J. Korean Math. Soc. **61** (2024), No. 4, pp. 743–760
https://doi.org/10.4134/JKMS.j230497
pISSN: 0304-9914 / eISSN: 2234-3008

AN INVARIANT FORTH-ORDER CURVE FLOW IN CENTRO-AFFINE GEOMETRY

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Abstract. In this paper, we are devoted to study a forth order curve flow for a smooth closed curve in centro-affine geometry. Firstly, a new evolutionary equation about this curve flow is proposed. Then the related geometric quantities and some meaningful conclusions are obtained through the equation. Next, we obtain finite order differential inequalities for energy by applying interpolation inequalities, Cauchy-Schwartz inequalities, etc. After using a completely new symbolic expression, the n-order differential inequality for energy is considered. Finally, by the means of energy estimation, we prove that the forth order curve flow has a smooth solution all the time for any closed smooth initial curve.

1. Introduction

In the past few decades, considerable researches centered on the curve shape evolution over time in Euclidean space have made great progress; see, for instance, [2,3,5–10,12–15,17–19,24], etc. Among them, the curve shortening flow (CSF) is the simplest curve evolution problem. It was used firstly by Mullins [15] to organize the model of the motion of grain boundaries. The curve evolution problem was also discussed by Gage-Hamilton [6] and Grayson [8, 9]. Their result expressed that the CSF with a convex closed curve in the plane as initial condition will shrink the curve to a point in finite time while evolving toward a circular shape at an infinite time limit. The book by Chou and Zhu [3] provided an excellent and unified account of many results related to flowing curves by curvature. In the Euclidean plane $\mathbb{R}^2$, a forth order curve flow

$$\frac{\partial C}{\partial t} = \left(-\kappa_{ss} + \kappa^3\right) N + 3\kappa\kappa_s T \tag{1.1}$$

was proposed by Yang and Fu [24]. Here T, N are the unit tangent vector and unit normal vector, respectively. κ is the Euclidean curvature of the curve C. It was shown that the flow (1.1) has a smooth solution for all time by

Received September 18, 2023; Revised November 13, 2023; Accepted December 7, 2023.
2020 *Mathematics Subject Classification.* Primary 51N10, 53A15, 53A55, 53E40.
Key words and phrases. Centro-affine geometry, energy estimation, smooth solution, forth order curve flow.

assuming some additional conditions. Liu and Jian [12] also studied a forth order Euclidean geometry heat equation

$$\frac{\partial C}{\partial t} = HN, \tag{1.2}$$

$$H = \kappa_{ss} - \frac{1}{2}\kappa^3 + \left(\frac{1}{2}c_0^2 + \lambda_1\right)\kappa + \lambda_2,$$

where c_0, λ_1, λ_2 are all constants. It means that for any smooth closed initial curve, the heat equation has a smooth solution that exists for all time as time goes to infinity. In addition, the problems of flows about preserving some geometric quantities are also concentrated by people. The special area-preserving and length-preserving flows were deeply studied in [10,13,17–19]. The analysis of fourth and higher order partial differential equations (PDEs) is indeed an area of increasing interest in mathematics. Researchers are actively studying the behavior of solutions to such equations and seeking to understand their properties. Geometric analysis techniques have been instrumental in obtaining many of the known results in this field. These methods utilize geometric structures and tools from differential geometry to analyze and study PDEs. They have provided valuable insights into the behavior of solutions and helped uncover deep connections between geometry and analysis. However, there are still significant challenges ahead. Generalizing these techniques to apply them to more classes of equations and understanding the behavior of solutions in complex settings remains an open question. Despite these ongoing questions, progress continues to be made in the analysis of fourth and higher order PDEs. New results are being obtained, and there is great anticipation about the potential applications of these techniques to other areas of mathematics and physics.

Affine differential geometry is based on Lie group $A(n, \mathbb{R}) = GL(n, \mathbb{R}) \times \mathbb{R}^n$ consisting of affine transformation $x \to Ax + b$, $A \in GL(n, \mathbb{R})$, $b \in \mathbb{R}^n$ acting on $x \in \mathbb{R}^n$, so everything is invariant under the correspondence. The affine curve shortening flow (ACSF) was firstly studied by Sapiro and Tannenbaum [21], and further prolonged by Angenent, Sapiro and Tannebunm [1]. It was turned out that the family of convex curves under the evolution equation

$$\frac{\partial C}{\partial t} = C_{ss} \tag{1.3}$$

converges to an elliptical point in the Hausdorff metric and finally to a point with absolute curvature converging to 2π. Centro-affine differential geometry refers to the subgroup of the affine transformation group that keeps the origin fixed, which is closely related to the geometry induced by the general linear group $x \mapsto Ax$, $A \in GL(n, \mathbb{R})$, $x \in \mathbb{R}^n$. Strictly speaking, centro-equiaffine differential geometry arises in connection with the subgroup $SL(n, \mathbb{R})$ of volume-preserving linear transformations. These cases are usually mentioned in books devoted to (equi-)affine geometry [22]. Qu and Yang [20] discussed the existence and uniqueness to a second-order centro-affine geometry flow equation

$$\frac{\partial C}{\partial t} = \left(\lambda + \int_0^{\xi} \varphi d\xi\right) C + \frac{\varphi}{2} C_{\xi}, \tag{1.4}$$

where λ is a constant, ξ is the centro-affine arc length element, and φ is centro-affine curvature. And they gave a classification to all closed self-similar solutions of the curve flow. Inspired by (1.4), Jiang et al. [11] found a flow that yielded a second-order nonlinear parabolic equation

$$\frac{\partial \varphi}{\partial t} = \frac{1}{2}\varphi_{\xi\xi} - \frac{1}{2}\varphi^3 + 2\varphi \tag{1.5}$$

for the centro-affine curvature, which had some analogous properties to the invariant heat flows in Euclidean, equi-affine and centro-equiaffine geometry. In view of the above consideration, they presented a study of the isoperimetric inequality in centro-affine plane geometry and additionally investigated the long-term behavior of an invariant plane curve flow. Then, the forward and backward limits over time were discussed. They also showed that a closed convex embedded curve may converge to an ellipse when evolving according to (1.5).

