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ANALYZING THE DURATION OF SUCCESS AND FAILURE IN MARKOV-MODULATED BERNOULLI PROCESSES

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Abstract. A Markov-modulated Bernoulli process is a generalization of a Bernoulli process in which the success probability evolves over time according to a Markov chain. It has been widely applied in various disciplines for modeling and analysis of systems in random environments. This paper focuses on providing analytical characterizations of the Markov-modulated Bernoulli process by introducing key metrics, including success period, failure period, and cycle. We derive expressions for the distributions and the moments of these metrics in terms of the model parameters.

1. Introduction

A Bernoulli process is a classical binary stochastic model that consists of a sequence of independent trials, where each trial results in one of two possible outcomes: 1 and 0, referred to as success and failure, respectively [19]. This model is often used in reliability analysis as a mathematical tool to describe the dynamics of a system that alternates between two dichotomous phases (e.g., working or disabling, conforming or non-conforming, acceptable or defective, etc.). A key advantage of the Bernoulli process, in addition to its broad applicability, is the simple characterization by a parameter $p \in (0, 1)$, representing the probability of obtaining a success in each trial. Notably, the parameter p in the Bernoulli process is assumed to be fixed as a constant.

A Markov-modulated Bernoulli process (MMBP), introduced by Özekici [15], generalizes the Bernoulli process by relaxing the assumption on the parameter p. In the MMBP, there exists an underlying Markov chain that evolves in a discrete-time setting. The state of this Markov chain at time n determines the success probability p for the nth trial. Consequently, the parameter p in the MMBP is not constrained to a constant but is allowed to vary over time. The

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Bernoulli process can be considered as a degenerate case of the MMBP in which the underlying Markov chain always results in the same success probability p regardless of its state. Furthermore, the MMBP can also be interpreted as a Markov chain with state-dependent rewards [3, 15].

The MMBP has been employed for modeling and analysis in a wide range of applications across multiple disciplines. They include, but are not limited to, the following: reliability analysis for systems in random environments [2,17,18]; modeling of random yields in production systems [14,23]; and analysis of active queue management algorithms in computer networks [1, 5, 13]. Recent studies also include the following: modeling of bursty and correlated traffic in modern telecommunication networks [10,22,24,25]; modeling of fading channels in wireless communications [4, 7, 8]; performance evaluation of scheduling algorithms in wireless networks [11]; and modeling of a missing data mechanism in linear systems [9]. These applications have a common feature in that the underlying Markov chain of the MMBP describes a randomly changing environment in which the considered system operates. Additionally, the success probability of the Bernoulli process within the system is affected by the current state of this environment.

Although the MMBP has found wide applications in various fields, its mathematical properties have received limited attention. Özekici [15] analyzed the transient and long-term behavior of the number of successes and the success times in the MMBP. Özekici and Soyer [16] focused on inferential issues in estimating the model parameters of the MMBP and presented a Bayesian analysis using Markov chain Monte Carlo methods. The main purpose of this paper is to analyze the mathematical properties of the MMBP from a different viewpoint by introducing new metrics. Our approach is based on the observation that any sample path of the MMBP exhibits an alternating pattern between state 1 and state 0, with random duration of sojourn times in each state. We formulate these sojourn times as two sorts of random variables: success periods and failure periods. Furthermore, to quantify the repetitive pattern observed in the MMBP, we introduce an additional metric, called a cycle, which consists of a success period followed by a failure period. In this paper, we analyze the probabilistic characteristics of these metrics using the first-step analysis, a well-established technique in Markov chain theory [12].

The motivation for this paper comes from the work by Kim [11], where the access delay experienced by mobile devices in a wireless network with CDF-based scheduling is analyzed using the MMBP. In this context, the access delay corresponds to the success period of the MMBP, given that a success occurs when a device is not selected by the scheduler. Furthermore, beyond the work [11], the metrics introduced in this paper can be applied to various scheduling algorithms in wireless networks where multiple users contend for access to a shared wireless medium. Typically, each user in a wireless network is subject to a randomly time-varying channel, which can be well described by a finite-state Markov chain [20]. This random environment affects the probability of a user being granted access to the channel, with the precise value determined by the specific scheduling algorithm employed. Accordingly, the MMBP can provide a framework for modeling the channel access process of a user, where success and failure indicate whether the user is granted access or not. Within this framework, a success period can represent transmission time of a user, while a failure period can represent inter-transmission time, thus impacting the throughput and latency experienced by the user. Moreover, a cycle, i.e., the sum of a pair of transmission time and inter-transmission time, can capture the duration between two consecutive transmissions by that user. This metric is essential for evaluating short-term fairness of a scheduling algorithm, as it can quantify the ability of the algorithm to grant access equitably among multiple users within a finite time frame [21]. Therefore, understanding the success period, failure period, and cycle of the MMBP can be useful from the perspective of applications, particularly in the analysis of scheduling algorithms in wireless networks.

The contributions of this paper are summarized as follows. (i) We introduce a set of metrics specific to the MMBP. These metrics are simple, yet useful in understanding and analyzing the behavior of the MMBP. (ii) We derive formulas for the probability distributions and the moments for these metrics. In our derivation, we express the model parameters of the MMBP using matrices and present the formulas in a concise matrix notation. This can facilitate the application, interpretation, and manipulation of the derived formulas.

The rest of this paper is organized as follows. In Section 2, we describe the MMBP model and introduce a set of definitions. In Section 3, we present preliminary lemmas. The main theorems for the distributions and the moments of the introduced metrics are presented in Sections 4 and 5, respectively.

2. Model description and formulation

In this section, we describe the MMBP model proposed in [15] and introduce a set of definitions that are specific to this model.

We consider a discrete-time system where the time axis is divided into slots of equal length L. Without loss of generality, we set $L = 1$. Hence, the nth slot corresponds to the time interval $[n - 1, n)$ for $n = 1, 2, 3, \ldots$. The system operates in an environment that undergoes random fluctuations over time. These fluctuations occur on a per-slot basis, meaning that the environment remains in a constant state within each slot but can transition to a different state between slots. Let $M(n)$ be a random variable that represents the state of the environment during the nth slot. We assume that $\{M(n), n \geq 1\}$ is a discrete-time Markov chain on state-space $\{1, 2, \ldots, m\}$ with the transition probability matrix given by

$$\mathbb{V} = \begin{bmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,m} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m,1} & v_{m,2} & \cdots & v_{m,m} \end{bmatrix},$$

where $v_{j,k} = \mathrm{P}(M(n+1) = k \,|\, M(n) = j)$ for $j, k \in \{1, 2, \ldots, m\}$. We further assume that $\{M(n), n \geq 1\}$ is an ergodic Markov chain having the limiting distribution $\boldsymbol{\pi} = [\pi_1, \pi_2, \ldots, \pi_m]$. One can obtain the distribution $\boldsymbol{\pi}$ by solving the following system of matrix equations:

$$\boldsymbol{\pi}\mathbb{V} = \boldsymbol{\pi}, \qquad \boldsymbol{\pi}\mathbf{1} = 1,$$

where $\mathbf{1}$ denotes the all-ones matrix of size $m \times 1$.

