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CENTRAL LIMIT THEOREMS FOR CONDITIONALLY STRONG MIXING AND CONDITIONALLY STRICTLY STATIONARY SEQUENCES OF RANDOM VARIABLES

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Abstract. From the ordinary notion of upper-tail quantitle function, a new concept called conditionally upper-tail quantitle function given a σ-algebra is proposed. Some basic properties of this terminology and further properties of conditionally strictly stationary sequences are derived. By means of these properties, several conditional central limit theorems for a sequence of conditionally strong mixing and conditionally strictly stationary random variables are established, some of which are the conditional versions corresponding to earlier results under non-conditional case.

1. Introduction

Let $(\Omega, \mathcal{A}, P)$ be a probability space on which all random variables under consideration are defined. For a given sequence $\{X_n, n \geq 1\}$ of random variables and $1 \leq j \leq l < \infty$, let $\mathcal{A}_j^l$ and $\mathcal{A}_l^\infty$, respectively, denote the σ-algebra generated by $\{X_i, j \leq i \leq l\}$ and $\{X_i, i \geq l\}$. Define the maximum measure of dependence between $\mathcal{A}_1^{\cdot}$ and $\mathcal{A}_{n+\cdot}^\infty$ at a distance of n indices in the sense that

$$\alpha(n) = \sup_{k \geq 1} \sup_{A \in \mathcal{A}_1^k, B \in \mathcal{A}_{n+k}^\infty} |P(A \cap B) - P(A) P(B)|$$

and say the sequence $\{X_n\}$ is strong mixing if $\alpha(n) \to 0$ as $n \to \infty$.

The notion of strong mixing was proposed in 1956 by Rosenblatt [14] to distinguish from a weaker type of "mixing" used in ergodic theory. Since that time, strong mixing condition has possessed a position of considerable importance in probability theory because of its tractability in the derivation of asymptotic

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properties of various functions of sequences of dependent random variables and has been successfully applied in maximal moment inequalities [20], moment bounds [21], central limit theorems [6, 11], functional central limit theorems [9], laws of iterated logarithm [22], large deviations [2], nonparametric kernel estimation [5], order statistics [19], robust estimators and bootstrap method [13] and so on.

Let $\mathcal{F}$ be a sub-σ-algebra contained in $\mathcal{A}$. We usually regard $\mathcal{F}$ as information available, for example, it may be the collection $\{\Omega, W, W^c, \emptyset\}$, where W represents an event of particular importance such as a massive disaster resulting from an earthquake or a hurricane. For the sake of convenience, we denote by $P^{\mathcal{F}}(A)$ the conditional probability $P(A|\mathcal{F})$ for $A \in \mathcal{A}$. The notion of strong mixing was extended to conditional case by Prakasa Rao [12] in the following way. Define

$$\alpha_{\mathcal{F}}(n) = \sup_{k\geq 1} \sup_{A\in\mathcal{A}_1^k, B\in\mathcal{A}_{k+n}^\infty} \left|P^{\mathcal{F}}(A\cap B) - P^{\mathcal{F}}(A)\,P^{\mathcal{F}}(B)\right| \text{ a.s.} \tag{1.1}$$

and say the sequence $\{X_n\}$ is strong mixing given $\mathcal{F}$ ($\mathcal{F}$-strong mixing, in short) if $\alpha_{\mathcal{F}}(n) \to 0$ a.s. as $n \to \infty$.

As a trivial set-theoretic observation, the sequence $\{\alpha_{\mathcal{F}}(n), n \geq 1\}$ of conditionally strong mixing coefficients is nonincreasing almost surely. So, from now on, when a random sequence $\{X_n, n \geq 1\}$ is called to be $\mathcal{F}$-strong mixing with coefficients $\{\alpha_{\mathcal{F}}(n), n \geq 1\}$, it means, for every $A \in \mathcal{A}_1^k$, $B \in \mathcal{A}_{k+n}^\infty$, and $k \geq 1$,

$$\left|P^{\mathcal{F}}(A\cap B) - P^{\mathcal{F}}(A)\,P^{\mathcal{F}}(B)\right| \leq \alpha_{\mathcal{F}}(n) \downarrow 0 \text{ a.s.},$$

and this convention will be tacitly understood and used freely.

The essence behind $\mathcal{F}$-strong mixing condition is that past and distant future are asymptotically $\mathcal{F}$-independent. Of course, $\mathcal{F}$-strong mixing condition, respectively, comes down to the ordinary strong mixing condition providing $\mathcal{F} = \{\emptyset,\ \Omega\}$ and $\mathcal{F}$-independence providing $\alpha_{\mathcal{F}}(n) \equiv 0$.

Some concrete examples have been obtained in Yuan and Lei [24] to show that the strong mixing property does not imply the conditionally strong mixing property, and vice versa. Hence one does have to derive results under conditioning if there is a need even though the results and proofs of such results may be analogous to those under the non-conditioning setup.

In the past few years, a lot of efforts have been dedicated to the extension of independent/dependent random variables to conditional case and have achieved many meaningful results. For example, Christofides and Hadjikyriakou [4] for conditional convex order, Ordóñez Cabrera et al. [10] for conditionally negative quadrant dependence, Yuan et al. [23] for conditionally negative association, Yuan and Xie [26] for conditionally linearly negative quadrant dependence, Yuan and Yang [27] for conditional association, Wang and Hu [18] for conditional mean convergence theorems, Sood and Yağan [17] for inhomogeneous K-out graphs. In particular, Khovansky and Zhylyevskyy [7] suggested a modification of GMM and proved its consistency when such a shock affects the data, Bulinski [3] studied arrays with rows consisting of conditionally independent random variables with respect to certain σ-algebras, Sheikhi et al. [16] looked at the perturbations of copulas via modification of the random variables under conditional independence structure and Lee and Song [8] established stable limit theorems for empirical processes under conditional neighborhood dependence. All of these outstanding achievements continue to inspire our interest in conditional independence/dependence.

It should be noted that the development of conditional independence/dependence is far from its maturity. One of the main reasons may be that rich theory and strong application have not yet been constructed up on a large scale due to starting the research in this area very late. In order not to lose ourselves in a too general conditioning setup, taking into account the basic work we did earlier in [24], the current paper is mainly focused on conditionally strong mixing random sequences.

The remainder of the paper is organized as follows. The definition and properties of conditionally upper-tail quantile function are displayed in Section 2 and the definition and properties of conditionally strict stationarity are established in Section 3. With the help of these properties, several conditional central limit theorems are developed in Section 4.

2. Conditionally upper-tail quantile function

Consider a random variable X and define its upper-tail quantile function $\mathcal{Q}_X^{\mathcal{F}} : \Omega \times (0,1) \to \mathbb{R}$ with respect to $\mathcal{F}$ ($\mathcal{F}$-upper-tail quantile function, in short) as follows:

$$\mathcal{Q}_X^{\mathcal{F}}(\omega, u) = \inf\left\{x \in \mathbb{R} : P^{\mathcal{F}}(X > x)(\omega) \le u\right\}, \ \omega \in \Omega, \ u \in (0,1).$$

Evidently, $\mathcal{F}$-upper-tail quantile function $\mathcal{Q}_X^{\mathcal{F}} : \Omega \times (0,1) \to \mathbb{R}$ reduces to the ordinary upper-tail quantile function $\mathcal{Q}_X : (0,1) \to \mathbb{R}$ if $\mathcal{F} = \{\emptyset, \Omega\}$. Let

$$A(\omega, u) = \left\{x \in \mathbb{R} : P^{\mathcal{F}}(X > x)(\omega) \le u\right\}.$$

Then $\mathcal{Q}_X^{\mathcal{F}}(\omega, u) = \inf A(\omega, u)$. For any $u \in (0,1)$ and almost all $\omega \in \Omega$, the set $A(\omega, u)$ not only is nonempty but also has a lower bound, the former because $\lim_{x\to\infty} P^{\mathcal{F}}(X > x) = 0$ a.s. and the latter because $\lim_{x\to-\infty} P^{\mathcal{F}}(X > x) = 1$ a.s., so that the function $u \mapsto \mathcal{Q}_X^{\mathcal{F}}(\cdot, u)$ is almost surely real-valued.

For any $u \in (0,1)$ and $x \in \mathbb{R}$, we have these easily proved claims:

(i) $A(\cdot, u)$ is almost surely a closed set;
(ii) $P^{\mathcal{F}}\left(X > \mathcal{Q}_X^{\mathcal{F}}(\cdot, u)\right)(\cdot) \le u$ a.s.;
(iii) $\left(\mathcal{Q}_X^{\mathcal{F}}(\cdot, u) \le x\right) = \left(P^{\mathcal{F}}(X > x)(\cdot) \le u\right)$ a.s.

It should be mentioned that claim (iii) above indicates that $\mathcal{Q}_X^{\mathcal{F}}(\cdot, u)$ is almost surely measurable for any $u \in (0,1)$. In the rest of this paper, the underlying probability space $(\Omega, \mathcal{A}, P)$ is tacitly assumed to be complete, so that $\mathcal{Q}_X^{\mathcal{F}}(\cdot, u)$ is a random variable. We omit the argument ω and denote $\mathcal{Q}_X^{\mathcal{F}}(\omega, u)$ by $\mathcal{Q}_X^{\mathcal{F}}(u)$ for simplicity if there is no confusion.

Proposition 2.1. *Assume that X is a nonnegative random variable and $A \in \mathcal{F}$. Then for any $u \in (0,1)$,*

$$\mathcal{Q}_{XI_A}^{\mathcal{F}}(u) = \mathcal{Q}_X^{\mathcal{F}}(u)\, I_A \text{ a.s.},$$

where I_A is the indicator function of the set A.

Proof. Obviously, $(XI_A > x) = (X > x) \cap A$ for any $x \in \mathbb{R}^+$, which yields

$$P^{\mathcal{F}}(XI_A > x) = P^{\mathcal{F}}(X > x)\, I_A \text{ a.s.},$$

and therefore

$$\begin{aligned} \mathcal{Q}_{XI_A}^{\mathcal{F}}(u) &= \inf\left\{x \in \mathbb{R}^+ : P^{\mathcal{F}}(XI_A > x) \le u\right\} \\ &= \inf\left\{x \in \mathbb{R}^+ : P^{\mathcal{F}}(X > x)\, I_A \le u\right\} \\ &= \inf\left\{x \in \mathbb{R}^+ : P^{\mathcal{F}}(X > x) \le u\right\} \cdot I_A \\ &= \mathcal{Q}_X^{\mathcal{F}}(u)\, I_A, \end{aligned}$$

which is just the desired result. □

There is no difficulty proving the following proposition by virtue of Proposition 2.1 and the definition of conditionally upper-tail quantile function.

Proposition 2.2. *Assume that X and Y are two random variables and $N \in \mathcal{F}$ is a P-null set. Then the following two statements are equivalent*:

(i) $P^{\mathcal{F}}(X > x)\, I_{N^c} \le P^{\mathcal{F}}(Y > x)\, I_{N^c}$ *for all* $x \in \mathbb{R}$;
(ii) $\mathcal{Q}_X^{\mathcal{F}}(u)\, I_{N^c} \le \mathcal{Q}_Y^{\mathcal{F}}(u)\, I_{N^c}$ *for all* $u \in (0,1)$.

Let us comment on statement (i) in Proposition 2.2. How to make that statement come true? For example, $X \le Y$ a.s. is a sufficient condition for it. To understand this implication, for every $r \in \mathbb{Q}$, the set of rational numbers, choose one version $P^{\mathcal{F}}(X > r)(\omega)$ of $P^{\mathcal{F}}(X > r)$ and one version $P^{\mathcal{F}}(Y > r)(\omega)$ of $P^{\mathcal{F}}(Y > r)$ separately. Since $\mathbb{Q}$ is countable set, there exists a P-null set $N \in \mathcal{F}$ such that $P^{\mathcal{F}}(X > r) \le P^{\mathcal{F}}(Y > r)$ on N^c for all $r \in \mathbb{Q}$. We next define

$$P^{\mathcal{F}}(X > x)(\omega) = \begin{cases} P(X > x), & \omega \in N, \\ \lim\limits_{r \in \mathbb{Q}, r \downarrow x} P^{\mathcal{F}}(X > r)(\omega), & \omega \in N^{\mathrm{c}} \end{cases}$$

and

$$P^{\mathcal{F}}(Y > x)(\omega) = \begin{cases} P(Y > x), & \omega \in N, \\ \lim\limits_{r \in \mathbb{Q}, r \downarrow x} P^{\mathcal{F}}(Y > r)(\omega), & \omega \in N^{\mathrm{c}}, \end{cases}$$

then for all $x \in \mathbb{R}$,

$$P^{\mathcal{F}}(X > x)\, I_{N^c} \le P^{\mathcal{F}}(Y > x)\, I_{N^c},$$

which is just statement (i) in Proposition 2.2.