Mainly motivated by [2, 5, 12, 16, 20, 24], the goal of this paper is to examine the evolution of a fourth-order curve flow in centro-affine geometry. We will begin by providing an overview of centro-affine geometry and the basics of curve flows. Next, we will discuss the specifics of fourth-order curve flows and their properties.

The paper aims to investigate the properties and behavior of the fourth-order curve flow in centro-affine geometry and provides theoretical results regarding the smoothness of its solutions. Now we summarize how the rest of the paper is laid out. In Section 2, we review some basic contents of centro-affine geometry and the known conclusions. At the same time, we give a centro-affine geometric flow equation. Section 3 calculates the centro-affine curvature in the form of energy estimation and obtain the n-order results according to the integral inequality. Combining the results of energy estimates, we ultimately give the proof of Theorem 1.1 in Section 4.

Theorem 1.1. *Assume $C(\cdot, t)$ is a solution of the centro-affine heat flow* (2.6) *in a maximal interval $[0, \omega)$, $\omega \leq \infty$, where the initial curve C_0 is a closed smooth embedded curve. Then the solution exists as long as the L^2-form of the curvature φ of $C(\cdot, t)$ is finite. Furthermore, when ω is finite,*

$$\oint_{C(\cdot,t)} \varphi^2 d\xi \geq D(\omega - t)^{-\frac{1}{4}}$$

for some constant D. When ω is infinity, the curvature φ of $C(\cdot, t)$ converges smoothly to a constant, that is, $C(\cdot, t)$ converges to an ellipse.

2. Preliminaries

In this section, we mainly review and summarize some fundamental concepts that will be useful for further discussions. For example, the affine geometry heat flow equation, arc length element, redefined metric and so on (for detail see [16]). In the rest of the section, we will give derivation of the evolution equations for quantities. The following facts are all based on the article [16,20].

2.1. Centro-affine differential geometry

One important result of affine differential geometry is the theory of centro-affine invariance. Here, "centro-affine" refers to a type of transformation that preserves both ratios of distances along lines and the barycenter of every set of points. If we express the centro-affine action of two-dimensional plane in the form of matrix product, then there will be the following form:

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = h \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix},$$

where $\det\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right) = 1$ and $h \neq 0$. Furthermore, the centro-affine transformation in the plane is defined as

$$\overline{X} = BX, \tag{2.1}$$

where X is a vector of the plane, B is a real 2×2 matrix whose determinant is not equal to zero. Here we take $B \in GL\,(2, \mathbb{R})$. When $B \in SL\,(2, \mathbb{R})$, we call (2.1) centro-equiaffine transformation.

In order to describe the operation relationship of the following formulas more conveniently, here and thereafter we use

$$[X, Y] = \det\,(X, Y)$$

to express the determinant of the 2×2 matrix whose columns are given by the two vectors X, Y. Moreover, we define

$$g = \sqrt{\epsilon \frac{[C_p, C_{pp}]}{[C, C_p]}}$$

as the centro-affine invariant metric. Here p is a free parameter of the curve C, $C_p = \frac{\mathrm{d}C}{\mathrm{d}p}$, $C_{pp} = \frac{\mathrm{d}^2 C}{\mathrm{d}p^2}$, and

$$\epsilon = \operatorname{sgn}\left(\frac{[C_p, C_{pp}]}{[C, C_p]}\right), \tag{2.2}$$

where sgn stands for symbolic function.

Therefore, the centro-affine arc length element is

$$\xi(p) = \int_{p_0}^{p} g\,(x)\,\mathrm{d}x.$$

Definition 2.1 ([20]). A curve $C(p) : I \to \mathbb{R}^2$ is said to be a star-shaped centro-affine curve if

$$K_0 = [C, C_p] \neq 0.$$

Definition 2.2 ([20]). A curve $C(p) : I \to \mathbb{R}^2$ is said to be a regular star-shaped centro-affine curve if

$$K_i = [C_i, C_{i+1}] \neq 0, \ i = 0, 1, \ldots, n,$$

where $C_0 = C$ and $C_i = \frac{d^i C}{dp^i}$, $i \geq 1$.

In this article, we focus on the regular curves defined by Definition 2.2. From (2.2), we wish to point out that the coefficient ϵ determines whether or not the vectors $C, C_{\xi\xi}$ located on the same direction of the vector C_ξ. By extension, the value of ϵ describes the bending directions of C. For a regular curve C, $\epsilon \equiv 1$ or $\epsilon \equiv -1$.

Lemma 2.3 ([16]). *If the curve C is closed, then $\epsilon = 1$ and the origin lies inside every simple closed loop of the curve C.*

Lemma 2.4 ([16]). *If the curve C is closed, then $\oint_C \varphi d\xi = 0$.*

2.2. A forth-order flow in centro-affine differential geometry

Definition 2.5 ([16]). Let $C : S^1 \times I \to R^2$ be a family of embedded smooth closed curves, $t \in I$ is a parameter for time and $p \in S^1$ is a free parameter. The centro-affine geometry heat flow equation is defined as the form

$$\frac{\partial C}{\partial t} = \beta\left(\varphi\left(\xi, t\right)\right) C_{\xi\xi}, \tag{2.3}$$

where ξ is centro-affine arc length element, φ is centro-affine curvature and β stands for the function related to φ. We assume the family of curves meet the condition

$$C\left(\xi, 0\right) = C_0\left(\xi\right), \tag{2.4}$$

where the initial curve C_0 is closed and convex.

For the convenience of the following narration, we use

$$C_{n\xi} = C_{\xi\cdots\xi} = \frac{\partial^n C}{\partial \xi^n}$$

to express the n-order partial derivative of the curve C to arc length element ξ. Because C_ξ and C do not constitute a parallel relationship, $C_{2\xi}$ is the linear representation of C_ξ, C. Equation (2.3) can be rewritten as

$$\frac{\partial C}{\partial t} = W C_\xi + U C, \tag{2.5}$$

where W, U are smooth functions.