At the beginning of each slot, a series of Bernoulli trials is conducted, where each trial can result in either a success or a failure. Let $B(n)$ be a random variable that represents the outcome of the nth Bernoulli trial, i.e.,

$$B(n) = \begin{cases} 1 & \text{if the } n\text{th trial is a success,} \\ 0 & \text{if the } n\text{th trial is a failure.} \end{cases}$$

We refer to the nth slot as a *success slot* if $B(n) = 1$, and as a *failure slot* if $B(n) = 0$. The probability of success or failure in each trial depends on the state of the environment at the time of the trial. Specifically, if the environment is in state j during the nth slot, then the nth Bernoulli trial results in a success with probability p_j or a failure with probability q_j $(= 1 - p_j)$, i.e.,

$$p_j = \mathrm{P}(B(n) = 1 \,|\, M(n) = j),$$
$$q_j = \mathrm{P}(B(n) = 0 \,|\, M(n) = j).$$

Furthermore, given the environmental process $\boldsymbol{M} = \{M(n), n \geq 1\}$, the Bernoulli trials are conditionally independent, i.e.,

$$\mathrm{P}(B(1) = i_1, B(2) = i_2, \ldots, B(n) = i_n \,|\, \boldsymbol{M}) = \prod_{k=1}^{n} \mathrm{P}(B(k) = i_k \,|\, \boldsymbol{M}) \tag{1}$$

for all $n = 1, 2, 3, \ldots$. In this manner, the process $\{B(n), n \geq 1\}$ is modulated by the Markov chain $\{M(n), n \geq 1\}$. Consequently, $\{B(n), n \geq 1\}$ is referred to as the MMBP, while $\{M(n), n \geq 1\}$ is referred to as its underlying Markov chain.

For later use, we define diagonal matrices $\mathbb{P}$ and $\mathbb{Q}$ of size $m \times m$ as

$$\mathbb{P} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_m \end{bmatrix}, \qquad \mathbb{Q} = \begin{bmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_m \end{bmatrix}.$$

We have $\mathbb{P} + \mathbb{Q} = \mathbb{I}$, where $\mathbb{I}$ is the identity matrix of size $m \times m$. Hence, an MMBP can be represented mathematically by the pair $(\mathbb{V}, \mathbb{P})$ or $(\mathbb{V}, \mathbb{Q})$.

Note that the MMBP in this paper is assumed to satisfy

$$0 < p_j < 1 \quad \text{(or equivalently, } 0 < q_j < 1) \tag{2}$$

for all $j = 1, 2, \ldots, m$. This aligns with the convention that the parameter p of a Bernoulli process is in the interval $(0, 1)$. Consequently, the MMBP with representation of the form $(\mathbb{V}, \mathbb{P}) = (\mathbb{V}, p\mathbb{I})$ or $(\mathbb{V}, \mathbb{Q}) = (\mathbb{V}, q\mathbb{I})$ simplifies to a conventional Bernoulli process with a parameter $p \in (0, 1)$ or $1 - q \in (0, 1)$, respectively.

A sample path of an MMBP always alternates between 0 and 1. In the case where $B(1) = 1$, it exhibits a sequence of success slots followed by failure slots. This pattern continues with another sequence of success slots followed by failure slots, forming a cyclic pattern that repeats indefinitely. A similar observation can be made in the case where $B(1) = 0$. In order to characterize this pattern mathematically, we introduce the following definitions.

Definition 2.1. We define a *success period* and a *failure period* as a time interval during which the MMBP remains in states 1 and 0, respectively. Specifically, a success period begins when a success slot follows a failure slot (indicating a transition of the MMBP from state 0 to state 1) and terminates when a failure slot appears for the first time after the success slot (indicating a transition from state 1 back to state 0). Similarly, a failure period begins with a transition of the MMBP from state 1 to state 0 and ends with its return to state 1. Let T^{s} and T^{f} be random variables denoting the length of a success and a failure period in the steady state, respectively. More precisely, T^{s} and T^{f} count the total number of slots comprising a success and a failure period in the steady state, respectively. They can be used to measure the duration of time that the MMBP persists in a specific state, once it enters that state.

Definition 2.2. We define a *cycle* as a pair consisting of a success period followed by a failure period. Let T^{c} be a random variable that represents the length of a cycle in the steady state. Similar to Definition 2.1, T^{c} counts the total number of slots comprising a cycle in the steady state. It can be used to measure the duration of time between the start of two successive success periods.

Example 2.1. Figure 1 shows a sample path of the MMBP $\{B(n), n \geq 1\}$ with the representation $(\mathbb{V}, \mathbb{P})$ given by

$$\mathbb{V} = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}, \qquad \mathbb{P} = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.6 \end{bmatrix}.$$

In Figure 1, we can observe that the interval $[3, 5)$ becomes the first success period, followed by the failure period $[5, 6)$. Hence, the interval $[3, 6)$ forms the first cycle. Similarly, the interval $[6, 7)$ becomes the second success period, followed by the failure period $[7, 8)$, which form the second cycle $[6, 8)$, and so

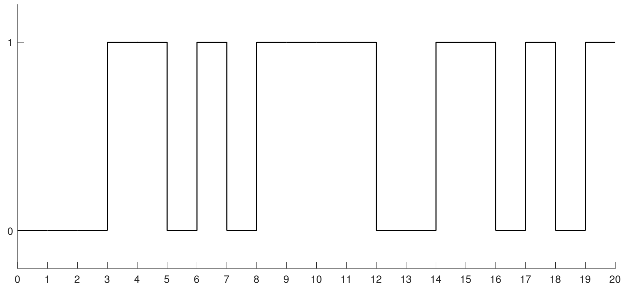

FIGURE 1. A sample path of the MMBP in Example 2.1.

on. If $T^{\mathrm{s}}(k)$, $T^{\mathrm{f}}(k)$ and $T^{\mathrm{c}}(k)$ denote the lengths of the kth success period, failure period and cycle, respectively, then

$$(T^{\mathrm{s}}(k), T^{\mathrm{f}}(k), T^{\mathrm{c}}(k)) = \begin{cases} (2,1,3) & k=1, \\ (1,1,2) & k=2, \\ (4,2,6) & k=3, \\ \quad\vdots & \end{cases}$$

The random variables T^{s}, T^{f} and T^{c} can be expressed as

$$(T^{\mathrm{s}}, T^{\mathrm{f}}, T^{\mathrm{c}}) = \lim_{k\to\infty} (T^{\mathrm{s}}(k), T^{\mathrm{f}}(k), T^{\mathrm{c}}(k)).$$

The aim of the paper is to derive expressions for the probability distributions and the moments of T^{s}, T^{f} and T^{c} in terms of the pair $(\mathbb{V}, \mathbb{P})$ or $(\mathbb{V}, \mathbb{Q})$. The resulting expressions will be presented in Sections 4 and 5 for the probability distributions and the moments, respectively.

3. Preliminary results

In this section, we present preliminary lemmas that serve as the basis for the analysis in Sections 4 and 5. We begin by defining a random vector

$$\mathbf{X}(n) = (B(n), M(n)), \qquad n \geq 1.$$

It represents the joint states of the Bernoulli process and the underlying Markov chain at each slot, from which we can fully describe the evolution of the MMBP. In Lemma 3.1, we show the characteristics of the process $\{\mathbf{X}(n), n \geq 1\}$.