For the probability distribution of a random variable, we can provide an alternative expression in terms of its $\mathcal{F}$-upper-tail quantile function.

Proposition 2.3. *Assume that X is any random variable and U is a random variable that is uniformly distributed on the unit interval $[0,1]$. Then the random variable $\mathcal{Q}_X^{\mathcal{F}}(\omega, U(\omega'))$ defined on $(\Omega\times\Omega, \mathcal{A}\times\mathcal{A}, P\times P)$ has the same distribution as the random variable X itself, that is,*

$$(P\times P)\circ\left[\mathcal{Q}_X^{\mathcal{F}}(\cdot, U(\cdot))\right]^{-1} = P\circ X^{-1}.$$

Proof. For any $x\in\mathbb{R}$, an appeal to Fubini's theorem gets that

$$\begin{aligned}
&(P\times P)\left\{(\omega,\omega')\in\Omega\times\Omega : \mathcal{Q}_X^{\mathcal{F}}(\omega, U(\omega'))\le x\right\}\\
&= \iint_{\Omega\times\Omega} I_{\left\{\mathcal{Q}_X^{\mathcal{F}}(\omega,U(\omega'))\le x\right\}}(\omega,\omega')\,(P\times P)(d\omega, d\omega')\\
&= \int_\Omega P(d\omega)\int_\Omega I_{\left\{\mathcal{Q}_X^{\mathcal{F}}(\omega,U(\omega'))\le x\right\}}(\omega')\,P(d\omega')\\
&= \int_\Omega P(d\omega)\int_\Omega I_{\{U(\omega')\ge P^{\mathcal{F}}\{X>x\}(\omega)\}}(\omega')\,P(d\omega')\\
&= \int_\Omega \left[1-P^{\mathcal{F}}(X>x)(\omega)\right]P(d\omega)\\
&= \int_\Omega P^{\mathcal{F}}(X\le x)(\omega)\,P(d\omega)\\
&= P(X\le x),
\end{aligned}$$

which leads to the desired formula. □

Related to the alternative expression of probability distribution of a random variable is the corresponding expression of moments.

Proposition 2.4. *Assume that X is a random variable and $p>0$. Then*

$$E^{\mathcal{F}}|X|^p = \int_0^1\left[\mathcal{Q}_{|X|}^{\mathcal{F}}(u)\right]^p du \ \ a.s., \tag{2.1}$$

where $E^{\mathcal{F}}\xi := E(\xi\,|\mathcal{F})$ is the conditional expectation (when it exists) of a random variable ξ with respect to the sub-σ-algebra $\mathcal{F}$.

Proof. Let $f:\mathbb{R}\to\mathbb{R}$ be a Borel function such that either $E|f(X)|<\infty$ or f is nonnegative (possible with $Ef(X)=\infty$). Then Proposition 2.3 guarantees that

$$Ef(X) = \int_\Omega dP(\omega)\int_0^1 f\left(\mathcal{Q}_X^{\mathcal{F}}(\omega,u)\right)du,$$

which together with Proposition 2.1 yields for any $p>0$ and $A\in\mathcal{F}$ that

$$\begin{aligned}
E\left(|X|^p I_A\right) &= \int_\Omega P(d\omega)\int_0^1\left[\mathcal{Q}_{|X|I_A}^{\mathcal{F}}(\omega,u)\right]^p du\\
&= \int_A P(d\omega)\int_0^1\left[\mathcal{Q}_{|X|}^{\mathcal{F}}(\omega,u)\right]^p du.
\end{aligned}$$

This means that (2.1) holds. □

Lemma 2.5. *Assume that* U *and* V *are two nonnegative random variables, and* ξ *is a random variable taking its values in the interval* $[0,1]$. *Then*

$$\int_0^\infty \int_0^\infty \min\left\{\xi, P^{\mathcal{F}}(U>s), P^{\mathcal{F}}(V>t)\right\} dsdt = \int_0^\xi \mathcal{Q}_U^{\mathcal{F}}(u)\, \mathcal{Q}_V^{\mathcal{F}}(u)\, du.$$

Proof. For each $\omega \in \Omega$, define the set $A(\omega)$ in $\mathbb{R}^3$ as follows:

$$A(\omega) = \big\{(u,s,t) \in (0,1) \times [0,\infty) \times [0,\infty) : u < \min\left\{\xi(\omega), P^{\mathcal{F}}(U>s)(\omega),\right.$$
$$\left. P^{\mathcal{F}}(V>t)(\omega)\right\}\big\},$$

which can be rewritten as

$$A(\omega) = \big\{(u,s,t) \in (0,1) \times [0,\infty) \times [0,\infty) : u < \xi(\omega), s < \mathcal{Q}_U^{\mathcal{F}}(\omega,u),$$
$$t < \mathcal{Q}_V^{\mathcal{F}}(\omega,u)\big\}.$$

Hence one has that

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \min\left\{\xi(\omega), P^{\mathcal{F}}(U>s)(\omega), P^{\mathcal{F}}(V>t)(\omega)\right\} dsdt \\
&= \int_0^\infty ds \int_0^\infty dt \int_0^{\min\left\{\xi(\omega), P^{\mathcal{F}}(U>s)(\omega), P^{\mathcal{F}}(V>t)(\omega)\right\}} du \\
&= \iiint_{(u,s,t)\in A(\omega)} dsdtdu \\
&= \int_0^{\xi(\omega)} dp \int_0^{\mathcal{Q}_U^{\mathcal{F}}(\omega,u)} ds \int_0^{\mathcal{Q}_V^{\mathcal{F}}(\omega,u)} dt \\
&= \int_0^{\xi(\omega)} \mathcal{Q}_U^{\mathcal{F}}(\omega,u)(\omega)\, \mathcal{Q}_V^{\mathcal{F}}(\omega,u)\, du,
\end{aligned}$$

which concludes the proof of Lemma 2.5. □

For an $\mathcal{F}$-strong mixing sequence, we can provide an $\mathcal{F}$-covariance inequality in terms of $\mathcal{F}$-upper-tail quantile functions.

Theorem 2.6. *Assume that* $\{X_n, n \geq 1\}$ *is a sequence of* $\mathcal{F}$*-strong mixing random variables with mixing coefficients* $\{\alpha_{\mathcal{F}}(n)\}$, *and* Y *and* Z *are, respectively,* $\mathcal{A}_1^k$*-measurable and* $\mathcal{A}_{k+n}^\infty$*-measurable random variables. If* $E^{\mathcal{F}}|Y| < \infty$ *a.s.,* $E^{\mathcal{F}}|Z| < \infty$ *a.s. and* $\int_0^1 \mathcal{Q}_{|Y|}^{\mathcal{F}}(u)\, \mathcal{Q}_{|Z|}^{\mathcal{F}}(u)\, du < \infty$ *a.s., then*

$$\left|E^{\mathcal{F}}YZ - E^{\mathcal{F}}Y \cdot E^{\mathcal{F}}Z\right| \leq 4\int_0^{\alpha_{\mathcal{F}}(n)} \mathcal{Q}_{|Y|}^{\mathcal{F}}(u)\, \mathcal{Q}_{|Z|}^{\mathcal{F}}(u)\, du \ \text{a.s.,} \tag{2.2}$$

$$E^{\mathcal{F}}|YZ| < \infty \ \text{a.s.} \tag{2.3}$$

Proof. Let U and V be, respectively, any two nonnegative $\mathcal{A}_1^k$-measurable and $\mathcal{A}_{k+n}^\infty$-measurable random variables. Then by Proposition 4.3 of Roussas [15] and Lemma 2.5,

$$\left|E^{\mathcal{F}}UV - E^{\mathcal{F}}U \cdot E^{\mathcal{F}}V\right| \tag{2.4}$$

$$
\begin{aligned}
&= \left| \int_0^\infty \int_0^\infty P^{\mathcal{F}} \left(U > s, V > t \right) - P^{\mathcal{F}} \left(U > s \right) P^{\mathcal{F}} \left(V > t \right) dsdt \right| \\
&\leq \int_0^\infty \int_0^\infty \left| P^{\mathcal{F}} \left(U > s, V > t \right) - P^{\mathcal{F}} \left(U > s \right) P^{\mathcal{F}} \left(V > t \right) \right| dsdt \\
&\leq \int_0^\infty \int_0^\infty \min \left\{ \alpha_{\mathcal{F}} \left(n \right), P^{\mathcal{F}} \left(U > s \right), P^{\mathcal{F}} \left(V > t \right) \right\} dsdt \\
&\leq \int_0^{\alpha_{\mathcal{F}}(n)} \mathcal{Q}_U^{\mathcal{F}} \left(u \right) \mathcal{Q}_V^{\mathcal{F}} \left(u \right) du \text{ a.s.}
\end{aligned}
$$

Setting $U = Y^+, V = Z^+$ in (2.4) and then using Proposition 2.2 to get

$$
\begin{aligned}
\left| E^{\mathcal{F}} Y^+ Z^+ - E^{\mathcal{F}} Y^+ \cdot E^{\mathcal{F}} Z^+ \right| &\leq \int_0^{\alpha_{\mathcal{F}}(n)} \mathcal{Q}_{Y^+}^{\mathcal{F}} \left(u \right) \mathcal{Q}_{Z^+}^{\mathcal{F}} \left(u \right) du \\
&\leq \int_0^{\alpha_{\mathcal{F}}(n)} \mathcal{Q}_{|Y|}^{\mathcal{F}} \left(u \right) \mathcal{Q}_{|Z|}^{\mathcal{F}} \left(u \right) du \text{ a.s.,}
\end{aligned}
$$

and analogous statements hold for Y^+ and Z^-, Y^- and Z^+, and Y^- and Z^-. Hence

$$
\begin{aligned}
&\left| E^{\mathcal{F}} YZ - E^{\mathcal{F}} Y \cdot E^{\mathcal{F}} Z \right| \\
\leq& \left| E^{\mathcal{F}} Y^+ Z^+ - E^{\mathcal{F}} Y^+ \cdot E^{\mathcal{F}} Z^+ \right| + \left| E^{\mathcal{F}} Y^+ Z^- - E^{\mathcal{F}} Y^+ \cdot E^{\mathcal{F}} Z^- \right| \\
&+ \left| E^{\mathcal{F}} Y^- Z^+ - E^{\mathcal{F}} Y^- \cdot E^{\mathcal{F}} Z^+ \right| + \left| E^{\mathcal{F}} Y^- Z^- - E^{\mathcal{F}} Y^- \cdot E^{\mathcal{F}} Z^- \right| \\
\leq& \, 4 \int_0^{\alpha_{\mathcal{F}}(n)} \mathcal{Q}_{|Y|}^{\mathcal{F}} \left(u \right) \mathcal{Q}_{|Z|}^{\mathcal{F}} \left(u \right) du \text{ a.s.}
\end{aligned}
$$

This completes the proof of (2.2). As for (2.3), employing Theorem 4.1 in [15] and using an analogous argument appeared in (2.4), one has that

$$
\begin{aligned}
&E^{\mathcal{F}} \left| YZ \right| \\
=& \, E^{\mathcal{F}} \int_0^\infty \int_0^\infty I \left(|Y| > s, |Z| > t \right) dsdt \\
=& \int_0^\infty \int_0^\infty P^{\mathcal{F}} \left(|Y| > s, |Z| > t \right) dsdt \\
\leq& \int_0^\infty \int_0^\infty \left| P^{\mathcal{F}} \left(|Y| > s, |Z| > t \right) - P^{\mathcal{F}} \left(|Y| > s \right) P^{\mathcal{F}} \left(|Z| > t \right) \right| dsdt \\
&+ \int_0^\infty \int_0^\infty P^{\mathcal{F}} \left(|Y| > s \right) P^{\mathcal{F}} \left(|Z| > t \right) dsdt \\
\leq& \int_0^\infty \int_0^\infty \min \left\{ \alpha_{\mathcal{F}} \left(n \right), P^{\mathcal{F}} \left(|Y| > s \right), P^{\mathcal{F}} \left(|Z| > t \right) \right\} dsdt + E^{\mathcal{F}} |Y| \cdot E^{\mathcal{F}} |Z| \\
\leq& \int_0^{\alpha_{\mathcal{F}}(n)} \mathcal{Q}_{|Y|}^{\mathcal{F}} \left(u \right) \mathcal{Q}_{|Z|}^{\mathcal{F}} \left(u \right) du + E^{\mathcal{F}} |Y| \cdot E^{\mathcal{F}} |Z| \\
<& \, \infty \text{ a.s.,}
\end{aligned}
$$

thereby proving (2.3). □

With the aid of this result the following $\mathcal{F}$-covariance inequality is within easy reach.

