varphi_1(\eta),x)^\sim,\dots,(\varphi_n(\eta),x)^\sim\big)d\gamma(\eta)\end{aligned}$$

for s-a.e. $x \in C_{a,b}[0,T]$, where (Q,Σ,γ) is any probability space, $\varphi_l(\eta) \equiv g_l$ for each $l \in \{1,\dots,n\}$, and $\theta(\eta;\cdot) = \widehat{\nu}(\cdot)$. Also, the condition (4.6) implies the condition

$$\begin{aligned} &\int_{Q\times\mathbb{R}^n} \exp\bigg\{\frac{\|a\|_{C'_{a,b}}}{\sqrt{2|q_0|}}\sum_{l=1}^n \|g_l\|_{C'_{a,b}}|v_l|\bigg\}d(|\nu_\eta|\times\gamma)(\eta,\vec{v})\\ &= \int_Q \bigg[\int_{\mathbb{R}^n} \exp\bigg\{\frac{\|a\|_{C'_{a,b}}}{\sqrt{2|q_0|}}\sum_{l=1}^n |v_l|\bigg\}d|\nu_\eta|(\vec{v})\bigg]d\gamma(\eta)\\ &= \int_{\mathbb{R}^n} \exp\bigg\{\frac{\|a\|_{C'_{a,b}}}{\sqrt{2|q_0|}}\sum_{l=1}^n |v_l|\bigg\}d|\nu|(\vec{v}) < +\infty.\end{aligned}$$

Thus, in view of Theorem 4.4 with these setting, equation (4.8) yields the formula (4.11) as desired. □

From (4.3) and (4.11) with $p = 1$, we have the following corollary.

Corollary 4.7. *Let q_0 and F be as in Corollary 4.6. Then, for any $q \in \mathbb{R}\setminus[-q_0,q_0]$, the generalized Feynman integral $E^{\mathrm{anf}_q}[F]$ exists and is given by the formula*

$$E_x^{\mathrm{anf}_q}[F(x)] = \int_{\mathbb{R}^n} \exp\bigg\{-\frac{i}{2q}\sum_{l=1}^n v_l^2 + i(-iq)^{-1/2}\sum_{l=1}^n v_l(g_l,a)_{C'_{a,b}}\bigg\}d\nu(\vec{v}).$$

5. Examples

In this section, we present various functions to apply our results in previous section. Let the linear operator S on $C'_{a,b}[0,T]$ be given by equation (3.6). Let

$$\psi(t) = \sqrt{3}b(T)^{-3/2}b(t),\ t \in [0,T]. \tag{5.1}$$

Using an integration by parts formula, we see that $\{S^*\psi\}$ is an orthonormal set in $C'_{a,b}[0,T]$, and using (3.8), we also have

$$\frac{1}{\sqrt{3}}b(T)^{3/2}(S^*\psi,x)^\sim = (S^*b,x)^\sim = \int_0^T x(t)Db(t)db(t) = \int_0^T x(t)db(t).$$

For given $\vec{m} = (m_1,\dots,m_n) \in \mathbb{R}^n$ and $\vec{\sigma^2} = (\sigma_1^2,\dots,\sigma_n^2) \in \mathbb{R}^n$ with $\sigma_l^2 > 0$, $l = 1,\dots,n$, let $\nu_{\vec{m},\vec{\sigma^2}}$ be the Gaussian measure given by

$$\nu_{\vec{m},\vec{\sigma^2}}(G) = \bigg(\prod_{l=1}^n 2\pi\sigma_l^2\bigg)^{-1/2}\int_G \exp\bigg\{-\sum_{l=1}^n\frac{(u_l-m_l)^2}{2\sigma_l^2}\bigg\}d\vec{u},\quad G \in \mathcal{B}(\mathbb{R}^n). \tag{5.2}$$

Then $\nu_{\vec{m},\vec{\sigma^2}} \in M(\mathbb{R}^n)$ and

$$\widehat{\nu_{\vec{m},\vec{\sigma^2}}}(\vec{u}) = \exp\bigg\{ -\frac{1}{2}\sum_{l=1}^{n}\sigma_l^2 u_l^2 + i\sum_{l=1}^{n} m_l u_l\bigg\}.$$