Lemma 3.1. *The process $\{\mathbf{X}(n), n \geq 1\}$ is a two-dimensional Markov chain on state-space $S = \{(i,j) \mid i = 0,1, j = 1,2,\ldots,m\}$ with the transition probability from state $(i_1, j_1) \in S$ to state $(i_2, j_2) \in S$ given by*

$$\mathrm{P}(\mathbf{X}(n+1) = (i_2, j_2) \mid \mathbf{X}(n) = (i_1, j_1)) = \begin{cases} v_{j_1,j_2} p_{j_2} & \text{if } i_2 = 1, \\ v_{j_1,j_2} q_{j_2} & \text{if } i_2 = 0. \end{cases} \tag{3}$$

Proof. We have, for all $(i_1, j_1), (i_2, j_2) \in S$,

$$\begin{aligned} &\mathrm{P}(\mathbf{X}(n+1) = (i_2, j_2) \,|\, \mathbf{X}(n) = (i_1, j_1), \mathbf{X}(n-1), \ldots, \mathbf{X}(1)) \\ &= \mathrm{P}(B(n+1) = i_2 \,|\, M(n+1) = j_2, \mathbf{X}(n) = (i_1, j_1), \mathbf{X}(n-1), \ldots, \mathbf{X}(1)) \\ &\quad \cdot \mathrm{P}(M(n+1) = j_2 \,|\, \mathbf{X}(n) = (i_1, j_1), \mathbf{X}(n-1), \ldots, \mathbf{X}(1)). \end{aligned} \tag{4}$$

The first factor on the right-hand side of (4) reduces to

$$\begin{aligned} &\mathrm{P}(B(n+1) = i_2 \,|\, M(n+1) = j_2, \mathbf{X}(n) = (i_1, j_1), \mathbf{X}(n-1), \ldots, \mathbf{X}(1)) \\ &= \mathrm{P}(B(n+1) = i_2 \,|\, M(n+1) = j_2) \\ &= \begin{cases} p_{j_2} & \text{if } i_2 = 1, \\ q_{j_2} & \text{if } i_2 = 0, \end{cases} \end{aligned}$$

where the first equality follows from (1), expressing the conditional independence of Bernoulli trials given the underlying Markov chain. Meanwhile, the second factor on the right-hand side of (4) reduces to

$$\begin{aligned} &\mathrm{P}(M(n+1) = j_2 \,|\, \mathbf{X}(n) = (i_1, j_1), \mathbf{X}(n-1), \ldots, \mathbf{X}(1)) \\ &= \mathrm{P}(M(n+1) = j_2 \,|\, M(n) = j_1) \\ &= v_{j_1, j_2}, \end{aligned}$$

where the first equality follows from the Markov property of $\{M(n), n \geq 1\}$. Hence, we have

$$\begin{aligned} &\mathrm{P}(\mathbf{X}(n+1) = (i_2, j_2) \,|\, \mathbf{X}(n) = (i_1, j_1), \mathbf{X}(n-1), \ldots, \mathbf{X}(1)) \\ &= \mathrm{P}(\mathbf{X}(n+1) = (i_2, j_2) \,|\, \mathbf{X}(n) = (i_1, j_1)) \\ &= \begin{cases} v_{j_1, j_2} p_{j_2} & \text{if } i_2 = 1, \\ v_{j_1, j_2} q_{j_2} & \text{if } i_2 = 0. \end{cases} \end{aligned}$$

This completes the proof. □

Based on Lemma 3.1, we can classify each success period according to the state of the underlying Markov chain of the MMBP. Specifically, we examine the initial slot of each success period and observe the state of the underlying Markov chain during that slot. If the observed state is $j \in \{1, 2, \ldots, m\}$, we classify the success period as a *j-success period*. Similarly, a failure period is classified as a *j-failure period* if the underlying Markov chain is in state j during the initial slot of the failure period. Let T_j^{s} and T_j^{f} be random variables representing the length of a j-success and a j-failure period, respectively. Then, we can express T^{s} and T^{f} in terms of T_j^{s} and T_j^{f} as follows:

$$T^{\mathrm{s}} = T_j^{\mathrm{s}} \qquad \text{with probability } \alpha_j, \qquad (j = 1, \ldots, m), \tag{5}$$

$$T^{\mathrm{f}} = T_j^{\mathrm{f}} \qquad \text{with probability } \beta_j, \qquad (j = 1, \ldots, m). \tag{6}$$

Here, α_j represents the steady-state probability that the underlying Markov chain of the MMBP is in state j over the initial slot of a success period, and β_j represents the corresponding probability for a failure period. Note that $\sum_{j=1}^{m} \alpha_j = \sum_{j=1}^{m} \beta_j = 1$. For later use, we define matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of size $1 \times m$ as

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m \end{bmatrix}, \tag{7}$$

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_m \end{bmatrix}. \tag{8}$$

Expanding on the previous argument, we classify a cycle as a *j-cycle* if the underlying Markov chain is in state $j \in \{1, 2, \ldots, m\}$ during the initial slot of the cycle. Let T_j^{c} be a random variable that represents the length of a j-cycle. According to Definition 2.2, there is a one-to-one correspondence between a cycle and a success period. Furthermore, the starting time of a cycle coincides with that of the corresponding success period. As a result, the probability that the underlying Markov chain of the MMBP is in state j over the initial slot of a cycle, is equal to α_j. Therefore, we can express T^{c} in terms of T_j^{c} as follows:

$$T^{\mathrm{c}} = T_j^{\mathrm{c}} \qquad \text{with probability } \alpha_j, \qquad (j = 1, \ldots, m). \tag{9}$$

Based on (5), (6) and (9), we proceed with the analysis through two stages. In the first stage, we focus on the random variables T_j^{s}, T_j^{f} and T_j^{c}, and examine their probabilistic properties using the first-step analysis [12] for the Markov chain $\{\mathbf{X}(n), n \geq 1\}$ of Lemma 3.1. In the second stage, we find the expressions for the probabilities α_j and β_j. Lemma 3.2 presents the detailed results of our analysis in the first stage.

Lemma 3.2. *For $1 \leq j \leq m$, the length of a j-success period is expressed as*

$$T_j^{\mathrm{s}} \overset{\mathrm{d}}{=} \begin{cases} 1 & \text{with probability } \sum_{k=1}^{m} v_{j,k} q_k, \\ 1 + T_k^{\mathrm{s}} & \text{with probability } v_{j,k} p_k, \quad (k = 1, \ldots, m), \end{cases} \tag{10}$$

where $\overset{\mathrm{d}}{=}$ denotes "equal in distribution." Similarly, the length of a j-failure period is expressed as

$$T_j^{\mathrm{f}} \overset{\mathrm{d}}{=} \begin{cases} 1 & \text{with probability } \sum_{k=1}^{m} v_{j,k} p_k, \\ 1 + T_k^{\mathrm{f}} & \text{with probability } v_{j,k} q_k, \quad (k = 1, \ldots, m). \end{cases} \tag{11}$$

Finally, the length of a j-cycle is expressed as

$$T_j^{\mathrm{c}} \overset{\mathrm{d}}{=} \begin{cases} 1 + T_k^{\mathrm{f}} & \text{with probability } v_{j,k} q_k, \\ 1 + T_k^{\mathrm{c}} & \text{with probability } v_{j,k} p_k, \end{cases} \quad (k = 1, \ldots, m). \tag{12}$$

Proof. For $1 \leq j \leq m$, let I_j^{s} denote a j-success period in the steady state. According to Definition 2.1, the period I_j^{s} consists of T_j^{s} (≥ 1) success slot(s), and we index its initial slot by $\hat{n}$. Following the first-step analysis, we observe the Markov chain $\{\mathbf{X}(n), n \geq 1\}$ at the ($\hat{n}$+1)st slot. Suppose that the observed state is $\mathbf{X}(\hat{n} + 1) = (i, k)$ for $(i, k) \in S$. Then, there are $2m$ possible outcomes, and we group them into two cases: (i) $i = 0$ and (ii) $i = 1$.