Corollary 2.7. *Assume that $\{X_n, n \geq 1\}$ is a sequence of $\mathcal{F}$-strong mixing random variables with mixing coefficients $\{\alpha_{\mathcal{F}}(n)\}$, and Y and Z are, respectively, $\mathcal{A}_1^k$-measurable and $\mathcal{A}_{k+n}^{\infty}$-measurable random variables. If*

$$E^{\mathcal{F}}|Y|^p < \infty \text{ a.s. and } E^{\mathcal{F}}|Z|^q < \infty \text{ a.s. for } p, q, r > 1 \text{ with } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,$$

then

$$\left|E^{\mathcal{F}}YZ - E^{\mathcal{F}}Y \cdot E^{\mathcal{F}}Z\right| \leq 4\alpha_{\mathcal{F}}^{1/r}(n)\left(E^{\mathcal{F}}|Y|^p\right)^{1/p}\left(E^{\mathcal{F}}|Z|^q\right)^{1/q} \text{ a.s.} \tag{2.5}$$

Proof. By Theorem 2.6, Hölder's inequality and (2.1) in turn, one has that

$$\begin{aligned}
&\left|E^{\mathcal{F}}YZ - E^{\mathcal{F}}Y \cdot E^{\mathcal{F}}Z\right| \\
&\leq 4\int_0^{\alpha_{\mathcal{F}}(n)} \mathcal{Q}_{|Y|}^{\mathcal{F}}(u)\,\mathcal{Q}_{|Z|}^{\mathcal{F}}(u)\,du \\
&= 4\int_0^1 \mathcal{Q}_{|Y|}^{\mathcal{F}}(u)\,\mathcal{Q}_{|Z|}^{\mathcal{F}}(u)\,I_{(0,\alpha_{\mathcal{F}}(n))}(u)du \\
&\leq 4\left[\int_0^1 \left[\mathcal{Q}_{|Y|}^{\mathcal{F}}(u)\right]^p du\right]^{1/p}\left[\int_0^1 \left[\mathcal{Q}_{|Z|}^{\mathcal{F}}(u)\right]^q du\right]^{1/q}\left[\int_0^1 I_{(0,\alpha_{\mathcal{F}}(n))}(u)du\right]^{1/r} \\
&\leq 4\alpha_{\mathcal{F}}^{1/r}(n)\left(E^{\mathcal{F}}|Y|^p\right)^{1/p}\left(E^{\mathcal{F}}|Z|^q\right)^{1/q} \text{ a.s.}
\end{aligned}$$

This completes the proof of (2.5). □

The following generalization of Corollary 2.7 to multivariate random variables is the basis of Corollary 2.9 below.

Corollary 2.8. *Assume that $\{X_n, n \geq 1\}$ is a sequence of $\mathcal{F}$-strong mixing random variables with mixing coefficients $\{\alpha_{\mathcal{F}}(n)\}$ and assume that integers s_j, t_j, $j = 1, 2, \ldots, n$ satisfy*

$$1 = s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n \text{ with } s_{j+1} - t_j \geq \tau,\ j = 1, 2, \ldots, n-1.$$

If Y_j is $\mathcal{A}_{s_j}^{t_j}$-measurable random variable such that

$$E^{\mathcal{F}}|Y_j|^{p_j} < \infty \text{ a.s. with } p_j > 1,\ j = 1, 2, \ldots, n \text{ and } \frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{r_n} < 1,$$

then

$$\left|E^{\mathcal{F}}\prod_{j=1}^n Y_j - \prod_{j=1}^n E^{\mathcal{F}}Y_j\right| \leq 4(n-1)\,\alpha_{\mathcal{F}}^{1-1/r_n}(\tau)\prod_{j=1}^n\left(E^{\mathcal{F}}|Y_j|^{p_j}\right)^{1/p_j} \text{ a.s.} \tag{2.6}$$

Proof. The desired result holds for $n = 2$ by means of Corollary 2.7. It follows that

$$\left| E^{\mathcal{F}} \prod_{j=1}^{n} Y_j - \prod_{j=1}^{n} E^{\mathcal{F}} Y_j \right| \leq \left| E^{\mathcal{F}} \left[\left(\prod_{j=1}^{n-1} Y_j \right) Y_n \right] - E^{\mathcal{F}} \prod_{j=1}^{n-1} Y_j \cdot E^{\mathcal{F}} Y_n \right| + E^{\mathcal{F}} \left| Y_n \right| \left| E^{\mathcal{F}} \prod_{j=1}^{n-1} Y_j - \prod_{j=1}^{n-1} E^{\mathcal{F}} Y_j \right|. \tag{2.7}$$

Assuming inequality (2.6) to be true for $n-1$, from this induction hypothesis one has that

$$\left| E^{\mathcal{F}} \prod_{j=1}^{n-1} Y_j - \prod_{j=1}^{n-1} E^{\mathcal{F}} Y_j \right| \leq 4\,(n-2)\,\alpha_{\mathcal{F}}^{1-1/r_{n-1}}\ (\tau) \prod_{j=1}^{n-1} \left(E^{\mathcal{F}} |Y_j|^{p_j} \right)^{1/p_j},$$

where $1/r_{n-1}\ = 1/p_1 + \cdots + 1/p_{n-1}$. By virtue of the estimates

$$E^{\mathcal{F}} \left| Y_n \right| \leq \left(E^{\mathcal{F}} |Y_n|^{p_n} \right)^{1/p_n}$$

and

$$\alpha_{\mathcal{F}}^{1-1/r_{n-1}}\ (\tau) \leq \alpha_{\mathcal{F}}^{1-1/r_n}\ (\tau)\,,$$

one has that

$$\begin{aligned} E^{\mathcal{F}} \left| Y_n \right| & \left| E^{\mathcal{F}} \prod_{j=1}^{n-1} Y_j - \prod_{j=1}^{n-1} E^{\mathcal{F}} Y_j \right| \\ & \leq 4\,(n-2)\,\alpha_{\mathcal{F}}^{1-1/r_n}\ (\tau) \prod_{j=1}^{n} \left(E^{\mathcal{F}} |Y_j|^{p_j} \right)^{1/p_j}. \end{aligned} \tag{2.8}$$

Next, applying Corollary 2.7 with $p = r_{n-1}$ and $q = p_n$, one has that

$$\begin{aligned} & \left| E^{\mathcal{F}} \left[\left(\prod_{j=1}^{n-1} Y_j \right) Y_n \right] - E^{\mathcal{F}} \prod_{j=1}^{n-1} Y_j \cdot E^{\mathcal{F}} Y_n \right| \\ & \leq 4\alpha_{\mathcal{F}}^{1-1/r_{n-1}-1/p_n}\ (\tau) \left(E^{\mathcal{F}} \left| \prod_{j=1}^{n-1} Y_j \right|^{r_{n-1}} \right)^{1/r_{n-1}} \left(E^{\mathcal{F}} |Y_n|^{p_n} \right)^{1/p_n} \\ & = 4\alpha_{\mathcal{F}}^{1-1/r_n}\ (\tau) \left(E^{\mathcal{F}} \left| \prod_{j=1}^{n-1} Y_j \right|^{r_{n-1}} \right)^{1/r_{n-1}} \left(E^{\mathcal{F}} |Y_n|^{p_n} \right)^{1/p_n}. \end{aligned} \tag{2.9}$$

Set $q_j = p_j / r_{n-1}$, $j = 1, 2, \ldots, n-1$, so that $1/q_1 + \cdots + 1/q_{n-1} = 1$. Then

$$E^{\mathcal{F}} \left| \prod_{j=1}^{n-1} Y_j \right|^{r_{n-1}} \leq \prod_{j=1}^{n-1} \left(E^{\mathcal{F}} |Y_j|^{r_{n-1} q_j} \right)^{1/q_j} = \prod_{j=1}^{n-1} \left(E^{\mathcal{F}} |Y_j|^{p_j} \right)^{1/q_j},$$

and therefore

$$\left(E^{\mathcal{F}}\left|\prod_{j=1}^{n-1} Y_j\right|^{r_{n-1}}\right)^{1/r_{n-1}} \leq \prod_{j=1}^{n-1}\left(E^{\mathcal{F}}|Y_j|^{p_j}\right)^{1/p_j}.$$

Inserting the last inequality into (2.9), we get

$$\left|E^{\mathcal{F}}\left[\left(\prod_{j=1}^{n-1} Y_j\right) Y_n\right] - E^{\mathcal{F}}\prod_{j=1}^{n-1} Y_j \cdot E^{\mathcal{F}} Y_n\right| \leq 4\alpha_{\mathcal{F}}^{1-1/r_n}(\tau)\prod_{j=1}^{n}\left(E^{\mathcal{F}}|Y_j|^{p_j}\right)^{1/p_j}. \tag{2.10}$$

Inserting (2.8) and (2.10) into (2.7) we obtain (2.6). □

Corollaries 2.7 and 2.8 can be further generalized to complex-valued random variables case, which will be applied in the proof of Theorem 4.3 below.

Corollary 2.9. *If Y and Z are complex-valued random variables, then* (2.5) *and* (2.6)*, respectively, turn into*

$$\left|E^{\mathcal{F}}YZ - E^{\mathcal{F}}Y \cdot E^{\mathcal{F}}Z\right| \leq 16\alpha_{\mathcal{F}}^{1/r}(n)\left(E^{\mathcal{F}}|Y|^p\right)^{1/p}\left(E^{\mathcal{F}}|Z|^q\right)^{1/q} \quad a.s.$$

and

$$\left|E^{\mathcal{F}}\prod_{j=1}^{n} Y_j - \prod_{j=1}^{n} E^{\mathcal{F}} Y_j\right| \leq 16(n-1)\alpha_{\mathcal{F}}^{1-1/r_n}(\tau)\prod_{j=1}^{n}\left(E^{\mathcal{F}}|Y_j|^{p_j}\right)^{1/p_j} \quad a.s.$$

3. Conditionally strict stationarity

The central limit theorem in Rosenblatt [14] was not restricted to strictly stationary sequences, but it evolved later on into a certain "basic" or "fundamental" form in many cases. Inspired by the above-mentioned evolutionary process, it is necessary for us to employ conditionally strict stationarity. A sequence $\{X_n,\ n \geq 1\}$ of random variables is called to be $\mathcal{F}$-strictly stationary if for all $1 \leq n_1 < n_2 < \cdots < n_k < \infty$ and $r \geq 1$, the joint distribution of $(X_{n_1}, X_{n_2}, \ldots, X_{n_k})$ conditioned on $\mathcal{F}$ is the same as that of $(X_{n_1+r}, X_{n_2+r}, \ldots, X_{n_k+r})$ conditioned on $\mathcal{F}$ almost surely.

In the case where the sequence $\{X_n,\ n \geq 1\}$ of random variables is $\mathcal{F}$-strictly stationary, the mixing coefficient defined in (1.1) can be put in a simpler form:

$$\alpha_{\mathcal{F}}(n) = \sup_{A \in \mathcal{A}_1^1, B \in \mathcal{A}_{n+1}^{\infty}}\left|P^{\mathcal{F}}(A \cap B) - P^{\mathcal{F}}(A)P^{\mathcal{F}}(B)\right| \text{ a.s.}$$

We establish the first proposition on conditionally strictly stationary sequence, which will be used frequently.

Proposition 3.1. *Assume that $\{X_n, n \geq 1\}$ is an $\mathcal{F}$-strictly stationary sequence of random variables with $E^{\mathcal{F}} X_1 = 0$ a.s. and $E^{\mathcal{F}} X_1^2 < \infty$ a.s. As usual, their partial sums are denoted by $S_n = \sum_{k=1}^{n} X_k,\ n \geq 1$.*

(i) *For each $n \geq 1$,*

$$E^{\mathcal{F}} S_n^2 = n E^{\mathcal{F}} X_1^2 + 2 \sum_{k=2}^{n} (n-k+1)\, E^{\mathcal{F}} X_1 X_k.$$

(ii) *If $E^{\mathcal{F}} X_1 X_n \to 0$ a.s. as $n \to \infty$, then $n^{-2} E^{\mathcal{F}} S_n^2 \to 0$ a.s. as $n \to \infty$.*

(iii) *If $\sum_{n=2}^{\infty} \left| E^{\mathcal{F}} X_1 X_n \right| < \infty$ a.s., then for every $n \geq 2$,*

$$n^{-1} E^{\mathcal{F}} S_n^2 \leq E^{\mathcal{F}} X_1^2 + 2 \sum_{n=2}^{\infty} \left| E^{\mathcal{F}} X_1 X_n \right|.$$

(iv) *If $\sum_{n=2}^{\infty} \left| E^{\mathcal{F}} X_1 X_n \right| < \infty$ a.s., then*

$$\sigma_{\mathcal{F}}^2 := E^{\mathcal{F}} X_1^2 + 2 \sum_{n=2}^{\infty} E^{\mathcal{F}} X_1 X_n \tag{3.1}$$

exists in $[0, \infty)$ almost surely, and one has that

$$\lim_{n \to \infty} n^{-1} E^{\mathcal{F}} S_n^2 = \sigma_{\mathcal{F}}^2 \ \text{a.s.}$$

(v) *If $\sum_{n=2}^{\infty} n \left| E^{\mathcal{F}} X_1 X_n \right| < \infty$ a.s. and $E^{\mathcal{F}} S_n^2 \to \infty$ a.s. as $n \to \infty$, then the random variable $\sigma_{\mathcal{F}}^2$ defined in (3.1) satisfies $\sigma_{\mathcal{F}}^2 > 0$ almost surely.*

Proof. Proof of part (i) is easy, and parts (ii) and (iii) follow quickly from (i). For each $n \geq 1$, one has by (i) that

$$n^{-1} E^{\mathcal{F}} S_n^2 = E^{\mathcal{F}} X_1^2 + 2 \sum_{k=2}^{n} \left(1 - \frac{k-1}{n}\right) E^{\mathcal{F}} X_1 X_k,$$

which together with dominated convergence theorem yields

$$\lim_{n \to \infty} n^{-1} E^{\mathcal{F}} S_n^2 = E^{\mathcal{F}} X_1^2 + 2 \sum_{k=2}^{\infty} E^{\mathcal{F}} X_1 X_k. \tag{3.2}$$

Of course the left-hand side of (3.2) is nonnegative almost surely, and therefore so is the right-hand side. This completes the proof of (iv).