Through a series of calculations in [16], we get

$$\frac{g_t}{g} = W_\xi + \frac{1}{2}\epsilon\varphi U_\xi - \frac{1}{2}\epsilon U_{2\xi}$$

and

$$\frac{\partial\varphi}{\partial t} = W\varphi_\xi + 2U_\xi - \frac{\epsilon}{2}\left(\varphi_\xi U_\xi + \varphi^2 U_\xi - U_{3\xi}\right).$$

In this paper, we are mainly concerned about the flow (2.5) with

$$U = -\varphi_\xi, \quad W = -\frac{1}{2}\varphi_{2\xi}.$$

Namely, from Lemma 2.3, we can see

$$\frac{\partial C}{\partial t} = -\frac{1}{2}\varphi_{2\xi}C_\xi - \varphi_\xi C \tag{2.6}$$

and

$$\frac{1}{g}\frac{\partial g}{\partial t} = -\frac{1}{2}\varphi\varphi_{2\xi}, \quad \frac{\partial\varphi}{\partial t} = -\frac{1}{2}\varphi_{4\xi} + \frac{1}{2}\varphi^2\varphi_{2\xi} - 2\varphi_{2\xi}. \tag{2.7}$$

It is obviously to see that, p and t are two independent variables, so $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial p}$ can commute during calculation. The relationship between the arc length ξ, the curve length L and time t are as follows.

Lemma 2.6. *The following relation holds*

$$\frac{\partial}{\partial t}\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\xi}\frac{\partial}{\partial t} = \frac{1}{2}\varphi\varphi_{2\xi}\frac{\partial}{\partial\xi}.$$

Proof. Straightforward checking shows

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial}{\partial\xi} &= \frac{\partial}{\partial t}\left(\frac{1}{g}\frac{\partial}{\partial p}\right)\\ &= \frac{1}{g}\frac{\partial}{\partial t}\frac{\partial}{\partial p} - \frac{1}{g^2}\frac{\partial g}{\partial t}\frac{\partial}{\partial p}\\ &= \frac{1}{g}\frac{\partial}{\partial p}\frac{\partial}{\partial t} - \frac{1}{g}\frac{\partial g}{\partial t}\frac{\partial}{\partial\xi}\\ &= \frac{\partial}{\partial\xi}\frac{\partial}{\partial t} + \frac{1}{2}\varphi\varphi_{2\xi}\frac{\partial}{\partial\xi}.\end{aligned}$$

□

Lemma 2.7. *The relationship between the curve length L and the time t is given by*

$$\frac{\partial L}{\partial t} = \frac{1}{2}\oint_C \varphi_\xi^2 d\xi. \tag{2.8}$$

Proof. From formula (2.7), a straightforward computation yields that

$$\frac{\partial L}{\partial t} = \frac{\partial}{\partial t}\oint_C d\xi = \frac{\partial}{\partial t}\oint_C g dp = \oint_C \frac{1}{g}\frac{\partial g}{\partial t}d\xi = \frac{1}{2}\oint_C \varphi_\xi^2 d\xi.$$

□

In the following, we assume that the existence and uniqueness of the local solution of (2.6) holds for each initial value $C(\cdot, 0) = C_0$.

3. The energy estimates of the centro-affine curvature

The isoperimetric inequality is a mathematical principle that relates the length L of a closed curve in a plane to the area A enclosed by the curve. It states that among all closed curves with the same enclosed area, a circle has the smallest perimeter. In other words, if we have different closed curves in a plane that enclose the same area, the one with the shortest perimeter or length will be the circle. This result demonstrates the efficiency of the circular shape in terms of maximizing the area within a given perimeter.

It attracts people's interest because of its intrinsic geometric characteristics and wide applications. In this section, the centro-affine isoperimetric inequality will be mentioned and we will also research the energy estimates of the centro-affine curvature.

We have the following isoperimetric inequality in Euclidean plane:

$$L^2 \leq 4\pi A,$$

where the equality holds when the curve is a circle. As is seen in [23], the equi-affine isoperimetric inequality in affine space is

$$L^3 \leq 8\pi^2 A,$$

and the equality holds only for the ellipse. In centro-affine geometry, the conclusion is

Proposition 3.1 ([11]). *In centro-affine geometry, for any smooth convex embedded closed curve C, the centro-affine perimeter of the curve*

$$L = \oint_C d\xi \leq 2\pi,$$

and equality holds if and only if C is an ellipse centered at the origin.

We will apply several inequalities in this manuscript, and the first one to mention is called the Young inequality [4]:

$$ab \leq \frac{a^{q_1}}{q_1} + \frac{b^{q_2}}{q_2}, \tag{3.1}$$

where a, b, q_1, q_2 are positive reals, and q_1, q_2 satisfy

$$\frac{1}{q_1} + \frac{1}{q_2} = 1.$$

With the same exponents as above, the Hölder inequality [4] is

$$\int_\Omega uv dx \leq \|u\|_{L^{q_1}(\Omega)} \|v\|_{L^{q_2}(\Omega)}. \tag{3.2}$$

As a consequence, we can get the following interpolation inequality [2]. For the periodic function u with zero mean, we can see the inequality

$$\|u^{(j)}\|_{L^q} \leq D_0 \|u\|_{L^{q_1}}^{1-\theta} \|u^{(k)}\|_{L^{q_2}}^{\theta}, \quad \theta \in (0,1) \tag{3.3}$$

is always true, where j, k are the order of the partial derivative of u. q, q_1, q_2, j, k satisfy q_1, q_2, $q > 1$, $j \geq 0$,

$$\frac{1}{q} = j + \theta \left(\frac{1}{q_2} - k \right) + (1 - \theta) \frac{1}{q_1},$$

and

$$\frac{j}{k} \leq \theta \leq 1.$$

The constant D_0 in (3.3) depends on q, q_1, q_2, j and k only.

Throughout the article, we always take for granted that $D_i, \epsilon_i, \eta_i, \zeta_i, \iota_i$, $i \in Z$ are defined as constants.