(i) If $i = 0$, the $(\hat{n}+1)$st slot is recognized as a failure slot, causing I_j^{s} to terminate at the end of the $\hat{n}$th slot regardless of the state k. Hence, we have for all $1 \le k \le m$:

$$T_j^{\mathrm{s}} = 1. \tag{13}$$

By (3) in Lemma 3.1, the probability of this event is

$$\sum_{k=1}^{m} \mathrm{P}(\mathbf{X}(\hat{n}+1) = (0,k) \,|\, \mathbf{X}(\hat{n}) = (1,j)) = \sum_{k=1}^{m} v_{j,k} q_k. \tag{14}$$

(ii) If $i = 1$, the $(\hat{n}+1)$st slot is recognized as a success slot, causing I_j^{s} to continue beyond the $\hat{n}$th slot. We note that, starting from the $(\hat{n}+1)$st slot, I_j^{s} behaves as if it were a newly generated success period due to the Markov property of $\{\mathbf{X}(n), n \ge 1\}$. In particular, the behavior of I_j^{s} from the $(\hat{n}+1)$st slot onward is stochastically identical to that of a k-success period, as the underlying Markov chain of the MMBP is in state k during the $(\hat{n}+1)$st slot. Hence, we have

$$T_j^{\mathrm{s}} \overset{\mathrm{d}}{=} 1 + T_k^{\mathrm{s}}, \qquad 1 \le k \le m. \tag{15}$$

By (3) in Lemma 3.1, the probability of this event is

$$\mathrm{P}(\mathbf{X}(\hat{n}+1) = (1,k) \,|\, \mathbf{X}(\hat{n}) = (1,j)) = v_{j,k} p_k, \qquad 1 \le k \le m. \tag{16}$$

Combining (13)–(16), we obtain (10).

By a similar argument as in the derivation of (10), we can prove (11). Due to similarities, we omit the details.

Now, we denote a j-cycle in the steady state by I_j^{c}. By Definition 2.2, the cycle I_j^{c} consists of $T_j^{\mathrm{c}}\,(\ge 2)$ slots. Without loss of generality, we can reuse $\hat{n}$ to index the initial slot of I_j^{c}, since the cycle I_j^{c} starts whenever the period I_j^{s} starts. Following the first-step analysis again, we suppose $\mathbf{X}(\hat{n}+1) = (i,k)$ for $(i,k) \in S$. As before, we consider two cases: (i) $i = 0$ and (ii) $i = 1$.

(i) If $i = 0$, the $(\hat{n}+1)$st slot is recognized as a failure slot, causing the j-success period that comprises I_j^{c} to terminate at the end of the $\hat{n}$th slot. Then, the k-failure period follows from the $(\hat{n}+1)$st slot. When this k-failure period terminates, the cycle I_j^{c} also terminates. Hence, we have

$$T_j^{\mathrm{c}} \overset{\mathrm{d}}{=} 1 + T_k^{\mathrm{f}}, \qquad 1 \le k \le m. \tag{17}$$

By (3) in Lemma 3.1, the probability of this event is

$$\mathrm{P}(\mathbf{X}(\hat{n}+1) = (0,k) \,|\, \mathbf{X}(\hat{n}) = (1,j)) = v_{j,k} q_k, \qquad 1 \le k \le m. \tag{18}$$

(ii) If $i = 1$, the $(\hat{n}+1)$st slot is recognized as a success slot. Since a cycle always starts with a success period, the behavior of I_j^{c} from the $(\hat{n}+1)$st slot onward is stochastically identical to that of a k-cycle. Hence, we have

$$T_j^{\mathrm{c}} \overset{\mathrm{d}}{=} 1 + T_k^{\mathrm{c}}, \qquad 1 \le k \le m. \tag{19}$$

The probability of this event is given by (16). Combining (16)–(19) leads to (12). This completes the proof. □

We now proceed to the second stage, where we derive expressions for the probabilities α_j and β_j for $1 \leq j \leq m$. The result is presented in Lemma 3.3.

Lemma 3.3. *The matrices* $\boldsymbol{\alpha}$ *in* (7) *and* $\boldsymbol{\beta}$ *in* (8) *are obtained by*

$$\boldsymbol{\alpha} = \frac{\boldsymbol{\pi}\mathbb{Q}\mathbb{V}\mathbb{P}}{\boldsymbol{\pi}\mathbb{Q}\mathbb{V}\mathbb{P}\mathbf{1}}, \qquad \boldsymbol{\beta} = \frac{\boldsymbol{\pi}\mathbb{P}\mathbb{V}\mathbb{Q}}{\boldsymbol{\pi}\mathbb{P}\mathbb{V}\mathbb{Q}\mathbf{1}}.$$

Proof. The proof of Lemma 3.3 follows a similar approach to the one used in the derivation of [11, (15)] (see the second bulleted part of [11, Section IV]). A key difference is that the approach in [11] involves an approximation for specific forms of $\mathbb{V}$ and $\mathbb{P}$, which are determined by the given application scenario. However, Lemma 3.3 considers general forms of $\mathbb{V}$ and $\mathbb{P}$ without approximation and is not limited to a specific application. Due to similarities, we omit the detailed proof. □

In Lemma 3.4, we establish the invertibility of the matrices $\mathbb{I} - \mathbb{V}\mathbb{P}$ and $\mathbb{I} - \mathbb{V}\mathbb{Q}$, which will be used in the proofs of Theorems 4.2, 5.1 and 5.2.