By (i) and (iv), one has the estimates

$$\begin{aligned} \left| n\sigma_{\mathcal{F}}^2 - E^{\mathcal{F}} S_n^2 \right| &= 2 \left| \sum_{k=2}^{n} (k-1)\, E^{\mathcal{F}} X_1 X_k + n \sum_{k=n+1}^{\infty} E^{\mathcal{F}} X_1 X_k \right| \\ &\leq 2 \sum_{n=2}^{\infty} n \left| E^{\mathcal{F}} X_1 X_n \right| < \infty \ \text{a.s.} \end{aligned}$$

Hence, if $\sigma_{\mathcal{F}}^2 = 0$, then $E^{\mathcal{F}} S_n^2 < \infty$, $n \geq 1$, this is in contradiction with assumption $E^{\mathcal{F}} S_n^2 \to \infty$ a.s. as $n \to \infty$. Thus (v) holds. $\square$

For a sequence $\{X_n, n \geq 1\}$ of random variables, the nth $\mathcal{F}$-variance of its partial sums will be denoted by $\sigma^2_{n,\mathcal{F}} = E^{\mathcal{F}} S_n^2$. It is always to be tacitly understood that $\sigma_{n,\mathcal{F}}$ denotes the nonnegative square root of $\sigma^2_{n,\mathcal{F}}$.

For each $n \geq 1$ and each $c > 0$, if one employs usual truncation as follows:

$$X'_{n,c} = X_n I\{|X_n| \leq c\} - E^{\mathcal{F}} X_n I\{|X_n| \leq c\},$$
$$X''_{n,c} = X_n I\{|X_n| > c\} - E^{\mathcal{F}} X_n I\{|X_n| > c\},$$

then $\{X'_{n,c}, n \geq 1\}$ and $\{X''_{n,c}, n \geq 1\}$ are each $\mathcal{F}$-strictly stationary and $\mathcal{F}$-centered, and

$$X_n = X'_{n,c} + X''_{n,c}$$

for each positive integer n providing that $E^{\mathcal{F}} X_n = 0$ a.s. If one additionally sets

$$S'_{n,c} = \sum_{k=1}^{n} X'_{k,c},\ S''_{n,c} = \sum_{k=1}^{n} X''_{k,c},$$
$$\left(\sigma'_{n,c,\mathcal{F}}\right)^2 = E^{\mathcal{F}}\left(S'_{n,c}\right)^2,\ \left(\sigma''_{n,c,\mathcal{F}}\right)^2 = E^{\mathcal{F}}\left(S''_{n,c}\right)^2,$$

then for each positive integer n,

$$S_n = S'_{n,c} + S''_{n,c}, \tag{3.3}$$

which together with conditional Minkowski's inequality yields

$$\left|\sigma_{n,\mathcal{F}} - \sigma'_{n,c,\mathcal{F}}\right| \leq \sigma''_{n,c,\mathcal{F}}. \tag{3.4}$$

The following two lemmas will play key roles in the proofs of Proposition 3.4 and Theorem 4.3 below.

Lemma 3.2. *Assume that $\{X_0, X_n, n \geq 1\}$ is a sequence of nonnegative $\mathcal{F}$-measurable random variables with $X_0 > 0$ a.s. and $X_n \to 0$ a.s. Then for any positive integer l, there exists $A_l \in \mathcal{F}$ with $P(A_l) > 1 - 2l^{-1}$ and a positive integer $n_0(l)$ depending on l such that for $n \geq n_0(l)$,*

$$X_n I_{A_l} \leq \frac{1}{4} X_0 I_{A_l}.$$

Proof. Noting that $\left(X_0 \geq n^{-1}\right) \uparrow (X_0 > 0)$ as $n \to \infty$, by assumption $X_0 > 0$ a.s., one has that

$$\lim_{n\to\infty} P\left(X_0 \geq n^{-1}\right) = 1.$$

Hence, for any positive integer l, there exists a positive integer $n'_0(l)$, such that for all $n \geq n'_0(l)$,

$$P\left(X_0 \geq n^{-1}\right) > 1 - l^{-1},$$

and, in particular,

$$P\left(X_0 \geq [n'_0(l)]^{-1}\right) > 1 - l^{-1}. \tag{3.5}$$

Let $n_0'(l)$ be the above-mentioned positive integer. By assumption $X_n \to 0$ a.s., one has that

$$\lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty}\left(X_k \le [4n_0'(l)]^{-1}\right)\right) = 1.$$

Hence, there exists a positive integer $n_0''(l)$ such that for all $n \ge n_0''(l)$,

$$P\left(\bigcap_{k=n}^{\infty}\left(X_k \le [4n_0'(l)]^{-1}\right)\right) > 1 - l^{-1}.$$

This, particularly, implies that

$$P\left(\bigcap_{n=n_0''(l)}^{\infty}\left(X_n \le [4n_0'(l)]^{-1}\right)\right) > 1 - l^{-1}. \tag{3.6}$$

Taking

$A_l = \left(X_0 \ge [n_0'(l)]^{-1}\right) \cap \left(\cap_{n=n_0''(l)}^{\infty}\left(X_n \le [4n_0'(l)]^{-1}\right)\right)$, $n_0(l) = n_0'(l) \vee n_0''(l)$,

then by (3.5) and (3.6), one has that

$$P(A_l) > 1 - 2l^{-1}$$

and for $n \ge n_0(l)$,

$$X_n I_{A_l} \le \frac{1}{4} X_0 I_{A_l}.$$

□

Lemma 3.3. *Let $p > 0$ and $\{X_n, n \ge 1\}$ be a sequence of random variables with $E^{\mathcal{F}}|X_n|^p \to 0$ a.s. Then $X_n \to 0$ in probability.*

Proof. For any $\varepsilon > 0$, by conditional Markov's inequality,

$$P^{\mathcal{F}}(|X_n| > \varepsilon) \le \varepsilon^{-p} E^{\mathcal{F}}|X_n|^p \to 0 \text{ as } n \to \infty,$$

which together with dominated convergence theorem yields

$$P(|X_n| > \varepsilon) \to 0 \text{ as } n \to \infty,$$

and consequently $X_n \to 0$ in probability as $n \to \infty$. □

We now set out to establish the second proposition on conditionally strictly stationary sequence.

Proposition 3.4. *In addition to the assumptions in Proposition 3.1, assume further that*

$$\liminf_{n\to\infty} n^{-1}\sigma_{n,\mathcal{F}}^2 = \xi_{\mathcal{F}}^2 \text{ a.s.} \tag{3.7}$$

and

$$\lim_{m\to\infty} \sup_{n\ge 1} n^{-1}\left(\sigma_{n,m,\mathcal{F}}''\right)^2 = 0 \text{ a.s.}, \tag{3.8}$$

where $\xi_{\mathcal{F}}$ is an $\mathcal{F}$-measurable random variable with $\xi_{\mathcal{F}} > 0$ a.s.

(i) *There exists an $\mathcal{F}$-measurable random variable $\eta_{\mathcal{F}}$ with $\eta_{\mathcal{F}} > 0$ a.s., which* *satisfies that for any positive integer* l, *there exist* $A_l \in \mathcal{F}$ *with* $P(A_l) > 1-2l^{-1}$ *and a positive integer* $m_0(l)$ *depending on* l *such that for* $m \geq m_0(l)$,

$$\inf_{n\geq 1} n^{-1/2}\, \sigma'_{n,m,\mathcal{F}} I_{A_l} \geq \eta_{\mathcal{F}} I_{A_l}. \tag{3.9}$$

(ii) *Assume that for every positive integer* m *that satisfies*

$$\inf\nolimits_{n\geq 1} n^{-1}\left(\sigma'_{n,m,\mathcal{F}}\right)^2 I_{A_l} \geq \eta_{\mathcal{F}} I_{A_l},$$

one has that

$$\frac{S'_{n,m}}{\sigma'_{n,m,\mathcal{F}}} \to N(0,1) \ \textit{ in distribution as } n \to \infty, \tag{3.10}$$

then

$$\frac{S_n}{\sigma_{n,\mathcal{F}}} \to N(0,1) \ \textit{ in distribution as } n \to \infty.$$

Proof. (i) Assumption (3.7) guarantees that

$$\sigma^2_{n,\mathcal{F}} \to \infty \text{ as } n \to \infty \tag{3.11}$$

and conditionally strict stationarity asserts that for all $n \geq 1$,

$$\sigma^2_{n,\mathcal{F}} > 0 \text{ a.s.} \tag{3.12}$$

In fact, if there exists some positive integer n_0 such that $P\left(\sigma^2_{n_0,\mathcal{F}} = 0\right) > 0$, then for any positive integer k,

$$\begin{aligned}
&\sigma^2_{kn_0,\mathcal{F}} \\
&= E^{\mathcal{F}}\left[\ (X_1 + \cdots + X_{n_0}) + (X_{n_0+1} + \cdots + X_{2n_0}) + \cdots + \left(X_{(k-1)n_0+1} + \right.\right. \\
&\qquad\quad \left.\left. \cdots + X_{kn_0}\right)\right]^2 \\
&\leq kE^{\mathcal{F}} S^2_{n_0} \\
&= k\sigma^2_{n_0,\mathcal{F}},
\end{aligned}$$

so that

$$P\left(\sigma^2_{kn_0,\mathcal{F}} = 0\right) \geq P\left(k\sigma^2_{n_0,\mathcal{F}} = 0\right) = P\left(\sigma^2_{n_0,\mathcal{F}} = 0\right) > 0,$$

which contradicts (3.12). Hence, (3.7) and (3.12) guarantee that

$$\varsigma_{\mathcal{F}} := \inf_{n\geq 1} n^{-1}\sigma^2_{n,\mathcal{F}} > 0 \text{ a.s.} \tag{3.13}$$

and it is obviously an $\mathcal{F}$-measurable random variable.

For any positive integer l, referring to (3.8), (3.13), and employing Lemma 3.2, there exist $A_l \in \mathcal{F}$ with $P(A_l) > 1 - 2l^{-1}$ and a positive integer $m_0(l)$ depending on l such that for $m \geq m_0(l)$,

$$\sup_{n\geq 1} n^{-1}\left(\sigma''_{n,m,\mathcal{F}}\right)^2 I_{A_l} \leq \frac{1}{4}\varsigma_{\mathcal{F}} I_{A_l},$$

which together with (3.13) and (3.4) yields for all $n \geq 1$ and $m \geq m_0(l)$,

$$n^{-1/2}\, \sigma'_{n,m,\mathcal{F}} I_{A_l} \geq n^{-1/2}\, \sigma_{n,\mathcal{F}} I_{A_l} - n^{-1/2}\, \sigma''_{n,m,\mathcal{F}} I_{A_l}$$

$$\geq \varsigma_{\mathcal{F}}^{1/2}\, I_{A_l} - \frac{1}{2}\varsigma_{\mathcal{F}}^{1/2}\, I_{A_l}$$
$$= \frac{1}{2}\varsigma_{\mathcal{F}}^{1/2}\, I_{A_l}.$$

Thus (3.9) holds with $\eta_{\mathcal{F}} = 2^{-1}\varsigma_{\mathcal{F}}^{1/2}$.

(ii) For any positive integer l, applying (i), let M be an integer such that $M \geq m_0(l)$. In what follows, the "truncation levels" m will be integer $\geq M$.