Utilizing the similar technology given in [2] and simultaneously (2.7), we reach

$$\begin{aligned}
\frac{\partial}{\partial t} \oint_C \varphi^2 d\xi &= \frac{\partial}{\partial t} \oint_C \varphi^2 g dp \\
&= \oint_C (2\varphi\varphi_t + \varphi^2 \frac{g_t}{g}) d\xi \\
&= \oint_C (-\varphi\varphi_{4\xi} + \varphi^3 \varphi_{2\xi} - 4\varphi\varphi_{2\xi} - \frac{1}{2}\varphi^3 \varphi_{2\xi}) d\xi \\
&= \oint_C -\varphi_{2\xi}^2 d\xi - \oint_C \frac{3}{2}\varphi^2 \varphi_\xi^2 d\xi + \oint_C 4\varphi_\xi^2 d\xi.
\end{aligned}$$

In accordance with (3.3), we conclude that

$$\left(\oint_C \varphi^4 d\xi \right)^{\frac{1}{2}} \leq D_1 \left(\oint_C \varphi^2 d\xi \right)^{\frac{7}{8}} \left(\oint_C \varphi_{2\xi}^2 d\xi \right)^{\frac{1}{8}},$$

$$\left(\oint_C \varphi_\xi^4 d\xi \right)^{\frac{1}{2}} \leq D_2 \left(\oint_C \varphi^2 d\xi \right)^{\frac{3}{8}} \left(\oint_C \varphi_{2\xi}^2 d\xi \right)^{\frac{5}{8}},$$

$$\left(\oint_C \varphi_\xi^2 d\xi \right)^{\frac{1}{2}} \leq D_3 \left(\oint_C \varphi^2 d\xi \right)^{\frac{1}{2}} \left(\oint_C \varphi_{2\xi}^2 d\xi \right)^{\frac{1}{2}}.$$

It follows that

$$\begin{aligned}
\oint_C \varphi^2 \varphi_\xi^2 d\xi &\leq \left(\oint_C \varphi^4 d\xi \right)^{\frac{1}{2}} \left(\oint_C \varphi_\xi^4 d\xi \right)^{\frac{1}{2}} \\
&\leq D_1 D_2 \left(\oint_C \varphi^2 d\xi \right)^{\frac{5}{4}} \left(\oint_C \varphi_{2\xi}^2 d\xi \right)^{\frac{3}{4}}.
\end{aligned}$$

Applying the Young inequality (3.1), we have

$$\begin{aligned}
\oint_C \varphi^2 \varphi_\xi^2 d\xi &\leq D_1 D_2 {\epsilon_1}^{-\frac{3}{4}} \left(\oint_C \varphi^2 d\xi \right)^{\frac{5}{4}} \left(\epsilon_1 \oint_C \varphi_{2\xi}^2 d\xi \right)^{\frac{3}{4}} \\
&\leq \epsilon_1 \oint_C \varphi_{2\xi}^2 d\xi + \frac{27}{256} (D_1 D_2)^4 \epsilon_1^{-3} \left(\oint_C \varphi^2 d\xi \right)^5.
\end{aligned}$$

Moreover, we can also get

$$\oint_C \varphi_\xi^2 d\xi \le \epsilon_2 \oint_C \varphi_{2\xi}^2 d\xi + \frac{D_3^2}{4\epsilon_2} \oint_C \varphi^2 d\xi.$$

Let

$$E = \oint_C \varphi^2 d\xi,$$

we deduce that

$$\frac{\partial}{\partial t} \oint_C \varphi^2 d\xi \le \left(-1 - \frac{3}{2}\epsilon_1 + 4\epsilon_2\right) \oint_C \varphi_{2\xi}^2 d\xi + P_1(E),$$

where $P_1(E) = -\frac{81}{512}(D_1D_2)^4 \epsilon_1^{-3} E^5 + \frac{D_3^2}{\epsilon_2} E$.

Choosing $-\frac{3}{2}\epsilon_1 + 4\epsilon_2 = 1$, we obtain

$$\frac{\partial E}{\partial t} \le D_4 \left(E + E^5\right). \tag{3.4}$$

From Lemma 2.6, one knows

$$\begin{aligned}
\frac{\partial}{\partial t} \oint_C \varphi_\xi^2 d\xi &= \oint_C (2\varphi_\xi \varphi_{\xi t} + \varphi_\xi^2 \frac{g_t}{g}) d\xi \\
&= \oint_C \left[2\varphi_\xi(-\frac{1}{2}\varphi_{5\xi} + \frac{3}{2}\varphi\varphi_\xi\varphi_{2\xi} + \frac{1}{2}\varphi^2\varphi_{3\xi} - 2\varphi_{3\xi}) - \frac{1}{2}\varphi\varphi_\xi^2\varphi_{2\xi}\right] d\xi \\
&= -\oint_C \varphi_\xi\varphi_{5\xi} d\xi + \frac{5}{2}\oint_C \varphi\varphi_\xi^2\varphi_{2\xi} d\xi + \oint_C \varphi^2\varphi_\xi\varphi_{3\xi} d\xi - 4\oint_C \varphi_\xi\varphi_{3\xi} d\xi \\
&= -\oint_C \varphi_{3\xi}^2 d\xi - \frac{1}{6}\int_C \varphi_\xi^4 d\xi - \oint_C \varphi^2\varphi_{2\xi}^2 d\xi + 4\oint_C \varphi_{2\xi}^2 d\xi.
\end{aligned}$$