Lemma 3.4. *The matrices* $\mathbb{I}-\mathbb{V}\mathbb{P}$ *and* $\mathbb{I}-\mathbb{V}\mathbb{Q}$ *are invertible, with their inverses given by* $(\mathbb{I}-\mathbb{V}\mathbb{P})^{-1} = \sum_{n=0}^{\infty}(\mathbb{V}\mathbb{P})^n$ *and* $(\mathbb{I}-\mathbb{V}\mathbb{Q})^{-1} = \sum_{n=0}^{\infty}(\mathbb{V}\mathbb{Q})^n$*, respectively.*

Proof. We first demonstrate that the matrix $\mathbb{V}\mathbb{P}$ is strictly sub-stochastic, i.e., it is non-negative and $\mathbb{V}\mathbb{P}\mathbf{1} < \mathbf{1}$. Clearly, $\mathbb{V}\mathbb{P}$ is non-negative. Since $\mathbb{P}$ is a diagonal matrix, the (i, j) entry of $\mathbb{V}\mathbb{P}$ is given by $(\mathbb{V}\mathbb{P})_{i,j} = v_{i,j}p_j$. Hence, the sum of the entries in the ith row of $\mathbb{V}\mathbb{P}$ is bounded above by

$$\sum_{j=1}^{m}(\mathbb{V}\mathbb{P})_{i,j} = \sum_{j=1}^{m} v_{i,j}p_j \leq \max_{1\leq j\leq m} p_j \cdot \sum_{j=1}^{m} v_{i,j} < \sum_{j=1}^{m} v_{i,j} = 1, \tag{20}$$

where the second inequality follows from (2), and the last equality holds because $\mathbb{V}$ is stochastic. Since (20) holds for all $i = 1, 2, \ldots, m$, we have $\mathbb{V}\mathbb{P}\mathbf{1} < \mathbf{1}$.

Next, we establish that $\rho(\mathbb{V}\mathbb{P}) < 1$, where $\rho(\mathbb{V}\mathbb{P})$ denotes the spectral radius of $\mathbb{V}\mathbb{P}$. As $\mathbb{V}\mathbb{P}$ is strictly sub-stochastic, the infinity norm of $\mathbb{V}\mathbb{P}$, defined as $\|\mathbb{V}\mathbb{P}\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{m} |(\mathbb{V}\mathbb{P})_{i,j}|$, is bounded above by $\|\mathbb{V}\mathbb{P}\|_\infty < 1$. This bound, combined with the well-known inequality $\rho(\mathbb{V}\mathbb{P}) \leq \|\mathbb{V}\mathbb{P}\|_\infty$, leads to $\rho(\mathbb{V}\mathbb{P}) \leq \|\mathbb{V}\mathbb{P}\|_\infty < 1$. Consequently, the inverse of $\mathbb{I}-\mathbb{V}\mathbb{P}$ exists and is expressed as $(\mathbb{I} - \mathbb{V}\mathbb{P})^{-1} = \sum_{n=0}^{\infty}(\mathbb{V}\mathbb{P})^n$ [6, Section 10.4, Facts 4(g)].

We can complete the proof of Lemma 3.4 by repeating the preceding argument but substituting $\mathbb{Q}$ and q_j for $\mathbb{P}$ and p_j, respectively. Due to similarities, we omit the details. □

4. Distributions

In this section, we derive explicit expressions for the probability mass functions (pmfs) and the complementary cumulative distribution functions

(CCDFs) of the lengths of a success period, a failure period and a cycle. Let

$$f(l) = \mathrm{P}(T^{\mathrm{s}} = l), \qquad l \geq 1,$$
$$g(l) = \mathrm{P}(T^{\mathrm{f}} = l), \qquad l \geq 1,$$
$$h(l) = \mathrm{P}(T^{\mathrm{c}} = l), \qquad l \geq 2.$$

The following theorem gives the expressions for these pmfs.

Theorem 4.1. *For the MMBP with representation* $(\mathbb{V}, \mathbb{P})$*, we have*

$$f(l) = \boldsymbol{\alpha}(\mathbb{VP})^{l-1}(\mathbb{VQ})\mathbf{1}, \qquad l \geq 1,$$
$$g(l) = \boldsymbol{\beta}(\mathbb{VQ})^{l-1}(\mathbb{VP})\mathbf{1}, \qquad l \geq 1,$$
$$h(l) = \boldsymbol{\alpha}\sum_{s=0}^{l-2}(\mathbb{VP})^{s}(\mathbb{VQ})^{l-1-s}(\mathbb{VP})\mathbf{1}, \qquad l \geq 2,$$

where $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *are given in Lemma 3.3.*

Proof. We first derive the pmf $f(l)$ of the length of a success period. Towards this end, we define a matrix $\boldsymbol{f}(l)$ of size $m \times 1$ whose jth element is the pmf $f_j(l) = \mathrm{P}(T_j^{\mathrm{s}} = l)$ of the length of a j-success period as follows:

$$\boldsymbol{f}(l) = \begin{bmatrix} f_1(l) \\ f_2(l) \\ \vdots \\ f_m(l) \end{bmatrix} = \begin{bmatrix} \mathrm{P}(T_1^{\mathrm{s}} = l) \\ \mathrm{P}(T_2^{\mathrm{s}} = l) \\ \vdots \\ \mathrm{P}(T_m^{\mathrm{s}} = l) \end{bmatrix}.$$

We first consider the case $l = 1$. Since each of the random variables $T_1^{\mathrm{s}}, T_2^{\mathrm{s}}, \ldots,$ T_m^{s} can take on positive integer values, (10) in Lemma 3.2 yields

$$f_j(1) = \sum_{k=1}^{m} v_{j,k}q_k. \tag{21}$$

Note that (21) holds for all $1 \leq j \leq m$. Hence, we can write (21) in matrix form as

$$\boldsymbol{f}(1) = \mathbb{VQ}\mathbf{1}. \tag{22}$$

Next, we consider the case $l \geq 2$. By (10) in Lemma 3.2 again, we have

$$f_j(l) = \sum_{k=1}^{m} v_{j,k}p_k f_k(l-1). \tag{23}$$

Note that (23) also holds for all $1 \leq j \leq m$. Hence, we can write (23) in matrix form as

$$\boldsymbol{f}(l) = \mathbb{VP}\boldsymbol{f}(l-1), \qquad l \geq 2. \tag{24}$$

Combining (22) and (24), we have

$$\boldsymbol{f}(l) = (\mathbb{VP})^{l-1}(\mathbb{VQ})\mathbf{1}, \qquad l \geq 1.$$

Using the law of total probability with (5), we can therefore obtain the pmf of the length of a success period as

$$f(l) = \sum_{j=1}^{m} \alpha_j f_j(l) = \boldsymbol{\alpha} \boldsymbol{f}(l) = \boldsymbol{\alpha}(\mathbb{V}\mathbb{P})^{l-1}(\mathbb{V}\mathbb{Q})\mathbf{1}, \qquad l \geq 1.$$

In a similar manner, we next derive the pmf $g(l)$ of the length of a failure period. We define a matrix $\boldsymbol{g}(l)$ of size $m \times 1$ whose jth element is the pmf $g_j(l) = \mathrm{P}(T_j^{\mathrm{f}} = l)$ of the length of a j-failure period as follows:

$$\boldsymbol{g}(l) = \begin{bmatrix} g_1(l) \\ g_2(l) \\ \vdots \\ g_m(l) \end{bmatrix} = \begin{bmatrix} \mathrm{P}(T_1^{\mathrm{f}} = l) \\ \mathrm{P}(T_2^{\mathrm{f}} = l) \\ \vdots \\ \mathrm{P}(T_m^{\mathrm{f}} = l) \end{bmatrix}.$$

Taking a similar approach as above, we can obtain

$$\boldsymbol{g}(l) = \begin{cases} \mathbb{V}\mathbb{P}\mathbf{1} & \text{if } l = 1, \\ \mathbb{V}\mathbb{Q}\boldsymbol{g}(l-1) & \text{if } l \geq 2. \end{cases}$$