By (3.9) and (3.10), one has that for every integer $m \geq M$, $S'_{n,m}/\sigma'_{n,m,\mathcal{F}} \to N(0,1)$ in distribution as $n \to \infty$, which implies that

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{S'_{n,m}}{\sigma'_{n,m,\mathcal{F}}} \leq x\right) - \Phi(x)\right| \to 0 \text{ as } n \to \infty,$$

where Φ is the distribution function of an $N(0,1)$ random variable. For $m \geq M$ as mentioned above, pick a positive integer J_m such that for $n \geq J_m$,

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{S'_{n,m}}{\sigma'_{n,m,\mathcal{F}}} \leq x\right) - \Phi(x)\right| \leq \frac{1}{m}. \tag{3.14}$$

Moreover, we may assume that

$$J_M < J_{M+1} < \cdots .$$

For each integer $n \geq J_M$, let $m(n)$ denote the integer such that $J_{m(n)} \leq n < J_{m(n)+1}$. Then $m(n) \geq M$, and hence by (3.14) with $m = m(n)$,

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{S'_{n,m(n)}}{\sigma'_{n,m(n),\mathcal{F}}} \leq x\right) - \Phi(x)\right| \leq \frac{1}{m(n)},$$

which together with the rather obvious fact that

$$m(n) \to \infty \text{ as } n \to \infty \tag{3.15}$$

yields

$$\frac{S'_{n,m(n)}}{\sigma'_{n,m(n),\mathcal{F}}} \to N(0,1) \text{ in distribution as } n \to \infty. \tag{3.16}$$

Let $\{A_l^*, l \geq 1\}$ be the disjoint version for $\{A_l, l \geq 1\}$, that is $A_1^* = A_1$, $A_l^* = A_1^c A_1^c \cdots A_{l-1}^c A_l$, $l \geq 2$. Then $\{A_l^*, l \geq 1\}$ is mutually exclusive with $\cup_{l=1}^{\infty} A_l^* = \cup_{l=1}^{\infty} A_l$ and the self-evident fact $P(\cup_{l=1}^{\infty} A_l^*) = 1$. By (3.8) and (3.15), one has that $\lim_{n\to\infty} n^{-1}(\sigma''_{n,m(n),\mathcal{F}})^2 = 0$ a.s. Also, by (3.9), (3.15) and $A_l^* \subset A_l$ one has that $\liminf_{n\to\infty} n^{-1}(\sigma'_{n,m(n),\mathcal{F}})^2 I_{A_l^*} \geq \eta_{\mathcal{F}} I_{A_l^*}$. Hence

$$E^{\mathcal{F}}\left(\frac{S''_{n,m(n)}}{\sigma'_{n,m(n),\mathcal{F}}}\right)^2 = \sum_{l=1}^{\infty} E^{\mathcal{F}}\left(\frac{S''_{n,m(n)}}{\sigma'_{n,m(n),\mathcal{F}}}\right)^2 I_{A_l^*}$$

$$= \sum_{l=1}^{\infty} \frac{n^{-1}\left(\sigma''_{n,m(n),\mathcal{F}}\right)^2}{n^{-1}\left(\sigma'_{n,m(n),\mathcal{F}}\right)^2 I_{A_l^*}}$$
$$\to 0 \text{ a.s. as } n \to \infty.$$

which together with Lemma 3.3 yields $S''_{n,m(n)}/\sigma'_{n,m(n),\mathcal{F}} \to 0$ in probability as $n \to \infty$. Hence by employing in turn (3.16), (3.3) and Slutzky's theorem in turn,

$$\frac{S_n}{\sigma'_{n,m(n),\mathcal{F}}} \to N(0,1) \text{ in distribution as } n \to \infty. \tag{3.17}$$

By (3.4) with $m = m(n)$, one has that for each $n \geq J_M$,

$$\left|\frac{\sigma_{n,\mathcal{F}}}{\sigma'_{n,m(n),\mathcal{F}}} - 1\right| \leq \frac{\sigma''_{n,m(n),\mathcal{F}}}{\sigma'_{n,m(n),\mathcal{F}}}.$$

However,

$$\frac{\sigma''_{n,m(n),\mathcal{F}}}{\sigma'_{n,m(n),\mathcal{F}}} = \sum_{l=1}^{\infty} \frac{\sigma''_{n,m(n),\mathcal{F}}}{\sigma'_{n,m(n),\mathcal{F}} I_{A_l^*}} \to 0 \text{ a.s. as } n \to \infty,$$

and therefore

$$\frac{\sigma_{n,\mathcal{F}}}{\sigma'_{n,m(n),\mathcal{F}}} \to 1 \text{ a.s.},$$

which together with (3.17) and Slutzky's theorem yields

$$\frac{S_n}{\sigma_{n,\mathcal{F}}} \to N(0,1) \text{ in distribution as } n \to \infty.$$

This completes the proof of part (ii). □

4. Conditional central limit theorems

We need to prove two lemmas prior to our conditional central limit theorems.

Lemma 4.1. *Assume that $\{X_n, n \geq 1\}$ is a sequence of $\mathcal{F}$-strong mixing and $\mathcal{F}$-strictly stationary random variables with mixing coefficients $\{\alpha_{\mathcal{F}}(n), n \geq 1\}$ satisfying*

$$n\alpha_{\mathcal{F}}(n) \to 0 \textit{ a.s.} \tag{4.1}$$

If $E^{\mathcal{F}} X_n = 0$ a.s. and $|X_n| \leq X_{\mathcal{F}}$ a.s., where $X_{\mathcal{F}}$ is an $\mathcal{F}$-measurable random variable, then

$$n^{-3} E^{\mathcal{F}} S_n^4 \to 0 \textit{ a.s.} \tag{4.2}$$

Proof. For each $n \geq 1$, one has that

$$E^{\mathcal{F}} S_n^4 = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} E^{\mathcal{F}} X_i X_j X_k X_l \leq 4! \sum_{1\leq i\leq j\leq k\leq l\leq n} \left|E^{\mathcal{F}} X_i X_j X_k X_l\right|,$$

and therefore to prove (4.2) it suffices to prove that

$$n^{-3} \sum_{1 \le i \le j \le k \le l \le n} \left| E^{\mathcal{F}} X_i X_j X_k X_l \right| \to 0 \text{ a.s.} \tag{4.3}$$

For each positive integer n and each $m \in \{0, 1, 2, \ldots, n-1\}$, define the following two sets:

$$Q(n,m) = \left\{ (i,j,k,l) \in \{1,2,\ldots,n\}^4 : i \le j \le k \le l \text{ and } j - i = m \ge l - k \right\},$$

$$R(n,m) = \left\{ (i,j,k,l) \in \{1,2,\ldots,n\}^4 : i \le j \le k \le l \text{ and } l - k = m \ge j - i \right\}.$$

For $m = (j-i) \vee (l-k)$, one has that either $(i,j,k,l) \in Q(n,m)$ or $(i,j,k,l) \in R(n,m)$. Hence, to prove (4.3), it suffices to prove that

$$n^{-3} \sum_{m=0}^{n-1} \sum_{(i,j,k,l) \in Q(n,m)} \left| E^{\mathcal{F}} X_i X_j X_k X_l \right| \to 0 \text{ a.s.} \tag{4.4}$$

and

$$n^{-3} \sum_{m=0}^{n-1} \sum_{(i,j,k,l) \in R(n,m)} \left| E^{\mathcal{F}} X_i X_j X_k X_l \right| \to 0 \text{ a.s.} \tag{4.5}$$

We only need to prove (4.4) because the proof of (4.5) is similar. For convenience, in the calculations that follow, we will use the notation

$$\alpha_{\mathcal{F}}(0) = \sup_{A \in \mathcal{A}_1^1, B \in \mathcal{A}_1^\infty} \left| P^{\mathcal{F}}(A \cap B) - P^{\mathcal{F}}(A) P^{\mathcal{F}}(B) \right|.$$

Of course $\alpha_{\mathcal{F}}(0) \le 1/4$ a.s. If $(i,j,k,l) \in Q(n,m)$, then by means of Corollary 2.7,

$$\begin{aligned} \left| E^{\mathcal{F}} X_i X_j X_k X_l \right| &= \left| E^{\mathcal{F}} X_i X_j X_k X_l - \left(E^{\mathcal{F}} X_i \right) \left(E^{\mathcal{F}} X_j X_k X_l \right) \right| \\ &\le 4 \alpha_{\mathcal{F}}(m) X_{\mathcal{F}}^4. \end{aligned} \tag{4.6}$$

For a given $n \ge 1$ and $0 \le m \le n-1$, there can be at most n choices for i since $1 \le i \le n$, followed by just one choice for j since $j = i + m$, followed by at most n choices for k since $1 \le k \le n$, followed by at most $m+1$ choices for l since $k \le l \le k + m$. In short, the set $Q(n,m)$ does not have more than $n^2(m+1)$ elements (i,j,k,l). Now applying (4.6), (4.1) and Toeplitz's lemma in turn, one has that

$$\begin{aligned} & n^{-3} \sum_{m=0}^{n-1} \sum_{(i,j,k,l) \in Q(n,m)} \left| E^{\mathcal{F}} X_i X_j X_k X_l \right| \\ &\le n^{-3} \sum_{m=0}^{n-1} n^2 (m+1) \cdot 4 \alpha_{\mathcal{F}}(m) X_{\mathcal{F}}^4 \\ &= 4 X_{\mathcal{F}}^4 n^{-1} \sum_{m=0}^{n-1} (m+1) \alpha_{\mathcal{F}}(m) \end{aligned}$$

$$\leq 4X_{\mathcal{F}}^4 n^{-1}\alpha_{\mathcal{F}}(0) + 8X_{\mathcal{F}}^4 n^{-1}\sum_{m=1}^{n} m\alpha_{\mathcal{F}}(m)$$
$$\to 0 \text{ a.s.}$$

This completes the proof of (4.4). □

Lemma 4.2. *Assume that $\{X_n, n \geq 1\}$ is a sequence of $\mathcal{F}$-independent random variables with $E^{\mathcal{F}}X_n = 0$ a.s. and $E^{\mathcal{F}}X_n^2 < \infty$ a.s. for every $n \geq 1$. If $\{X_n\}$ satisfies the $\mathcal{F}$-Lyapunov's condition, that is, there exists $\delta > 0$ such that*

$$\frac{1}{\sigma_{n,\mathcal{F}}^{2+\delta}}\sum_{j=1}^{n} E^{\mathcal{F}}|X_j|^{2+\delta} \to 0 \text{ a.s.}, \tag{4.7}$$

then

$$\frac{S_n}{\sigma_{n,\mathcal{F}}} \to N(0,1) \text{ in distribution as } n \to \infty.$$

Proof. For any $\varepsilon > 0$, (4.7) guarantees that

$$\begin{aligned}\frac{1}{\sigma_{n,\mathcal{F}}^2}\sum_{j=1}^{n} E^{\mathcal{F}}X_j^2 I(|X_j| > \varepsilon\sigma_{n,\mathcal{F}}) &\leq \frac{1}{\varepsilon^\delta\sigma_{n,\mathcal{F}}^{2+\delta}}\sum_{j=1}^{n} E^{\mathcal{F}}|X_j|^{2+\delta} I(|X_j| > \varepsilon\sigma_{n,\mathcal{F}})\\ &\leq \frac{1}{\varepsilon^\delta\sigma_{n,\mathcal{F}}^{2+\delta}}\sum_{j=1}^{n} E^{\mathcal{F}}|X_j|^{2+\delta} \to 0 \text{ a.s.},\end{aligned}$$

which implies that $\{X_n\}$ satisfies the $\mathcal{F}$-Lindeberg's condition. Hence by Theorem 4.1 in Yuan et al. [25],

$$E^{\mathcal{F}}\exp\left(\frac{itS_n}{\sigma_{n,\mathcal{F}}}\right) \to e^{-\frac{t^2}{2}} \text{ as } n \to \infty,$$

which implies the desired result. □

Our first conditional central limit theorem in subsequent considerations is a conditional version of Theorem 10.3 in Bradley [1], which extends the strong mixing and strictly stationary sequence of random variables to conditional case.