Now, doing the same thing as above, we can easily check that the following inequalities are valid:

$$\oint_C \varphi_\xi^4 d\xi \le D_5 \left(\oint_C \varphi^2 d\xi\right)^{\frac{7}{6}} \left(\oint_C \varphi_{3\xi}^2 d\xi\right)^{\frac{5}{6}},$$

$$\left(\oint_C \varphi^4 d\xi\right)^{\frac{1}{2}} \le D_6 \left(\oint_C \varphi^2 d\xi\right)^{\frac{11}{12}} \left(\oint_C \varphi_{3\xi}^2 d\xi\right)^{\frac{1}{12}},$$

$$\left(\oint_C \varphi_{2\xi}^4 d\xi\right)^{\frac{1}{2}} \le D_7 \left(\oint_C \varphi^2 d\xi\right)^{\frac{1}{4}} \left(\oint_C \varphi_{3\xi}^2 d\xi\right)^{\frac{3}{4}},$$

$$\oint_C \varphi_{2\xi}^2 d\xi \le D_8 \left(\oint_C \varphi^2 d\xi\right)^{\frac{1}{3}} \left(\oint_C \varphi_{3\xi}^2 d\xi\right)^{\frac{2}{3}}.$$

Thus,

$$\oint_C \varphi_\xi^4 d\xi \le \epsilon_3 \oint_C \varphi_{3\xi}^2 d\xi + \frac{1}{6} D_5^6 \left(\frac{6}{5}\epsilon_3\right)^{-5} \left(\oint_C \varphi^2 d\xi\right)^7,$$

$$\oint_C \varphi^2 \varphi_{2\xi}^2 d\xi \le D_6 D_7 \left(\oint_C \varphi^2 d\xi\right)^{\frac{7}{6}} \left(\oint_C \varphi_{3\xi}^2 d\xi\right)^{\frac{5}{6}}$$
$$\le \epsilon_4 \oint_C \varphi_{3\xi}^2 d\xi + \frac{1}{6}\left(\frac{6}{5}\epsilon_4\right)^{-5} (D_6 D_7)^{12} \left(\oint_C \varphi^2 d\xi\right)^7,$$

$$\oint_C \varphi_{2\xi}^2 d\xi \le \epsilon_5 \oint_C \varphi_{3\xi}^2 d\xi + \frac{1}{3}\left(\frac{3}{2}\epsilon_5\right)^{-\frac{2}{3}} D_8^3 \oint_C \varphi^2 d\xi.$$

Using the integral inequalities, we see

$$\frac{\partial}{\partial t}\oint_C \varphi_\xi^2 d\xi \le D_9\left(E + E^7\right). \tag{3.5}$$

Finally,

$$\frac{\partial}{\partial t}\oint_C \varphi_{2\xi}^2 d\xi$$
$$= -\oint_C \varphi_{4\xi}^2 d\xi - \oint_C \varphi^2 \varphi_{3\xi}^2 d\xi + \frac{3}{2}\oint_C \varphi_\xi^2 \varphi_{2\xi}^2 d\xi + \oint_C \varphi \varphi_{2\xi}^3 d\xi + 4\oint_C \varphi_{3\xi}^2 d\xi.$$

Proceeding to the next step, we introduce

$$\oint_C \varphi^2 \varphi_{3\xi}^2 d\xi \le D_{10}\left(\oint_C \varphi^2 d\xi\right)^{\frac{9}{8}} \left(\oint_C \varphi_{4\xi}^2 d\xi\right)^{\frac{7}{8}},$$

$$\oint_C \varphi_\xi^2 \varphi_{2\xi}^2 d\xi \le D_{11}\left(\oint_C \varphi^2 d\xi\right)^{\frac{9}{8}} \left(\oint_C \varphi_{4\xi}^2 d\xi\right)^{\frac{7}{8}},$$

$$\oint_C \varphi \varphi_{2\xi}^3 d\xi \le D_{12}\left(\oint_C \varphi^2 d\xi\right)^{\frac{9}{8}} \left(\oint_C \varphi_{4\xi}^2 d\xi\right)^{\frac{7}{8}},$$

$$\oint_C \varphi_{3\xi}^2 d\xi \le D_{13}\left(\oint_C \varphi^2 d\xi\right)^{\frac{1}{4}} \left(\oint_C \varphi_{4\xi}^2 d\xi\right)^{\frac{3}{4}},$$

and ultimately we acquire

$$\frac{\partial}{\partial t}\oint_C \varphi_{2\xi}^2 d\xi \le D_{14}\left(E + E^9\right). \tag{3.6}$$

For facilitating the calculations, we assume $P_m^n(\omega)$ is any linear combination of the type $\partial_\xi^{i_1}\omega * \cdots * \partial_\xi^{i_m}\omega$ with universal constant coefficients, where $n = i_1 + \cdots + i_m$ is the total number of derivatives, then we observe

$$P_v^\mu(\omega) * P_\beta^\alpha(\omega) = P_{v+\beta}^{\mu+\alpha}(\omega), \quad \partial_\xi P_v^\mu(\omega) = P_v^{\mu+1}(\omega).$$

By a further arrangement, we can rewrite derivative of the curvature with respect to time t as

$$\varphi_t = -\frac{1}{2}\varphi_{4\xi} + P_3^2(\varphi) + P_1^2(\varphi), \quad \varphi_{\xi t} = -\frac{1}{2}\varphi_{5\xi} + P_3^3(\varphi) + P_1^3(\varphi). \tag{3.7}$$

Lemma 3.2. *The evolution of $(\varphi_{k\xi})_t$ is given by*

$$(\varphi_{k\xi})_t = -\varphi_{(k+4)\xi} + P_3^{k+2}(\varphi) + P_1^{k+2}(\varphi)$$

for any nonnegative integer k.