Due to similarities, we omit the details. It then follows that

$$\boldsymbol{g}(l) = (\mathbb{V}\mathbb{Q})^{l-1}(\mathbb{V}\mathbb{P})\mathbf{1}, \qquad l \geq 1. \tag{25}$$

Using the law of total probability with (6), we can obtain the pmf of the length of a failure period as

$$g(l) = \sum_{j=1}^{m} \beta_j g_j(l) = \boldsymbol{\beta} \boldsymbol{g}(l) = \boldsymbol{\beta}(\mathbb{V}\mathbb{Q})^{l-1}(\mathbb{V}\mathbb{P})\mathbf{1}, \qquad l \geq 1.$$

Finally, we derive the pmf $h(l)$ of the length of a cycle. Unlike the pmfs $f(l)$ and $g(l)$, for the pmf $h(l)$, we consider $l \geq 2$ since a cycle consists of at least two slots. For each of $l \geq 2$, we define a matrix $\boldsymbol{h}(l)$ of size $m \times 1$ whose jth element is the pmf $h_j(l) = \mathrm{P}(T_j^{\mathrm{c}} = l)$ of the length of a j-cycle as follows:

$$\boldsymbol{h}(l) = \begin{bmatrix} h_1(l) \\ h_2(l) \\ \vdots \\ h_m(l) \end{bmatrix} = \begin{bmatrix} \mathrm{P}(T_1^{\mathrm{c}} = l) \\ \mathrm{P}(T_2^{\mathrm{c}} = l) \\ \vdots \\ \mathrm{P}(T_m^{\mathrm{c}} = l) \end{bmatrix}.$$

From (12) in Lemma 3.2, we have

$$h_j(l) = \sum_{k=1}^{m} v_{j,k} q_k g_k(l-1) + \sum_{k=1}^{m} v_{j,k} p_k h_k(l-1). \tag{26}$$

Note that, since $l \geq 2$ is considered, the first term on the right-hand side of (26) reduces to

$$\sum_{k=1}^{m} v_{j,k} q_k g_k(l-1) = g_j(l),$$

where we used (11) in Lemma 3.2 (similarly to the one in (23)). Hence, we can rewrite (26) as

$$h_j(l) = g_j(l) + \sum_{k=1}^{m} v_{j,k} p_k h_k(l-1).$$

Since the above relation holds for all $1 \le j \le m$, it gives rise to the matrix expression

$$\boldsymbol{h}(l) = \boldsymbol{g}(l) + \mathbb{VP}\boldsymbol{h}(l-1) = (\mathbb{VQ})^{l-1}(\mathbb{VP})\mathbf{1} + \mathbb{VP}\boldsymbol{h}(l-1),$$

where the second equality comes from (25). By repeating the procedure, we then have

$$\boldsymbol{h}(l) = \sum_{s=0}^{l-2} (\mathbb{VP})^s (\mathbb{VQ})^{l-1-s} (\mathbb{VP})\mathbf{1}, \qquad l \ge 2.$$

Using the law of total probability with (9), we can therefore obtain the pmf of the length of a cycle as

$$h(l) = \sum_{j=1}^{m} \alpha_j h_j(l) = \boldsymbol{\alpha}\boldsymbol{h}(l) = \boldsymbol{\alpha} \sum_{s=0}^{l-2} (\mathbb{VP})^s (\mathbb{VQ})^{l-1-s} (\mathbb{VP})\mathbf{1}, \qquad l \ge 2.$$

This completes the proof. □

Now we derive the CCDFs of the lengths of a success period, a failure period and a cycle by using the corresponding pmfs obtained in Theorem 4.1. Let

$$F(l) = \mathrm{P}(T^{\mathrm{s}} > l), \qquad l \ge 0,$$
$$G(l) = \mathrm{P}(T^{\mathrm{f}} > l), \qquad l \ge 0,$$
$$H(l) = \mathrm{P}(T^{\mathrm{c}} > l), \qquad l \ge 1.$$

The following theorem gives the expressions for these CCDFs.

Theorem 4.2. *For the MMBP with representation* $(\mathbb{V}, \mathbb{P})$*, we have*

$$F(l) = \boldsymbol{\alpha}(\mathbb{VP})^l \mathbf{1}, \qquad l \ge 0,$$
$$G(l) = \boldsymbol{\beta}(\mathbb{VQ})^l \mathbf{1}, \qquad l \ge 0,$$
$$H(l) = \boldsymbol{\alpha} \sum_{s=0}^{l} (\mathbb{VP})^s (\mathbb{VQ})^{l-s} \mathbf{1}, \qquad l \ge 1,$$

where $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *are given in Lemma 3.3.*

Proof. We first derive the CCDF $F(l)$ of the length of a success period using the pmf $f(\cdot)$ in Theorem 4.1 as follows:

$$F(l) = \sum_{i=l+1}^{\infty} f(i) = \boldsymbol{\alpha} \sum_{i=l+1}^{\infty} (\mathbb{VP})^{i-1} (\mathbb{VQ})\mathbf{1}. \tag{27}$$

Since $\mathbb{Q} = \mathbb{I} - \mathbb{P}$, we can rewrite the factor $(\mathbb{V}\mathbb{Q})\mathbf{1}$ as

$$(\mathbb{V}\mathbb{Q})\mathbf{1} = \mathbb{V}(\mathbb{I} - \mathbb{P})\mathbf{1} = \mathbb{V}\mathbf{1} - \mathbb{V}\mathbb{P}\mathbf{1} = \mathbf{1} - \mathbb{V}\mathbb{P}\mathbf{1} = (\mathbb{I} - \mathbb{V}\mathbb{P})\mathbf{1}, \tag{28}$$

where the third equality follows as $\mathbb{V}$ is a stochastic matrix with each row summing to 1. Substituting (28) into (27) and applying Lemma 3.4 gives

$$F(l) = \boldsymbol{\alpha} \sum_{i=l+1}^{\infty} (\mathbb{V}\mathbb{P})^{i-1}(\mathbb{I} - \mathbb{V}\mathbb{P})\mathbf{1} = \boldsymbol{\alpha}(\mathbb{V}\mathbb{P})^{l}\mathbf{1}, \qquad l \geq 0.$$

In a similar manner, we next derive the CCDF $G(l)$ of the length of a failure period using the pmf $g(\cdot)$ in Theorem 4.1 as follows:

$$G(l) = \sum_{i=l+1}^{\infty} g(i) = \boldsymbol{\beta} \sum_{i=l+1}^{\infty} (\mathbb{V}\mathbb{Q})^{i-1}(\mathbb{V}\mathbb{P})\mathbf{1} = \boldsymbol{\beta}(\mathbb{V}\mathbb{Q})^{l}\mathbf{1}, \qquad l \geq 0.$$

Due to similarities, we omit the details.