Theorem 4.3. *Assume that $\{X_n, n \geq 1\}$ is a sequence of $\mathcal{F}$-strong mixing and $\mathcal{F}$-strictly stationary random variables with mixing coefficients $\{\alpha_{\mathcal{F}}(n), n \geq 1\}$ satisfying (4.1). Assume also that $E^{\mathcal{F}}X_1 = 0$ a.s. and $|X_1| \leq X_{\mathcal{F}}$ a.s., where $X_{\mathcal{F}}$ is an $\mathcal{F}$-measurable random variable. If*

$$\sum_{n=2}^{\infty}\left|E^{\mathcal{F}}X_1X_n\right| < \infty \text{ a.s.}, \tag{4.8}$$

then $\sigma_{\mathcal{F}}^2 := E^{\mathcal{F}}X_1^2 + 2\sum_{n=2}^{\infty} E^{\mathcal{F}}X_1X_n$ exists in $[0,\infty)$ almost surely, the sum being absolutely convergent. If $\sigma_{\mathcal{F}}^2 > 0$ almost surely, then

$$\frac{S_n}{\sqrt{n}\sigma_{\mathcal{F}}} \to N(0,1) \text{ in distribution as } n \to \infty. \tag{4.9}$$

Proof. The previous part follows directly from part (iv) of Proposition 3.1. The next task is to prove (4.9) and it will involve a blocking argument and be divided into four steps similar to that in the proof of Theorem 10.3 in Bradley [1].

Step 1. The parameters. For any positive integer l, since (4.2) and $\sigma_{\mathcal{F}}^2 > 0$ a.s., for any positive integer l, exactly similar to the proof of Lemma 3.2, there exist $A_l \in \mathcal{F}$ satisfying $P(A_l) \geq 1 - l^{-1}$, positive integers $n_1(l)$ and $n_2(l)$ satisfying $n_2(l) \geq n_1(l) \geq l$, such that

$$n^{-3}\left(E^{\mathcal{F}} S_n^4\right) I_{A_l} \leq \frac{1}{n_1(l)} \sigma_{\mathcal{F}}^2 I_{A_l},\ n \geq n_2(l). \tag{4.10}$$

Also, one can assume that A_l is nondecreasing as l increases. For each real number $x \geq 1$, define

$$n_1(x) = n_1(\lceil x \rceil),$$

where $\lceil x \rceil$ denotes the maximum integer which does not exceed x. Keeping the above-mentioned positive integers $n_1(x)$ and $n_2(l)$ in the mind, let γ : $[1, \infty) \to (0, \infty)$ be defined by

$$\gamma(x) = \max\left\{1/x^{1/2}, 1/[n_1(x)]^{1/3}\right\},$$

then (4.10) can be rewritten as

$$\left(E^{\mathcal{F}} S_n^4\right) I_{A_l} \leq n^3 [\gamma(l)]^3,\ n \geq n_2(l). \tag{4.11}$$

For reference, some properties of $\gamma(x)$ are given as follows:

$$x\gamma(x) \geq 1, \tag{4.12}$$

$$x\gamma(x) \to \infty \text{ as } x \to \infty, \tag{4.13}$$

$$\gamma(x) \text{ is nonincreasing as } x \text{ increases in } [1, \infty) \tag{4.14}$$

and

$$\gamma(x) \to 0 \text{ as } x \to \infty. \tag{4.15}$$

According to the definition of $\gamma(x)$, we may assume

$$\gamma\left(l^{1/2}\right) < 1. \tag{4.16}$$

For each integer $n \geq n_2(l)$, define the integers $k_{n,l}$ and $q_{n,l}$ by

$$k_{n,l} = q_{n,l} = \left\lceil n^{1/2}\, \gamma\left(l^{1/2}\right)\right\rceil, \tag{4.17}$$

which together with (4.12) implies that these integers $k_{n,l}$ and $q_{n,l}$ are positive. For each integer $n \geq n_2(l)$, let $p_{n,l}$ be the integers such that

$$k_{n,l}(p_{n,l} - 1 + q_{n,l}) < n \leq k_{n,l}(p_{n,l} + q_{n,l}). \tag{4.18}$$

Since $k_{n,l}q_{n,l} \le n\big[\gamma\left(l^{1/2}\right)\big]^2 < n$ by (4.17) and (4.16), the integer $p_{n,l}$ defined in (4.18) is positive. By (4.13), one has that $n^{1/2}\,\gamma\left(l^{1/2}\right) \ge l^{1/2}\,\gamma\left(l^{1/2}\right) \to \infty$ as $l \to \infty$, which together with (4.17) yields

$$k_{n,l} = q_{n,l} \sim n^{1/2}\,\gamma\left(l^{1/2}\right) \to \infty \text{ as } l \to \infty, \tag{4.19}$$

and then by means of (4.19) and (4.15),

$$k_{n,l}q_{n,l} = o\left(n\right) \text{ as } l \to \infty. \tag{4.20}$$

Hence by (4.18), one has that

$$k_{n,l}p_{n,l} \sim n \text{ as } l \to \infty, \tag{4.21}$$

which together with (4.19) and (4.17) yields

$$p_{n,l} \sim n^{1/2}\,/\gamma\left(l^{1/2}\right) \to \infty \text{ as } l \to \infty. \tag{4.22}$$

Also, by (4.22) and (4.16), there exists a positive integer l_0 such that

$$p_{n,l} > n^{1/2}\,,\ l \ge l_0. \tag{4.23}$$

Throughout the rest of the proof of (4.9), the only values of n and l that will be dealt with are the ones satisfying $n \ge n_2\left(l\right)$ and $l > l_0$.

Step 2. The blocks. In what follows, positive integers n, l are always assumed to satisfy $n \ge n_2\left(l\right) \ge n_1\left(l\right) \ge l \ge l_0$, where l_0 is defined in (4.23). The sum $S_n = \sum_{j=1}^{n} X_j$ will be broken into an alternating sequence of big blocks and small blocks. The big blocks will each use $p_{n,l}$ indices. The small blocks will each use $q_{n,l}$ indices (except perhaps for a leftover small block at the end).

Recalling the positive integer $k_{n,l}$ defined in (4.17), for each $k = 1, 2, \dots, k_{n,l}$, define the sets

$$G\left(n,k\right) = \left\{j \ge 1 : \left(k-1\right)p_{n,l} + \left(k-1\right)q_{n,l} + 1 \le j \le kp_{n,l} + \left(k-1\right)q_{n,l}\right\}, \tag{4.24}$$

and (if $k_{n,l} \ge 2$)

$$H\left(n,k\right) = \begin{cases} \left\{j \ge 1 : kp_{n,l} + \left(k-1\right)q_{n,l} + 1 \le j \le kp_{n,l} + kq_{n,l}\right\}, & k = 1, 2, \dots, k_{n,l} - 1, \\ \left\{j \ge 1 : k_{n,l}p_{n,l} + \left(k_{n,l}-1\right)q_{n,l} + 1 \le j \le n\right\}, & k = k_{n,l}. \end{cases} \tag{4.25}$$

By (4.17) and the first inequality in (4.18),

$$\begin{aligned} n &\ge k_{n,l}\left(p_{n,l} - 1 + q_{n,l}\right) + 1 = k_{n,l}p_{n,l} + k_{n,l}q_{n,l} - k_{n,l} + 1 \\ &= k_{n,l}p_{n,l} + k_{n,l}q_{n,l} - q_{n,l} + 1 = k_{n,l}p_{n,l} + \left(k_{n,l}-1\right)q_{n,l} + 1, \end{aligned}$$

and therefore the set $H\left(n, k_{n,l}\right)$ is nonempty. Furthermore, by the second inequality in (4.18), the cardinality of $H\left(n, k_{n,l}\right)$

$$\begin{aligned} \#H\left(n, k_{n,l}\right) &= n - \left[k_{n,l}p_{n,l} + \left(k_{n,l}-1\right)q_{n,l}\right] \\ &\le k_{n,l}\left(p_{n,l} + q_{n,l}\right) - \left[k_{n,l}p_{n,l} + \left(k_{n,l}-1\right)q_{n,l}\right] = q_{n,l}. \end{aligned} \tag{4.26}$$

It is easy to check that these blocks $G(n,k)$, $H(n,k)$, $1 \le k \le k_{n,l}$ are disjoint, and

$$\{1,2,\ldots,n\} = G(n,1) \cup H(n,1) \cup G(n,2) \cup H(n,2) \cup \cdots \cup G(n,k_{n,l}) \cup H(n,k_{n,l}).$$

For $k = 1,2,\ldots,k_{n,l}$, define the big blocks

$$V_k^{(n)} = \sum_{j \in G(n,k)} X_j \tag{4.27}$$

and (if $k_{n,l} \ge 2$) the small blocks

$$W_k^{(n)} = \begin{cases} \displaystyle\sum_{j \in H(n,k)} X_j, \ k = 1,\ldots,k_{n,l}-1, \\ \displaystyle\sum_{j \in H(n,k_{n,l})} X_j, \ k = k_{n,l}, \end{cases} \tag{4.28}$$

so that

$$S_n = V_1^{(n)} + W_1^{(n)} + V_2^{(n)} + W_2^{(n)} + \cdots + V_{k_{n,l}}^{(n)} + W_{k_{n,l}}^{(n)}. \tag{4.29}$$

Step 3. Negligibility of the small blocks. In what follows, positive integers n, l are always assumed to satisfy $n \ge n_2(l) \ge n_1(l) \ge l \ge l_0$. Put

$$H(n) = H(n,1) \cup H(n,2) \cup \cdots \cup H(n,k_{n,l}).$$

Then $\#H(n) \le k_{n,l} q_{n,l}$ by (4.25) and (4.26). By (4.28) and conditional strict stationarity,

$$\begin{aligned} E^{\mathcal{F}}\left(\sum_{j=1}^{k_{n,l}} W_j^{(n)}\right)^2 &= E^{\mathcal{F}}\left(\sum_{i \in H(n)} X_i\right)^2 \\ &= \sum_{i \in H(n)} E^{\mathcal{F}} X_i^2 + 2 \sum_{i \in H(n)} \sum_{\substack{j \in H(n) \\ j > i}} E^{\mathcal{F}} X_i X_j \\ &\le \#H(n)\, E^{\mathcal{F}} X_1^2 + 2 \sum_{i \in H(n)} \sum_{j=i+1}^{\infty} \left| E^{\mathcal{F}} X_i X_j \right| \\ &\le \#H(n)\, E^{\mathcal{F}} X_1^2 + 2 \sum_{i \in H(n)} \sum_{j=2}^{\infty} \left| E^{\mathcal{F}} X_1 X_j \right| \\ &= \#H(n) \left(E^{\mathcal{F}} X_1^2 + \sum_{j=2}^{\infty} \left| E^{\mathcal{F}} X_1 X_j \right| \right) \\ &\le k_{n,l} q_{n,l} \left(E^{\mathcal{F}} X_1^2 + \sum_{j=2}^{\infty} \left| E^{\mathcal{F}} X_1 X_j \right| \right), \end{aligned}$$

which together with (4.20) and (4.8) yields

$$n^{-1}E^{\mathcal{F}}\left(\sum_{j=1}^{k_{n,l}} W_j^{(n)}\right)^2 \to 0 \text{ a.s. as } n\to\infty. \tag{4.30}$$

By Lemma 3.3, (4.30) guarantees that

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{k_{n,l}} W_j^{(n)} \to 0 \text{ in probability as } n\to\infty,$$

which together with assumption $\sigma_{\mathcal{F}}^2>0$ yields

$$\frac{1}{\sqrt{n}\sigma_{\mathcal{F}}}\sum_{j=1}^{k_{n,l}} W_j^{(n)} \to 0 \text{ in probability as } n\to\infty.$$

Step 4. Asymptotic normality of S_n. In what follows, positive integers n,l are always assumed to satisfy $n\ge n_2(l)\ge n_1(l)\ge l\ge l_0$. According to (4.24), (4.27) and conditionally strict stationarity, the random variables $V_1^{(n)},V_2^{(n)},\ldots,V_{k_{n,l}}^{(n)}$ each have the same distribution conditioned on $\mathcal{F}$ as the random variable $S_{p_{n,l}}$ conditioned on $\mathcal{F}$. Enlarging the probability space if necessary, for each $n\ge n_2(l)$, let $\tilde V_1^{(n)},\tilde V_2^{(n)},\ldots,\tilde V_{k_{n,l}}^{(n)}$ be $\mathcal{F}$-independent random variables, and each having the same distribution conditioned on $\mathcal{F}$ as $S_{p_{n,l}}$ conditioned on $\mathcal{F}$. Hence $E^{\mathcal{F}}\tilde V_1^{(n)}=0$ and

$$E^{\mathcal{F}}\left[\sum_{j=1}^{k_{n,l}}\tilde V_j^{(n)}\right]^2 = k_{n,l}E^{\mathcal{F}}\left[\tilde V_1^{(n)}\right]^2 = k_{n,l}\sigma^2_{p_{n,l},\mathcal{F}}.$$