Proof. In order to get the conclusion, we use the mathematical induction. Taking $k=0$, by means of (3.7), we can prove that the equation holds for φ_t. Now we assume the conclusion is true for $k=n$. Then, when $k=n+1$, we can easily verify that

$$\begin{aligned}\varphi_{(n+1)\xi t} &= \varphi_{(n\xi)t\xi} - \frac{1}{g}\frac{\partial g}{\partial t}\varphi_{(n+1)\xi}\\ &= \frac{\partial}{\partial \xi}\left[-\varphi_{(n+4)\xi} + P_3^{n+2}(\varphi) + P_1^{n+2}(\varphi)\right] + P_3^{n+3}(\varphi)\\ &= -\varphi_{(n+5)\xi} + P_3^{n+3}(\varphi) + P_1^{n+3}(\varphi).\end{aligned}$$

This ends the proof of the lemma. □

Proposition 3.3. *Let $C: I \to \mathbb{R}^2$ be a smooth closed curve. For any $P_v^\mu(\varphi)$, which includes only derivatives of φ of order at most $l-1$, one has*

$$\oint_C |P_v^\mu(\varphi)|d\xi \leq \eta_0 \oint_C \left(\partial_\xi^l(\varphi)\right)^2 d\xi + D_{16}\eta_0^{\frac{\mu+\frac{v}{2}-1}{\mu+\frac{v}{2}-1-2l}} \left(\oint_C \varphi^2 d\xi\right)^{\frac{vl-\mu-\frac{v}{2}+1}{2l-\mu-\frac{v}{2}+1}},$$

where $\eta_0 \geq 0$, $v \geq 2$.

Proof. Consider $P_v^\mu(\varphi) = \partial_\xi^{i_1}(\varphi)\cdots\partial_\xi^{i_v}(\varphi)$, where $i_1+\cdots+i_v=\mu$. Through the Hölder inequality, we reach

$$\begin{aligned}\oint_C P_v^\mu(\varphi)\,d\xi &\leq \left[\oint_C \left(\partial_\xi^{i_1}(\varphi)\right)^v d\xi\right]^{\frac{1}{v}}\cdots\left[\oint_C \left(\partial_\xi^{i_v}(\varphi)\right)^v d\xi\right]^{\frac{1}{v}}\\ &= \prod_{j=1}^{v}\left[\oint_C \left(\partial_\xi^{i_j}(\varphi)\right)^v d\xi\right]^{\frac{1}{v}}\\ &= \prod_{j=1}^{v}||\partial_\xi^{i_j}(\varphi)||_{L^v}.\end{aligned}$$

By interpolation inequality (3.3), one has

$$||\partial_\xi^{i_j}(\varphi)||_{L^v} \leq D_{15}||\varphi||_{L^2}^{1-\theta_j}||\partial_\xi^l(\varphi)||_{L^2}^{\theta_j},$$

where $\theta_j = \frac{1}{l}\left(i_j + \frac{1}{2} - \frac{1}{v}\right)$. Therefore,

$$\begin{aligned}\oint_C P_v^\mu(\varphi)\,d\xi &\leq D_{15}||\varphi||_{L^2}^{1-\theta_1}||\partial_\xi^l(\varphi)||_{L^2}^{\theta_1}\cdots D_{15}||\varphi||_{L^2}^{1-\theta_v}||\partial_\xi^l(\varphi)||_{L^2}^{\theta_v}\\ &= D_{15}^v||\varphi||_{L^2}^{v-(\theta_1+\cdots+\theta_v)}||\partial_\xi^l(\varphi)||_{L^2}^{\theta_1+\cdots+\theta_v}\\ &= D_{15}^v||\varphi||_{L^2}^{v-\frac{1}{l}(\mu+\frac{v}{2}-1)}||\partial_\xi^l(\varphi)||_{L^2}^{\frac{1}{l}\left(\mu+\frac{v}{2}-1\right)}.\end{aligned}$$

According to (3.1), we can prove that

$$\begin{aligned}\oint_C P_v^{\mu}(\varphi)\,d\xi &\le \eta_0 \oint_C \left(\partial_\xi^l(\varphi)\right)^2 d\xi + \left(1-\frac{\mu+\frac{v}{2}-1}{2k}\right) D_{15}^{\frac{v}{1-\frac{1}{2k}\left(\mu+\frac{v}{2}-1\right)}} \\ &\quad\times\left(\frac{2l\eta_0}{\mu+\frac{v}{2}-1}\right)^{\frac{\mu+\frac{v}{2}-1}{\mu+\frac{v}{2}-1-2l}} \left(\int_C \varphi^2 d\xi\right)^{\frac{vl-\mu-\frac{v}{2}+1}{2l-\mu-\frac{v}{2}+1}} \\ &= \eta_0 \oint_C \left(\partial_\xi^l(\varphi)\right)^2 d\xi + D_{16}\eta_0^{\frac{\mu+\frac{v}{2}-1}{\mu+\frac{v}{2}-1-2l}} \left(\oint_C \varphi^2 d\xi\right)^{\frac{vl-\mu-\frac{v}{2}+1}{2l-\mu-\frac{v}{2}+1}}. \qquad \square\end{aligned}$$

Proposition 3.4. *For the centro-affine heat flow, the following inequality holds*:

$$\frac{\partial}{\partial t}\oint_C \varphi_{n\xi}^2 d\xi \le D_{19}\left(E+E^{2n+5}\right). \tag{3.8}$$

Proof. Lemma 3.2 gives

$$\begin{aligned}\frac{\partial}{\partial t}\oint_C \varphi_{k\xi}^2 d\xi &= \oint_C 2\varphi_{k\xi}\varphi_{k\xi t} + \varphi_{k\xi}^2\frac{g_t}{g} d\xi \\ &= -\oint_C \varphi_{(k+2)\xi}^2 d\xi + \oint_C P_4^{2k+2}(\varphi)\,d\xi + \oint_C P_2^{2k+2}(\varphi)\,d\xi.\end{aligned}$$

Applying Proposition 3.3 to $P_4^{2k+2}(\varphi)$, and combining with $v=4$, $\mu=2k+2$, $l=k+2$, we arrive at

$$\oint_C P_4^{2k+2}(\varphi)\,d\xi \le \eta_1 \oint_C \varphi_{(k+2)\xi}^2 d\xi + D_{16}\eta_1^{-(2k+3)}\left(\oint_C \varphi^2 d\xi\right)^{2k+5}.$$

In addition, we also have

$$\oint_C P_2^{2k+2}(\varphi)\,d\xi \le \eta_2 \oint_C \varphi_{(k+2)\xi}^2 d\xi + D_{16}\eta_2^{-(k+1)}\oint_C \varphi^2 d\xi.$$