Finally, we derive the CCDF $H(l)$ of the length of a cycle using the pmf $h(\cdot)$ in Theorem 4.1. By the same reason as used in the derivation of $h(\cdot)$, we consider $l \geq 1$ for $H(l)$ unlike $F(l)$ and $G(l)$. For $l \geq 1$, we have

$$H(l) = \sum_{i=l+1}^{\infty} h(i) = \boldsymbol{\alpha} \sum_{i=l+1}^{\infty} \sum_{s=0}^{i-2} (\mathbb{V}\mathbb{P})^{s}(\mathbb{V}\mathbb{Q})^{i-1-s}(\mathbb{I} - \mathbb{V}\mathbb{Q})\mathbf{1},$$

where the last equality follows from (28). Since the summand in the expression of $H(l)$ is non-negative, we can change the order of summation to obtain the following:

$$\begin{aligned} H(l) = {} & \boldsymbol{\alpha} \sum_{s=0}^{l-1} \sum_{i=l+1}^{\infty} (\mathbb{V}\mathbb{P})^{s}(\mathbb{V}\mathbb{Q})^{i-1-s}(\mathbb{I} - \mathbb{V}\mathbb{Q})\mathbf{1} \\ & + \boldsymbol{\alpha} \sum_{s=l}^{\infty} \sum_{i=s+2}^{\infty} (\mathbb{V}\mathbb{P})^{s}(\mathbb{V}\mathbb{Q})^{i-1-s}(\mathbb{I} - \mathbb{V}\mathbb{Q})\mathbf{1}. \end{aligned} \tag{29}$$

Based on Lemma 3.4, we can simplify the first term on the right-hand side of (29) as

$$\sum_{s=0}^{l-1} \sum_{i=l+1}^{\infty} (\mathbb{V}\mathbb{P})^{s}(\mathbb{V}\mathbb{Q})^{i-1-s}(\mathbb{I} - \mathbb{V}\mathbb{Q})\mathbf{1} = \sum_{s=0}^{l-1} (\mathbb{V}\mathbb{P})^{s}(\mathbb{V}\mathbb{Q})^{l-s}\mathbf{1}, \tag{30}$$

while simplifying the second term as

$$\sum_{s=l}^{\infty} \sum_{i=s+2}^{\infty} (\mathbb{V}\mathbb{P})^{s}(\mathbb{V}\mathbb{Q})^{i-1-s}(\mathbb{I} - \mathbb{V}\mathbb{Q})\mathbf{1} = \sum_{s=l}^{\infty} (\mathbb{V}\mathbb{P})^{s}(\mathbb{V}\mathbb{Q})\mathbf{1} = (\mathbb{V}\mathbb{P})^{l}\mathbf{1}. \tag{31}$$

Substituting (30) and (31) into (29) gives

$$H(l) = \boldsymbol{\alpha} \sum_{s=0}^{l-1} (\mathbb{V}\mathbb{P})^{s}(\mathbb{V}\mathbb{Q})^{l-s}\mathbf{1} + \boldsymbol{\alpha}(\mathbb{V}\mathbb{P})^{l}\mathbf{1} = \boldsymbol{\alpha} \sum_{s=0}^{l} (\mathbb{V}\mathbb{P})^{s}(\mathbb{V}\mathbb{Q})^{l-s}\mathbf{1}, \qquad l \geq 1.$$

This completes the proof. □

5. Moments

In this section, we derive recursive expressions for the nth moments of the lengths of a success period, a failure period and a cycle. Let

$$\mu_n = \mathrm{E}[(T^{\mathrm{s}})^n], \qquad n \geq 1,$$
$$\eta_n = \mathrm{E}[(T^{\mathrm{f}})^n], \qquad n \geq 1,$$
$$\psi_n = \mathrm{E}[(T^{\mathrm{c}})^n], \qquad n \geq 1.$$

Similarly to the derivation of Theorem 4.1, we introduce matrices $\boldsymbol{\mu}_n$, $\boldsymbol{\eta}_n$ and $\boldsymbol{\psi}_n$, each of size $m \times 1$. The jth element of these matrices, denoted by $\mu_{j,n}$, $\eta_{j,n}$ and $\psi_{j,n}$, respectively, is defined as the nth moment of the length of a j-success period, a j-failure period and a j-cycle, respectively, as specified below:

$$\boldsymbol{\mu}_n = \begin{bmatrix} \mu_{1,n} \\ \mu_{2,n} \\ \vdots \\ \mu_{m,n} \end{bmatrix} = \begin{bmatrix} \mathrm{E}[(T_1^{\mathrm{s}})^n] \\ \mathrm{E}[(T_2^{\mathrm{s}})^n] \\ \vdots \\ \mathrm{E}[(T_m^{\mathrm{s}})^n] \end{bmatrix}, \qquad n \geq 1,$$

$$\boldsymbol{\eta}_n = \begin{bmatrix} \eta_{1,n} \\ \eta_{2,n} \\ \vdots \\ \eta_{m,n} \end{bmatrix} = \begin{bmatrix} \mathrm{E}[(T_1^{\mathrm{f}})^n] \\ \mathrm{E}[(T_2^{\mathrm{f}})^n] \\ \vdots \\ \mathrm{E}[(T_m^{\mathrm{f}})^n] \end{bmatrix}, \qquad n \geq 1,$$

$$\boldsymbol{\psi}_n = \begin{bmatrix} \psi_{1,n} \\ \psi_{2,n} \\ \vdots \\ \psi_{m,n} \end{bmatrix} = \begin{bmatrix} \mathrm{E}[(T_1^{\mathrm{c}})^n] \\ \mathrm{E}[(T_2^{\mathrm{c}})^n] \\ \vdots \\ \mathrm{E}[(T_m^{\mathrm{c}})^n] \end{bmatrix}, \qquad n \geq 1.$$

Then, we can obtain the following by applying the law of total probability with (5), (6) and (9) for μ_n, η_n and ψ_n, respectively:

$$\mu_n = \sum_{j=1}^{m} \alpha_j \mu_{j,n} = \boldsymbol{\alpha}\boldsymbol{\mu}_n,$$
$$\eta_n = \sum_{j=1}^{m} \beta_j \eta_{j,n} = \boldsymbol{\beta}\boldsymbol{\eta}_n,$$
$$\psi_n = \sum_{j=1}^{m} \alpha_j \psi_{j,n} = \boldsymbol{\alpha}\boldsymbol{\psi}_n,$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are given in Lemma 3.3.

In the rest of this section, we derive expressions for the matrices $\boldsymbol{\mu}_n$, $\boldsymbol{\eta}_n$ and $\boldsymbol{\psi}_n$, considering the cases $n = 1$ and $n \geq 2$ sequentially. Theorem 5.1 addresses the case where $n = 1$.

Theorem 5.1. *For the MMBP with representation* $(\mathbb{V}, \mathbb{P})$*, we have*

$$\boldsymbol{\mu}_1 = (\mathbb{I} - \mathbb{V}\mathbb{P})^{-1}\mathbf{1}, \tag{32}$$

$$\boldsymbol{\eta}_1 = (\mathbb{I} - \mathbb{V}\mathbb{Q})^{-1}\mathbf{1}, \tag{33}$$

$$\boldsymbol{\psi}_1 = (\mathbb{I} - \mathbb{V}\mathbb{P})^{-1}(\mathbb{I} - \mathbb{V}\mathbb{Q})^{-1}\mathbf{1}. \tag{34}$$

Proof. We take expectations on both sides of (10) in Lemma 3.2 to obtain the expression for $\boldsymbol{\mu}_1$:

$$\mu_{j,1} = 1 + \sum_{k=1}^{m} v_{j,k} p_k \mu_{k,1}. \tag{35}$$

Since (35) holds for all $1 \leq j \leq m$, we can write it in matrix form as

$$\boldsymbol{\mu}_1 = \mathbf{1} + \mathbb{V}\mathbb{P}\boldsymbol{\mu}_1.$$

By applying Lemma 3.4, we obtain (32).