By virtue of (4.11),

$$\begin{aligned}\left(\frac{1}{k_{n,l}^2\sigma^4_{p_{n,l},\mathcal{F}}}\sum_{j=1}^{k_{n,l}}E^{\mathcal{F}}\left[\tilde V_j^{(n)}\right]^4\right)I_{A_l} &= \left(\frac{1}{k_{n,l}^2\sigma^4_{p_{n,l},\mathcal{F}}}\sum_{j=1}^{k_{n,l}}E^{\mathcal{F}}S^4_{p_{n,l}}\right)I_{A_l} \\ &= \frac{1}{k_{n,l}\sigma^4_{p_{n,l},\mathcal{F}}}\left(E^{\mathcal{F}}S^4_{p_{n,l}}\right)I_{A_l} \le \frac{1}{k_{n,l}\sigma^4_{p_{n,l},\mathcal{F}}}p^3_{n,l}[\gamma(l)]^3 I_{A_l}.\end{aligned} \tag{4.31}$$

According to (4.8), part (iv) of Proposition 3.1 guarantees that

$$\sigma^2_{n,\mathcal{F}}\sim n\sigma^2_{\mathcal{F}} \text{ a.s.}, \tag{4.32}$$

and therefore

$$\frac{1}{k_{n,l}\sigma^4_{p_{n,l},\mathcal{F}}}p^3_{n,l}[\gamma(l)]^3 \sim \frac{1}{\sigma^4_{\mathcal{F}}}\cdot\frac{p_{n,l}}{k_{n,l}}[\gamma(l)]^3. \tag{4.33}$$

By (4.19), (4.22), (4.14), and (4.15), one has that

$$\frac{p_{n,l}}{k_{n,l}}[\gamma(l)]^3 \sim \frac{n^{1/2}/\gamma\left(l^{1/2}\right)}{n^{1/2}\gamma\left(l^{1/2}\right)}[\gamma(l)]^3 = \frac{[\gamma(l)]^3}{\left[\gamma\left(l^{1/2}\right)\right]^2}$$

$$\leq \frac{\left[\gamma\left(l^{1/2}\right)\right]^3}{\left[\gamma\left(l^{1/2}\right)\right]^2} = \gamma\left(l^{1/2}\right) \to 0 \text{ as } l \to \infty,$$

which together with (4.31) and (4.33) yields

$$\left(\frac{1}{k_{n,l}^2 \sigma_{p_{n,l},\mathcal{F}}^4} \sum_{j=1}^{k_{n,l}} E^{\mathcal{F}} \left[\tilde{V}_j^{(n)}\right]^4\right) I_{A_l} \to 0 \text{ as } l \to \infty.$$

For any $\varepsilon > 0$, by $P(A_l) > 1 - l^{-1}$ and the monotonicity of $P(A_l)$, one has that

$$\lim_{m\to\infty} P\left\{\bigcup_{l=m}^{\infty} \left| \frac{1}{k_{n,l}^2 \sigma_{p_{n,l},\mathcal{F}}^4} \sum_{j=1}^{k_{n,l}} E^{\mathcal{F}} \left[\tilde{V}_j^{(n)}\right]^4 - \left(\frac{1}{k_{n,l}^2 \sigma_{p_{n,l},\mathcal{F}}^4} \sum_{j=1}^{k_{n,l}} E^{\mathcal{F}} \left[\tilde{V}_j^{(n)}\right]^4\right) I_{A_l} \right| > \varepsilon \right\}$$

$$\leq \lim_{m\to\infty} P\left(\bigcup_{l=m}^{\infty} A_l^c\right) = \lim_{m\to\infty} P\left(A_m^c\right) = 0,$$

which implies that

$$\frac{1}{k_{n,l}^2 \sigma_{p_{n,l},\mathcal{F}}^4} \sum_{j=1}^{k_{n,l}} E^{\mathcal{F}} \left[\tilde{V}_j^{(n)}\right]^4 - \left(\frac{1}{k_{n,l}^2 \sigma_{p_{n,l},\mathcal{F}}^4} \sum_{j=1}^{k_{n,l}} E^{\mathcal{F}} \left[\tilde{V}_j^{(n)}\right]^4\right) I_{A_l} \to 0 \text{ a.s. as } l \to \infty,$$

and therefore

$$\frac{1}{k_{n,l}^2 \sigma_{p_{n,l},\mathcal{F}}^4} \sum_{j=1}^{k_{n,l}} E^{\mathcal{F}} \left[\tilde{V}_j^{(n)}\right]^4 \to 0 \text{ a.s. as } l \to \infty. \tag{4.34}$$

The last expression together with Lemma 4.2 yields

$$\frac{1}{k_{n,l}^{1/2}\, \sigma_{p_{n,l},\mathcal{F}}} \sum_{j=1}^{k_{n,l}} \tilde{V}_j^{(n)} \to N(0,1) \text{ in distribution as } l \to \infty. \tag{4.35}$$

However, by (4.32) and (4.21),

$$k_{n,l}^{1/2}\, \sigma_{p_{n,l},\mathcal{F}} \sim k_{n,l}^{1/2}\, p_{n,l}^{1/2}\, \sigma_{\mathcal{F}} \sim n^{1/2}\, \sigma_{\mathcal{F}} \text{ as } l \to \infty,$$

so that (4.35) can be rewritten as

$$\frac{1}{\sqrt{n}\sigma_{\mathcal{F}}} \sum_{j=1}^{k_{n,l}} \tilde{V}_j^{(n)} \to N(0,1) \text{ in distribution as } l \to \infty. \tag{4.36}$$

It follows from Properties 2.3 and 2.4(ii) in [24] that the random sequence $\left\{V_j/\left(k_{n,l}^{1/2}\, \sigma_{p_{n,l},\mathcal{F}}\right), 1 \leq j \leq k_{n,l}\right\}$ is strong mixing. By Corollary 2.9 and (4.1),

$$\left| E^{\mathcal{F}} \exp\left(\frac{it}{\sqrt{n}\sigma_{\mathcal{F}}} \sum_{j=1}^{k_{n,l}} V_j\right) - E^{\mathcal{F}} \exp\left(\frac{it}{\sqrt{n}\sigma_{\mathcal{F}}} \sum_{j=1}^{k_{n,l}} \tilde{V}_j\right) \right|$$

$$= \left| E^{\mathcal{F}} \prod_{j=1}^{k_{n,l}} \exp\left(\frac{it}{\sqrt{n}\sigma_{\mathcal{F}}} V_j\right) - \prod_{j=1}^{k_{n,l}} E^{\mathcal{F}} \exp\left(\frac{it}{\sqrt{n}\sigma_{\mathcal{F}}} V_j\right) \right|$$
$$\leq 16\left(k_{n,l}-1\right)\alpha_{\mathcal{F}}\left(q_{n,l}+1\right) \leq 16 k_{n,l}\alpha_{\mathcal{F}}\left(q_{n,l}\right)$$
$$= 16 q_{n,l}\alpha_{\mathcal{F}}\left(q_{n,l}\right) \to 0 \text{ a.s. as } l \to \infty,$$

which together with dominated convergence theorem yields

$$E\exp\left(\frac{it}{\sqrt{n}\sigma_{\mathcal{F}}}\sum_{j=1}^{k_{n,l}} V_j\right) - E\exp\left(\frac{it}{\sqrt{n}\sigma_{\mathcal{F}}}\sum_{j=1}^{k_{n,l}} \tilde{V}_j\right) \to 0 \text{ as } l \to \infty.$$

The last expression together with (4.34) yields

$$\frac{1}{\sqrt{n}\sigma_{\mathcal{F}}}\sum_{j=1}^{k_{n,l}} V_j \to N(0,1) \text{ in distribution as } l \to \infty. \tag{4.37}$$

By (4.29), one has that

$$\frac{S_n}{\sqrt{n}\sigma_{\mathcal{F}}} = \frac{1}{\sqrt{n}\sigma_{\mathcal{F}}}\sum_{j=1}^{k_{n,l}} V_j^{(n)} + \frac{1}{\sqrt{n}\sigma_{\mathcal{F}}}\sum_{j=1}^{k_{n,l}} W_j^{(n)}.$$

Hence by (4.37), (4.36) and the previous convention $n \geq n_2(l) \geq l$, one has that

$$\frac{S_n}{\sqrt{n}\sigma_{\mathcal{F}}} \to N(0,1) \text{ in distribution as } n \to \infty.$$

This completes the proof of (4.9). □

Corollary 4.4. *Assume that* $\{X_n, n \geq 1\}$ *is a sequence of* $\mathcal{F}$*-strong mixing and* $\mathcal{F}$*-strictly stationary random variables with mixing coefficients* $\{\alpha_{\mathcal{F}}(n), n \geq 1\}$. *Assume also that* $E^{\mathcal{F}}X_1 = 0$ *a.s.*, $E^{\mathcal{F}}X_1^2 < \infty$ *a.s., and each of the following three conditions holds*:

(a) $\sum_{n=1}^{\infty}\alpha_{\mathcal{F}}(n) < \infty$ *a.s.*

(b) $\sigma_{\mathcal{F}}^2 := E^{\mathcal{F}}X_1^2 + 2\sum_{n=2}^{\infty}E^{\mathcal{F}}X_1X_n$ *exists in* $(0,\infty)$ *almost surely, the sum being absolutely convergent.*

(c) $\lim_{m\to\infty}\sum_{n=2}^{\infty}\left|Cov^{\mathcal{F}}\left(X_1 I(|X_1|>m), X_n I(|X_n|>m)\right)\right| = 0$ *a.s., where* $Cov^{\mathcal{F}}(\xi,\eta)$ *is the conditional covariance of* ξ *and* η *given* $\mathcal{F}$.

Then (4.9) *holds.*

Proof. Noting the monotonicity of $\alpha_{\mathcal{F}}(n)$, assumption (a) implies that (4.1) holds. In fact,

$$n\alpha_{\mathcal{F}}(n) \leq 2\sum_{j=\lceil n/2\rceil+1}^{n}\alpha_{\mathcal{F}}(n) \leq 2\sum_{j=\lceil n/2\rceil+1}^{n}\alpha_{\mathcal{F}}(j) \tag{4.38}$$
$$\leq 2\sum_{j=\lceil n/2\rceil+1}^{\infty}\alpha_{\mathcal{F}}(j) \to 0 \text{ as } n \to \infty.$$

Noting that for each positive integer m,

$$E^{\mathcal{F}}\left(X''_{1,m}\right)^2=E^{\mathcal{F}}X_1^2 I\left(|X_1|>m\right)-\left[E^{\mathcal{F}}X_1 I\left(|X_1|>m\right)\right]^2,$$

and then using assumption $E^{\mathcal{F}}X_1^2<\infty$ and conditionally dominated convergence theorem to get

$$\lim_{m\to\infty} E^{\mathcal{F}}\left(X''_{1,m}\right)^2=0 \text{ a.s.},$$

which together with assumption (c) yields

$$E^{\mathcal{F}}\left(X''_{1,m}\right)^2+2\sum_{n=2}^{\infty}\left|E^{\mathcal{F}}X''_{1,m}X''_{n,m}\right|\to 0 \text{ a.s. as } m\to\infty.$$

Hence by part (iii) of Proposition 3.1,

$$\lim_{m\to\infty}\sup_{n\geq 1} n^{-1}\left(\sigma''_{n,m,\mathcal{F}}\right)^2=0 \text{ a.s.} \tag{4.39}$$

Also, by part (iv) of Proposition 3.1 and hypothesis (b),

$$\lim_{n\to\infty} n^{-1}\sigma^2_{n,\mathcal{F}}=\sigma^2_{\mathcal{F}}>0 \text{ a.s.} \tag{4.40}$$

By (4.39), (4.40) and part (i) of Proposition 3.4, there exists an $\mathcal{F}$-measurable random variable $\eta_{\mathcal{F}}$ with $\eta_{\mathcal{F}}>0$ a.s., which satisfies that for any positive integer l, there exist $A_l\in\mathcal{F}$ with $P\left(A_l\right)>1-2l^{-1}$ and a positive integer $m_0\left(l\right)$ depending on l such that for $m\geq m_0\left(l\right)$,

$$\inf_{n\geq 1} n^{-1/2}\,\sigma'_{n,m,\mathcal{F}}I_{A_l}\geq \eta_{\mathcal{F}}I_{A_l}. \tag{4.41}$$

Let m be any positive integer such that (4.41) holds. Then the sequence $\left\{X'_{n,m},n\geq 1\right\}$ of $\mathcal{F}$-strong mixing and $\mathcal{F}$-strictly stationary random variables satisfies the hypothesis of Theorem 4.3. Hence

$$\tau^2_{m,\mathcal{F}}=E^{\mathcal{F}}\left(X'_{1,m}\right)^2+2\sum_{n=2}^{\infty}X'_{1,m}X'_{n,m} \tag{4.42}$$

exists in $[0,\infty)$ almost surely, the sum being absolutely convergent. By part (iv) of Proposition 3.1,

$$\lim_{n\to\infty} n^{-1}\left(\sigma'_{n,m,\mathcal{F}}\right)^2=\tau^2_{m,\mathcal{F}} \text{ a.s.}$$

Hence by (4.41), $\tau^2_{m,\mathcal{F}}I_{A_l}\geq\eta_{\mathcal{F}}I_{A_l}$ for $m\geq m_0\left(l\right)$, which together with Theorem 4.3 yields for $m\geq m_0\left(l\right)$,

$$\frac{S'_{n,m}}{\sqrt{n}\tau_{m,\mathcal{F}}}=\frac{S'_{n,m}I_{A_l}}{\sqrt{n}\tau_{m,\mathcal{F}}I_{A_l}}\to N\left(0,1\right) \text{ in distribution as } n\to\infty. \tag{4.43}$$

The last equation together with (4.42) implies for $m\geq m_0\left(l\right)$,

$$\frac{S'_{n,m}}{\sigma'_{n,m,\mathcal{F}}}\to N\left(0,1\right) \text{ in distribution as } n\to\infty.$$

What we have just shown is that for every positive integer m such that (4.41) holds, one has that (4.43) holds. Hence by (4.41) and part (ii) of Proposition 3.4,

$$\frac{S_n}{\sqrt{n}\sigma_{n,\mathcal{F}}} \to N(0,1) \text{ in distribution as } n \to \infty,$$

which together with (4.40) implies that (4.9) holds. □

Our second conditional central limit theorem in subsequent considerations is a conditional version of Theorem 10.19 in Bradley [1].