Let $\eta_1+\eta_2=1$, we get

$$\frac{d}{dt}\oint_C \varphi_{k\xi}^2 d\xi \le D_{17}\left(E+E^{2k+5}\right),$$

which means the inequality (3.8) holds on for $n=k$. When $n=k+1$,

$$\oint_C P_4^{2k+4}(\varphi)\,d\xi \le \eta_3 \oint_C \varphi_{(k+3)\xi}^2 d\xi + D_{16}\eta_3^{-(2k+5)}\left(\oint_C \varphi^2 d\xi\right)^{2k+7},$$

$$\oint_C P_2^{2k+4}(\varphi)\,d\xi \le \eta_4 \oint_C \varphi_{(k+3)\xi}^2 d\xi + D_{16}\eta_4^{-(k+2)}\oint_C \varphi^2 d\xi.$$

By the same way, we can see

$$\frac{\partial}{\partial t}\oint_C \varphi_{(k+1)\xi}^2 d\xi \le D_{18}(E+E^{2k+7}).$$

Therefore, according to the mathematical induction, we can get the conclusion. $\square$

4. Proof of Theorem 1.1

We start with the following lemmas and propositions which encourage us present several conclusions which play a key role to complete the final proof.

Lemma 4.1. *The infinite integral $\int_0^{+\infty} E(t)\,dt$ converges.*

Proof. Integrating (2.8) with respect to t, and putting $t=0$, $t=t$ leads to

$$\frac{1}{2}\int_0^t dt \oint_C \varphi_\xi^2(\xi,t)\,d\xi = L(t) - L(0).$$

When t goes to infinity, due to Proposition 3.1, we can easily check that

$$\int_0^{+\infty} dt \oint_C \varphi_\xi^2(\xi,t)\,d\xi \le L_0$$

holds for some constant L_0. Furthermore, with the Wirtinger inequality, we can show that

$$\int_0^{+\infty} dt \oint_C \varphi^2(\xi,t)\,d\xi \le L_1$$

for constant L_1.

From the above inequalities and the monotone bounded principle, we can gather that $\int_0^{+\infty} E(t)\,dt$ converges. □

Lemma 4.2. $\int_0^{+\infty} dt \oint_C \varphi_\xi^2(\xi,t)d\xi$ *converges.*

Proof. Using monotone bounded principle and Lemma 4.1 gives the conclusion. □

Proposition 4.3. *The energy $E(t)$ is uniformly bounded on the interval $[0,+\infty)$.*

Proof. Employing Lemma 4.1 and Cauchy convergence principle, it is easy to obtain that for arbitrary $\zeta_0>0$, there exists $j_0\ge 0$ such that when $j\ge j_0$, we have

$$\int_j^{j+1} E(t)\,dt \le \zeta_0.$$

Moreover, mean value theorem of integrals implies

$$E(t_j) = \int_j^{j+1} E(t)\,dt \le \zeta_0,$$

where $t_j\in[j,j+1]$. Considering the time interval $[t_j,t_j+2]$, it can be easily seen that

$$1+E^4(t)\le S_j$$

holds because of the boundedness for the continuous function $E(t)$, where S_j is a constant. Therefore, by (3.4),

$$\frac{\partial E(t)}{\partial t} \le D_4E(t)\left[1+E^4(t)\right] \le D_4S_jE(t).$$

Motivated by these results, for the case that $D_4 \le \frac{1}{4S_j}$, we reach

$$\frac{\partial E\left(t\right)}{\partial t} \le \frac{1}{4} E\left(t\right). \tag{4.1}$$

Straightforward checking shows

$$\begin{aligned} E(t) &= E\left(t_j\right) + E'\left(\xi_1\right)\left(t - t_j\right) \\ &\le E\left(t_j\right) + \frac{1}{2} E\left(\xi_1\right), \ \forall \xi_1 \in \left(t_j, t\right), \end{aligned}$$

$$\begin{aligned} E\left(t\right) &\le \frac{3}{2} E\left(t_j\right) + \frac{1}{2}\left[E\left(\xi_1\right) - E\left(t_j\right)\right] \\ &\le \frac{3}{2} E\left(t_j\right) + \frac{1}{2^2} E\left(\xi_2\right), \ \forall \xi_2 \in \left(t_j, \xi_1\right), \end{aligned}$$

$$E\left(t\right) \le \left(1 + \frac{1}{2} + \frac{1}{2^2}\right) E\left(t_j\right) + \frac{1}{2^3} E\left(\xi_3\right), \ \forall \xi_3 \in \left(t_j, \xi_2\right),$$

$$\cdots\cdots,$$

$$\begin{aligned} E(t) &\le \left(\sum_{k=0}^{n-1} \frac{1}{2^k}\right) E\left(t_j\right) + \frac{1}{2^n} E\left(\xi_n\right) \\ &\le 2E\left(t_j\right) + \frac{1}{2^n}\left(S_j - 1\right)^{\frac{1}{4}} \\ &< 2\zeta_0 + \frac{1}{2^n}(S_j - 1)^{\frac{1}{4}}, \ \forall \xi_n \in \left(t_j, \xi_{n-1}\right). \end{aligned}$$

On the other hand, when $D_4 \ge \frac{1}{4S_j}$, we hereby begin by fixing time t into $\left(t_j, \, t_j + \frac{1}{2D_4S_j}\right]$. Similar to the previous calculation process, we have

$$\begin{aligned} E\left(t\right) &\le E\left(t_j\right) + D_4 S_j E\left(\xi_1\right)\left(t - t_j\right) \\ &\le E\left(t_j\right) + \frac{1}{2} E\left(\xi_1\right), \ \forall \xi_1 \in \left(t_j, t\right), \end{aligned}$$

$$\begin{aligned} E\left(t\right) &\le \frac{3}{2} E\left(t_j\right) + \frac{1}{2}\left[E\left(\xi_1\right) - E\left(t_j\right)\right] \\ &\le \frac{3}{2} E\left(t_j\right) + \frac{1}{2} D_4 S_j \cdot E\left(\xi_2\right) \cdot \frac{1}{2D_4S_j} \\ &= \frac{3}{2} E\left(t_j\right) + \frac{1}{2^2} E\left(\xi_2\right), \ \forall \xi_2 \in \left(t_j, \xi_1\right), \end{aligned}$$

$$\cdots\cdots,$$

$$E\left(t\right) < 2\zeta_0 + \frac{1}{2^n}\left(S_j - 1\right)^{\frac{1}{4}} < \zeta_1.$$