Under these setting, we can apply our results in previous section to the function having the form

$$F_6(x) = \exp\bigg\{ -\frac{1}{2}\sum_{l=1}^{n}\sigma_l^2[(g_l,x)^{\sim}]^2 + i\sum_{l=1}^{n} m_l (g_l,x)^{\sim}\bigg\},$$

where $\{g_1,\dots,g_n\}$ is an orthonormal set of functions in $C'_{a,b}[0,T]$. For instance, taking $n=1$, $g_1 = S^*\psi$, $\vec{m}=m_1=0$ and $\vec{\sigma^2}=\sigma_1^2 = 2b(T)^3/3$ in F_6, we have

$$F_7(x) = \exp\bigg\{ -\bigg(\int_0^T x(t)db(t)\bigg)^2\bigg\}. \tag{5.3}$$

Using (5.2), the Fubini theorem and the integration formula [10, equation (2.15)], it follows that for each nonzero real number q,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \exp\bigg\{\frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q|}}\sum_{l=1}^{n}|v_l|\bigg\} d|\nu_{\vec{m},\vec{\sigma^2}}|(\vec{v})\\
&= \prod_{l=1}^{n}\bigg[(2\pi\sigma_l^2)^{-1/2}\int_{-\infty}^{0}\exp\bigg\{-\frac{v_l^2}{2\sigma_l^2} + \bigg(\frac{m_l}{\sigma_l^2} - \frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q|}}\bigg)v_l - \frac{m_l^2}{2\sigma_l^2}\bigg\}dv_l\\
&\qquad + (2\pi\sigma_l^2)^{-1/2}\int_{0}^{+\infty}\exp\bigg\{-\frac{v_l^2}{2\sigma_l^2} + \bigg(\frac{m_l}{\sigma_l^2} + \frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q|}}\bigg)v_l - \frac{m_l^2}{2\sigma_l^2}\bigg\}dv_l\bigg]\\
&< \prod_{l=1}^{n}\bigg[(2\pi\sigma_l^2)^{-1/2}\int_{\mathbb{R}}\exp\bigg\{-\frac{v_l^2}{2\sigma_l^2} + \bigg(\frac{m_l}{\sigma_l^2} - \frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q|}}\bigg)v_l - \frac{m_l^2}{2\sigma_l^2}\bigg\}dv_l\\
&\qquad + (2\pi\sigma_l^2)^{-1/2}\int_{\mathbb{R}}\exp\bigg\{-\frac{v_l^2}{2\sigma_l^2} + \bigg(\frac{m_l}{\sigma_l^2} + \frac{\|a\|_{C'_{a,b}}}{\sqrt{|2q|}}\bigg)v_l - \frac{m_l^2}{2\sigma_l^2}\bigg\}dv_l\bigg]\\
&= \prod_{l=1}^{n}\bigg[\exp\bigg\{\frac{\|a\|^2_{C'_{a,b}}}{|2q|} - \frac{m_l\|a\|_{C'_{a,b}}}{\sqrt{|2q|}}\bigg\} + \exp\bigg\{\frac{\|a\|^2_{C'_{a,b}}}{|2q|} + \frac{m_l\|a\|_{C'_{a,b}}}{\sqrt{|2q|}}\bigg\}\bigg]\\
&< +\infty.
\end{aligned}$$

Thus for all $q\in\mathbb{R}\setminus\{0\}$, $T_q^{(p)}(F_6)$ (and hence $T_q^{(p)}(F_7)$) exists by Corollary 4.6. Also, we can apply Corollary 4.7 to obtain the generalized Feynman integrals $E^{\mathrm{anf}_q}[F_6]$ and $E^{\mathrm{anf}_q}[F_7]$.

The function

$$F_8(x) = \exp\bigg\{ i\int_0^T x(t)db(s)\bigg\} \tag{5.4}$$

also is a function under our consideration, because

$$F_8(x) = \exp\{i(S^*b, x)^\sim\} = \exp\left\{\frac{i}{\sqrt{3}}b(T)^{3/2}(S^*\psi, x)^\sim\right\}$$
$$= \int_{\mathbb{R}} \exp\{i(S^*\psi, x)^\sim v\} d\delta_1(v) = \widehat{\delta_1}((S^*\psi, x)^\sim)$$

where ψ is given by (5.1) and δ_1 is the Dirac measure concentrated at $v = b(T)^{3/2}/\sqrt{3}$ in $\mathbb{R}$. Clearly, δ_1 satisfies condition (4.10) with ν replaced with δ_1, for all $q_0 \in \mathbb{R} \setminus \{0\}$.

The functions given by equations (5.3) and (5.4) arise naturally in quantum mechanics.

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