Theorem 4.5. *Assume that* $\{X_n, n \geq 1\}$ *is a sequence of* $\mathcal{F}$*-strong mixing and* $\mathcal{F}$*-strictly stationary random variables with mixing coefficients* $\{\alpha_{\mathcal{F}}(n), n \geq 1\}$*. Assume also that* $E^{\mathcal{F}} X_1 = 0$ *a.s.,* $E^{\mathcal{F}} X_1^2 < \infty$ *a.s., and*

$$\sum_{n=1}^{\infty} \int_0^{\alpha_{\mathcal{F}}(n)} \left[\mathcal{Q}_{|X_1|}^{\mathcal{F}}(u)\right]^2 du < \infty \text{ a.s.} \tag{4.44}$$

Then

$$\sigma_{\mathcal{F}}^2 := E^{\mathcal{F}} X_1^2 + 2\sum_{n=2}^{\infty} E^{\mathcal{F}} X_1 X_n$$

exists in $[0,\infty)$ *almost surely, the sum being absolutely convergent. If* $\sigma_{\mathcal{F}}^2 > 0$ *almost surely, then* (4.9) *holds.*

Proof. By Theorem 2.6 and conditionally strict stationarity,

$$\begin{aligned}\sum_{n=2}^{\infty} \left|Cov^{\mathcal{F}}(X_1, X_n)\right| &= \sum_{n=1}^{\infty} \left|Cov^{\mathcal{F}}(X_1, X_{n+1})\right| \\ &\leq 4\sum_{n=1}^{\infty} \int_0^{\alpha_{\mathcal{F}}(n)} \left[\mathcal{Q}_{|X_1|}^{\mathcal{F}}(u)\right]^2 du \\ &< \infty \text{ a.s.,}\end{aligned}$$

which together with part (iv) of Proposition 3.1 completes the proof of the previous part.

For every given $\omega \in \Omega$, taking $g(n) = \alpha_{\mathcal{F}}(\omega, n)$ and $f(u) = \mathcal{Q}_{|X_1|}^{\mathcal{F}}(\omega, u)$, one has by (4.44) and part (iv) of Proposition 10.18 in [1] that

$$\sum_{n=1}^{\infty} \alpha_{\mathcal{F}}(n) < \infty \text{ a.s.} \tag{4.45}$$

Our next task is to verify assumption (c) in Corollary 4.4. For each $c > 0$, define the random variable $W(c)$ by

$$W(c) = |X_1| I(|X_1| > c),$$

then $0 \leq W(c) \leq |X_1|$ a.s. Hence by Proposition 2.2, one has that

$$\mathcal{Q}_{W(c)}^{\mathcal{F}}(u) \leq \mathcal{Q}_{|X_1|}^{\mathcal{F}}(u) \text{ a.s., } u \in (0,1). \tag{4.46}$$

For any $c > 0$ and $u \in (0,1)$ such that $P^{\mathcal{F}}(|X_1| > c) \leq u$, one has that $P^{\mathcal{F}}(W(c) > 0) \leq u$, thereby $\mathcal{Q}^{\mathcal{F}}_{W(c)}(u) = 0$. Further,

$$\lim_{c\to\infty} \mathcal{Q}^{\mathcal{F}}_{W(c)}(u) = 0 \text{ a.s., } u \in (0,1). \tag{4.47}$$

Noting that (2.1) and $E^{\mathcal{F}} X_1^2 < \infty$ a.s. guarantee that $\int_0^1 \left[\mathcal{Q}^{\mathcal{F}}_{|X_1|}(u)\right]^2 du < \infty$ a.s., one has that by (4.46), (4.47) and dominated convergence theorem,

$$\lim_{c\to\infty} \int_0^1 \left[\mathcal{Q}^{\mathcal{F}}_{W(c)}(u)\right]^2 du = 0 \text{ a.s.,}$$

and consequently $\lim_{c\to\infty} \int_0^{\alpha_{\mathcal{F}}(n)} \left[\mathcal{Q}^{\mathcal{F}}_{W(c)}(u)\right]^2 du = 0$ a.s. for each $n \geq 1$. Also by (4.46), one has that for each $n \geq 1$,

$$\int_0^{\alpha_{\mathcal{F}}(n)} \left[\mathcal{Q}^{\mathcal{F}}_{W(c)}(u)\right]^2 du \leq \int_0^{\alpha_{\mathcal{F}}(n)} \left[\mathcal{Q}^{\mathcal{F}}_{|X_1|}(u)\right]^2 du,$$

which together with (4.44) and dominated convergence theorem yields

$$\lim_{c\to\infty} \sum_{n=1}^{\infty} \int_0^{\alpha_{\mathcal{F}}(n)} \left[\mathcal{Q}^{\mathcal{F}}_{W(c)}(u)\right]^2 du = 0 \text{ a.s.} \tag{4.48}$$

Now for each $c > 0$ and each $n \geq 1$, by (2.2) and conditionally strict stationarity,

$$\left|Cov^{\mathcal{F}}\left(X_1 I(|X_1| > c), X_{n+1} I(|X_{n+1}| > c)\right)\right| \leq 4 \int_0^{\alpha_{\mathcal{F}}(n)} \left[\mathcal{Q}^{\mathcal{F}}_{W(c)}(u)\right]^2 du \text{ a.s.,}$$

which together with (4.48) yields

$$\lim_{c\to\infty} \sum_{n=2}^{\infty} \left|Cov^{\mathcal{F}}\left(X_1 I(|X_1| > c), X_{n+1} I(|X_{n+1}| > c)\right)\right| = 0 \text{ a.s.}$$

That is, assumption (c) in Corollary 4.4 is fulfilled.

In summary, by (4.45) and assumption $\sigma^2_{\mathcal{F}} > 0$, all assumptions in Corollary 4.4 are fulfilled and the proof of Theorem 4.5 is complete. □

Our third conditional central limit theorem, as a corollary of Theorem 4.5 here, partially generalizes Theorem 4.2 in [24].

Corollary 4.6. *Assume that $\{X_n, n \geq 1\}$ is a sequence of $\mathcal{F}$-strong mixing and $\mathcal{F}$-strictly stationary random variables. Assume also that $E^{\mathcal{F}} X_1$ a.s., and*

$$E^{\mathcal{F}} |X_1|^{2+\delta} < \infty \text{ a.s.} \tag{4.49}$$

for some $\delta > 0$. If the sequence of mixing coefficients $\{\alpha_{\mathcal{F}}(n)\}$ satisfies

$$\sum_{n=1}^{\infty} \alpha_{\mathcal{F}}^{\delta/(2+\delta)}(n) < \infty \text{ a.s.,} \tag{4.50}$$

then $\sigma^2_{\mathcal{F}} := E^{\mathcal{F}} X_1^2 + 2\sum_{n=2}^{\infty} E^{\mathcal{F}} X_1 X_n$ *exists in* $[0, \infty)$ *almost surely, the sum being absolutely convergent. If* $\sigma^2_{\mathcal{F}} > 0$ *almost surely, then* (4.9) *holds.*

Proof. By part (i) of Theorem 10.18 in [1], it suffices to prove that

$$\int_0^1 \alpha_{\mathcal{F}}^{-1}(u) \Big[\mathcal{Q}^{\mathcal{F}}_{|X_1|}(u) \Big]^2 du < \infty \text{ a.s.,} \tag{4.51}$$

where

$$\alpha_{\mathcal{F}}^{-1}(u) := \alpha_{\mathcal{F}}^{-1}(\cdot, u) = \max\{n \geq 1 : \alpha_{\mathcal{F}}(n)(\cdot) > u\}, \ u \in (0,1).$$

By Hölder's inequality,

$$\begin{aligned} &\int_0^1 \alpha_{\mathcal{F}}^{-1}(u) \Big[\mathcal{Q}^{\mathcal{F}}_{|X_1|}(u) \Big]^2 du \\ &\leq \left\{ \int_0^1 \left[\alpha_{\mathcal{F}}^{-1}(u) \right]^{(2+\delta)/\delta} du \right\}^{\delta/(2+\delta)} \left\{ \int_0^1 \Big[\mathcal{Q}^{\mathcal{F}}_{|X_1|}(u) \Big]^{2+\delta} du \right\}^{2/(2+\delta)}. \end{aligned}$$

However, by Proposition 2.4 and (4.49),

$$\int_0^1 \Big[\mathcal{Q}^{\mathcal{F}}_{|X_1|}(u) \Big]^{2+\delta} du = E^{\mathcal{F}} |X_1|^{2+\delta} < \infty \text{ a.s.,}$$

and therefore, to prove (4.51), it suffices to show that

$$\int_0^1 \left[\alpha_{\mathcal{F}}^{-1}(u) \right]^{(2+\delta)/\delta} du < \infty \text{ a.s.} \tag{4.52}$$

Without loss of generality, we assume that $\alpha_{\mathcal{F}}(0) = 1$ a.s. Then for all $n = 0, 1, 2, \ldots$ and all $u \in [\alpha_{\mathcal{F}}(n+1)(\omega), \alpha_{\mathcal{F}}(n)(\omega)]$ (if $\alpha_{\mathcal{F}}(n+1)(\omega) < \alpha_{\mathcal{F}}(n)(\omega)$), one has that $\alpha_{\mathcal{F}}^{-1}(u) = n$. Hence

$$\begin{aligned} \int_0^1 \left[\alpha_{\mathcal{F}}^{-1}(u) \right]^{(2+\delta)/\delta} du &= \sum_{n=0}^{\infty} \int_{\alpha_{\mathcal{F}}(n+1)}^{\alpha_{\mathcal{F}}(n)} n^{(2+\delta)/\delta} du \\ &= \sum_{n=0}^{\infty} n^{(2+\delta)/\delta} \left[\alpha_{\mathcal{F}}(n) - \alpha_{\mathcal{F}}(n+1) \right] \\ &= \sum_{k=1}^{\infty} \alpha_{\mathcal{F}}(k) \left[k^{(2+\delta)/\delta} - (k-1)^{(2+\delta)/\delta} \right] \\ &= \sum_{k=1}^{\infty} \alpha_{\mathcal{F}}(k) \left[\frac{2+\delta}{\delta} \int_{k-1}^{k} x^{2/\delta} dx \right] \\ &\leq \frac{2+\delta}{\delta} \sum_{k=1}^{\infty} k^{2/\delta} \alpha_{\mathcal{F}}(k), \end{aligned}$$

so that, in order to get (4.52), we only need to show that

$$\sum_{n=1}^{\infty} n^{2/\delta} \alpha_{\mathcal{F}}(n) < \infty \text{ a.s.} \tag{4.53}$$

Exactly similar to the proof of (4.38), one has by (4.50) that

$$n\alpha_{\mathcal{F}}^{\delta/(2+\delta)}(n) \le 2 \sum_{j=\lceil n/2 \rceil+1}^{\infty} \alpha_{\mathcal{F}}^{\delta/(2+\delta)}(j) \to 0 \text{ as } n \to \infty. \tag{4.54}$$

Assume that $N \in \mathcal{F}$ is the exceptional set on which (4.50) and (4.54) do not hold. Then

$$\sum_{n=1}^{\infty} n^{2/\delta}\, \alpha_{\mathcal{F}}(n)\, I_{N^c} = \sum_{n=1}^{\infty} \left[n\alpha_{\mathcal{F}}^{\delta/(2+\delta)}(n)\right]^{2/\delta} \alpha_{\mathcal{F}}^{\delta/(2+\delta)}(n) I_{N^c} \to 0,$$

which completes the proof of (4.53). □

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