Taking into account the next time interval $\left(t_j + \frac{1}{2D_4S_j}, \, t_j + \frac{1}{D_4S_j}\right]$, one can show that

$$E\left(t\right) \le 2\zeta_1 + \frac{1}{2^n} E\left(\xi_n\right) < \zeta_2.$$

We can go faster in the computation since the strategy is the same as above. Thus, a straightforward computation gives us

$$\begin{aligned} E\left(t\right) &\leq \left(\sum_{k=0}^{n-1}\frac{1}{2^k}\right)E\left(t_j+\frac{k-1}{2D_4S_j}\right)+\frac{1}{2^n}E\left(\xi_n\right) \\ &\leq 2\zeta_{k-1}+\frac{1}{2^n}E\left(\xi_n\right) \\ &< \zeta_k \end{aligned}$$

for any $t\in\left(t_j+\frac{k-1}{2D_4S_j},t_j+\frac{k}{2D_4S_j}\right]$, and $\zeta_k\geq 2\zeta_{k-1}\geq\cdots\geq 2^k\zeta_0$.

In summary, we shall employ the following statement: the energy function $E(t)$ is uniformly bounded on the interval $[0,+\infty)$ because of the arbitrariness of j. □

Proposition 4.4. $\lim\limits_{t\to+\infty}\oint_C\varphi^2(\xi,t)d\xi=0.$

Proof. Combining Proposition 4.3 and inequality (3.4) with the fact $\frac{\partial E}{\partial t}$ is uniformly bounded, we assume

$$\frac{\partial E}{\partial t}\leq M_1$$

for $t\in[0,+\infty)$, and M_1 is a positive constant. By the differential mean value theorem, $E(t)$ is uniformly continuous on $[0,+\infty)$.

Taking $\delta_1=\frac{\iota_1}{M_1}$, one can confirm that for any $\iota_1>0$, $t_1,t_2\in[0,+\infty)$, when $|t_1-t_2|<\delta_1$, we attain

$$|E\left(t_1\right)-E\left(t_2\right)|\leq M_1|t_1-t_2|.$$

At the same time, by Lemma 4.1, for above δ_1, there exists $t\geq 0$, for arbitrary $t_2>t_1\geq t$,

$$\int_{t_1}^{t_2}E\left(\tau\right)d\tau\leq\frac{\delta_1^2}{2}.$$

Hence,

$$\begin{aligned} E\left(t\right) &= \left|E\left(t\right)-\frac{2}{\delta_1}\int_{t_3}^{t_3+\frac{\delta_1}{2}}E\left(\tau\right)d\tau+\frac{2}{\delta_1}\int_{t_3}^{t_3+\frac{\delta_1}{2}}E\left(\tau\right)d\tau\right| \\ &\leq \frac{2}{\delta_1}\int_{t_3}^{t_3+\frac{\delta_1}{2}}|E\left(t\right)-E\left(\tau\right)|\,d\tau+\frac{2}{\delta_1}\left|\int_{t_3}^{t_3+\frac{\delta_1}{2}}E\left(\tau\right)d\tau\right| \\ &< \left(1+\frac{1}{M_1}\right)\iota_1. \end{aligned}$$

Taking the limit as $t\to+\infty$, we write

$$\lim_{t\to+\infty}\oint_C\varphi^2\left(\xi,t\right)d\xi=0.$$

□

Proposition 4.5. $\lim_{t\to+\infty}\oint_C \varphi_\xi^2(\xi,t)\,d\xi = 0.$

Proof. We use the same idea as we did in the preceding proposition. Let

$$F(t) = \oint_C \varphi_\xi^2(\xi,t)\,d\xi.$$

Taking $\frac{\partial F}{\partial t} \le M_2$ and $\delta_2 = \frac{\iota_2}{M_2}$, one knows

$$\begin{aligned} F(t) &\le \frac{1}{\delta_2}\int_{t_3}^{t_3+\delta_2} |F(t) - F(\tau)|\,d\tau + \frac{1}{\delta_2}\left|\int_{t_3}^{t_3+\delta_2} F(\tau)\,d\tau\right| \\ &\le \iota_2 + \frac{1}{\delta_2}\frac{\delta_2^2}{2} \\ &< \left(1 + \frac{1}{2M_2}\right)\iota_2, \end{aligned}$$

where M_2 is a positive constant. □

We now back to the main part.

Proof of Theorem 1.1. If $E(t)$ is uniformly bounded on $[0,T)$, by integrating (3.5) and (3.6), we obtain that it implies a uniform bound on the L^2-norm of the first and the second derivatives of the curvature φ with respect to the centro-affine arc length ξ. What is more, it follows from (3.8) and parabolic regularity that all spatial and time derivatives of φ are uniformly bounded. In fact, similar to [2], one can acquire a uniform H^k-bound on the curve with a prescribed k. Hence, the local existence of solution for the forth order flow equation (2.6) can be employed to extend the flow beyond T. Nevertheless, according to Proposition 4.3, $E(t)$ is uniformly bounded on $[0,+\infty)$. So the existence of solution can be proved. If we take T to be ω, it can be concluded that $E(t)$ must become unbounded as a finite ω is approached. When ω is finite and t is close to ω, which implies $E(t) > 1$, by integrating (3.4) from t to ω, we have

$$E(t) \ge \frac{1}{8D_4}(\omega - t)^{-\frac{1}{4}}.$$

This gives the desired lower bound for the blow-up rate. Finally, in view of Proposition 4.4, Proposition 4.5, combined with Section 6 in [11], we can obtain that the curvature φ smoothly converges to zero, namely that the curve C will eventually shrink to an ellipse with the passage of time. As stated above, our certification is complete. □

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