Similarly, we can derive (33) from (11) in Lemma 3.2. Due to similarities, we omit the details.

Finally, we derive the expression for $\boldsymbol{\psi}_1$ by taking expectations on both sides of (12) in Lemma 3.2. Then, we obtain

$$\psi_{j,1} = 1 + \sum_{k=1}^{m} v_{j,k} q_k \eta_{k,1} + \sum_{k=1}^{m} v_{j,k} p_k \psi_{k,1} = \eta_{j,1} + \sum_{k=1}^{m} v_{j,k} p_k \psi_{k,1}, \tag{36}$$

where, in the second equality, we used the identity

$$\eta_{j,1} = 1 + \sum_{k=1}^{m} v_{j,k} q_k \eta_{k,1},$$

which can be obtained from (11) in Lemma 3.2 using a similar argument as in the derivation of (35). Furthermore, (36) holds for all $1 \leq j \leq m$, leading to the matrix expression

$$\boldsymbol{\psi}_1 = \boldsymbol{\eta}_1 + \mathbb{V}\mathbb{P}\boldsymbol{\psi}_1.$$

Applying Lemma 3.4 again, we obtain

$$\boldsymbol{\psi}_1 = (\mathbb{I} - \mathbb{V}\mathbb{P})^{-1}\boldsymbol{\eta}_1 = (\mathbb{I} - \mathbb{V}\mathbb{P})^{-1}(\mathbb{I} - \mathbb{V}\mathbb{Q})^{-1}\mathbf{1},$$

where the second equality follows from (33). This shows (34), completing the proof of Theorem 5.1. □

Now, we derive recursive expressions for the matrices $\boldsymbol{\mu}_n$, $\boldsymbol{\eta}_n$ and $\boldsymbol{\psi}_n$ for the case where $n \geq 2$. Theorem 5.2 presents the result.

Theorem 5.2. *For the MMBP with representation* $(\mathbb{V}, \mathbb{P})$*, we have*

$$\boldsymbol{\mu}_n = (\mathbb{I} - \mathbb{V}\mathbb{P})^{-1}\left\{\mathbf{1} + \mathbb{V}\mathbb{P}\sum_{s=1}^{n-1}\binom{n}{s}\boldsymbol{\mu}_s\right\}, \qquad n \geq 2, \tag{37}$$

$$\boldsymbol{\eta}_n = (\mathbb{I} - \mathbb{V}\mathbb{Q})^{-1} \left\{ \mathbf{1} + \mathbb{V}\mathbb{Q} \sum_{s=1}^{n-1} \binom{n}{s} \boldsymbol{\eta}_s \right\}, \qquad n \geq 2, \tag{38}$$

$$\boldsymbol{\psi}_n = (\mathbb{I} - \mathbb{V}\mathbb{P})^{-1} \left\{ \boldsymbol{\eta}_n + \mathbb{V}\mathbb{P} \sum_{s=1}^{n-1} \binom{n}{s} \boldsymbol{\psi}_s \right\}, \qquad n \geq 2. \tag{39}$$

Consequently, for a fixed $n \geq 2$, the matrices $\boldsymbol{\mu}_n$, $\boldsymbol{\eta}_n$ and $\boldsymbol{\psi}_n$ can be obtained recursively using the initial terms $\boldsymbol{\mu}_1$, $\boldsymbol{\eta}_1$ and $\boldsymbol{\psi}_1$ presented in Theorem 5.1.

Proof. Let $n \geq 2$. We first derive the expression for $\boldsymbol{\mu}_n$. By raising both sides of (10) to the power of n and taking expectations on them, we have

$$\begin{aligned} \mu_{j,n} &= \sum_{k=1}^{m} v_{j,k} q_k + \sum_{k=1}^{m} v_{j,k} p_k \left\{ 1 + \sum_{s=1}^{n} \binom{n}{s} \mu_{k,s} \right\} \\ &= 1 + \sum_{k=1}^{m} v_{j,k} p_k \left\{ \sum_{s=1}^{n} \binom{n}{s} \mu_{k,s} \right\}. \end{aligned} \tag{40}$$

Since (40) holds for all $1 \leq j \leq m$, we can write it using matrices as

$$\boldsymbol{\mu}_n = \mathbf{1} + \mathbb{V}\mathbb{P} \left\{ \sum_{s=1}^{n} \binom{n}{s} \boldsymbol{\mu}_s \right\} = \mathbf{1} + \mathbb{V}\mathbb{P} \left\{ \sum_{s=1}^{n-1} \binom{n}{s} \boldsymbol{\mu}_s \right\} + \mathbb{V}\mathbb{P}\boldsymbol{\mu}_n.$$

Leveraging the invertibility of the matrix $\mathbb{I} - \mathbb{V}\mathbb{P}$ established in Lemma 3.4, we can solve for $\boldsymbol{\mu}_n$, yielding (37).

Similarly, we can derive (38) from (11) in Lemma 3.2. Due to similarities, we omit the details.

Finally, we derive the expression for $\boldsymbol{\psi}_n$. As above, we raise both sides of (12) to the power of n and take expectations on them. Then, we have

$$\begin{aligned} \psi_{j,n} &= \sum_{k=1}^{m} v_{j,k} q_k \left\{ 1 + \sum_{s=1}^{n} \binom{n}{s} \eta_{k,s} \right\} + \sum_{k=1}^{m} v_{j,k} p_k \left\{ 1 + \sum_{s=1}^{n} \binom{n}{s} \psi_{k,s} \right\} \\ &= \eta_{j,n} + \sum_{k=1}^{m} v_{j,k} p_k \left\{ \sum_{s=1}^{n} \binom{n}{s} \psi_{k,s} \right\}, \end{aligned} \tag{41}$$

where, in the second equality, we used the identity

$$\eta_{j,n} = 1 + \sum_{k=1}^{m} v_{j,k} q_k \left\{ \sum_{s=1}^{n} \binom{n}{s} \eta_{k,s} \right\},$$

which can be obtained from (11) in Lemma 3.2 using reasoning similar to the derivation of (40). Since (41) holds for all $1 \leq j \leq m$, we can write it using matrices as

$$\boldsymbol{\psi}_n = \boldsymbol{\eta}_n + \mathbb{V}\mathbb{P} \left\{ \sum_{s=1}^{n} \binom{n}{s} \boldsymbol{\psi}_s \right\} = \boldsymbol{\eta}_n + \mathbb{V}\mathbb{P} \left\{ \sum_{s=1}^{n-1} \binom{n}{s} \boldsymbol{\psi}_s \right\} + \mathbb{V}\mathbb{P}\boldsymbol{\psi}_n.$$

Applying Lemma 3.4 again, we can obtain (39). This completes the proof. □

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