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APPROXIMATION OF HELIX BY G^2 CUBIC POLYNOMIAL CURVES

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ABSTRACT. In this paper we present the approximation method of the circular helix by G^2 cubic polynomial curves. The approximants are G^1 Hermite interpolation of the circular helix and their approximation order is four. We obtain numerical examples to illustrate the geometric continuity and the approximation order of the approximants. The method presented in this paper can be extended to approximating the elliptical helix. Using the property of affine transformation invariance we show that the approximant has G^2 continuity and the approximation order four. The numerical examples are also presented to illustrate our assertions.

1. INTRODUCTION

Approximation of trigonometric function curves by parametric polynomial curves is an important task in the fields of CAGD (Computer Aided Geometric Design) and CAD/CAM [1, 2, 3, 4, 5]. Many methods for circle approximation and several methods for helix approximation have been developed using geometric Hermite interpolation.

The geometric Hermite interpolation (GHI) has been used to elevate the approximation order and the order of geometric continuity of approximation curves. For the plane curve, the optimal approximation order of GHI by polynomial curve of degree n is $2n$ [6, 7, 8, 9, 10]. The cubic G^2 Hermite interpolation of plane curves has the approximation order six [11]. For circular arcs and conic sections, G^{k+1} approximation by the polynomial curve of odd degree $n = 2k+1$ has approximation order $2n$ [12], and the Chebyshev approximation by the polynomial curve of any degree $n \geq 2$ has also approximation order $2n$ [13, 14, 15]. A lot of approximation methods of circular arcs and conic sections by polynomial curves of low degree n with approximation order $2n$ have been developed [1, 2, 16, 17, 18, 19].

For spatial curves in $\mathbb{R}^3$, the optimal approximation order of GHI by polynomial curve of degree n is $n + 1 + [(n - 1)/2]$ [6, 7, 8, 9], where $[x]$ is the greatest integer less than or equal to x. The cubic G^1 Hermite interpolation of spatial curves with interpolating a third point has the approximation order five [6]. Several methods for helix approximation by rational Bézier curves have been developed. In these methods [20, 21, 22, 23, 24] the rational approximation

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curves lie on the cylinder containing the helix, which is an important merit on the helix approximation by rational approximants. Mick and Röschel [20] presented G^1 Hermite interpolations of the helix by cubic and quartic rational Bézier curves. The quartic approximant has the same tangent direction with the helix at the midpoint additionally. Juhász [21] proposed two approximation methods of the helix by cubic rational curves with error estimation. One method is G^1 Hermite interpolation and the other is G^2 approximation. Seemann [22] obtained the geometric Hermite interpolation by rational Bézier curves of degrees from four to six, using geometric conditions at both endpoints and midpoints and using free parameters coming from the degree elevation. Yang [23] achieved high accuracy approximation by quintic rational Bézier and polynomial curves. The helix approximations by G^1 conic and quadratic Bézier curves exist having the closed-form sharp error bound [24]. It can be proved that their approximation order is three, which is the optimal order. The circular arc and helix curves have been approximated by the two-point Taylor polynomial curves [25] and the Maclaurin polynomial curves [26] of odd degree n.

In this paper the approximation of the circular helix by G^2 cubic polynomial curves is proposed. The cubic polynomial curves satisfy G^1 Hermite interpolation of the circular helix. Although the approximant does not have the optimal approximation order, it has G^2 continuity and G^1 Hermite interpolation of the circular helix simultaneously. We show that the approximant has the approximation order four. Our method can be extended to approximating the elliptical helix. Using the affine transformation property, we also show that the approximant for the elliptical helix has the approximation order four and G^2 continuity. For given error tolerance, our approximation method can find the smallest number of subdivision of the helix with the approximation error less than the tolerance and can yield the cubic polynomial curves whose curvature vectors are continuous. We present numerical examples to illustrate our assertions.

The remainder of this paper is constructed as follows. In Section 2, the notions for the geometric continuity and geometric interpolation are given. The approximation methods of the circular helix and elliptical helix by G^2 cubic polynomial curves are presented and their properties are analyzed in Sections 3 and 4. We summarize our study in Section 5.

2. Preliminaries

Let $\mathbf{c} : [t_0, t_1] \to \mathbb{R}^3$ and $\hat{\mathbf{c}} : [s_0, s_1] \to \mathbb{R}^3$ be regular curves with $\mathbf{c}(\tau) = \hat{\mathbf{c}}(\sigma)$ for some $\tau \in [t_0, t_1]$ and $\sigma \in [s_0, s_1]$. Two curves $\mathbf{c}$ and $\hat{\mathbf{c}}$ are G^k continuous [27, 28] at $\mathbf{c}(\tau) = \hat{\mathbf{c}}(\sigma)$ if there exists a regular reparameterization $\rho : [t_0, t_1] \to [s_0, s_1]$ such that $\rho' > 0$, $\rho(\tau) = \sigma$ and

$$\frac{d^j \mathbf{c}}{dt^j}(\tau) = \frac{d^j(\hat{\mathbf{c}} \circ \rho)}{dt^j}(\tau), \quad j = 0, 1, \ldots, k.$$

It is well-known [29, 30] that $\mathbf{c}$ and $\hat{\mathbf{c}}$ are G^2 continuous at a point if and only if they have common unit tangent and curvature vectors at the point

Let $\mathbf{b} : [0,1] \to \mathbb{R}^3$ be a parametric polynomial curve. If there exists a regular bijective reparameterization $\rho : [0,1] \to [t_0, t_1]$ with $\rho' > 0$, such that

$$\frac{d^j\mathbf{b}}{dt^j}(t) = \frac{d^j(\mathbf{c} \circ \rho)}{dt^j}(t), \quad j = 0, 1, \ldots, k, \quad t = 0, 1,$$

then we say that $\mathbf{b}$ is a G^k Hermite interpolation of $\mathbf{c}$, [31].

3. Approximation of circular helix by G^2 cubic polynomial curve

Let $\mathbf{c}$ be the circular helix with the angle α represented by

$$\mathbf{c}(t) = (a\cos t, a\sin t, ct)^T, \qquad t \in [0, \alpha],$$

for positive real numbers a, c. Let $\mathbf{b}$ be a cubic Bézier curve approximating the circular helix, and

$$\mathbf{b}(t) = \sum_{i=0}^{3} \mathbf{b}_i B_i^3(t),$$

where $\mathbf{b}_i$, $i = 0, 1, 2, 3$, are control points of $\mathbf{b}$, and $B_i^3(t) = \frac{3!}{i!(3-i)!} t^3 (1-t)^{3-i}$, $i = 0, 1, 2, 3$, are cubic Bernstein polynomials. The control points satisfy

$$\mathbf{b}_0 = \mathbf{c}(0), \;\; \mathbf{b}_1 = \mathbf{c}(0) + d\mathbf{c}'(0), \;\; \mathbf{b}_2 = \mathbf{c}(\alpha) - d\mathbf{c}'(\alpha), \;\; \mathbf{b}_3 = \mathbf{c}(\alpha), \tag{3.1}$$

for some positive real number $d > 0$, if and only if the approximant $\mathbf{b}$ is a G^1 Hermite interpolation of $\mathbf{c}$. We determine the parameter d such that the cubic Bézier approximant has G^2 continuity with the consecutive cubic approximant. Let $\hat{\mathbf{b}}$ be the consecutive cubic approximant for the circular helix $\mathbf{c}$ defined on the interval $[-\alpha, 0]$. By the same way, the control points of $\hat{\mathbf{b}}$ are

$$\hat{\mathbf{b}}_0 = \mathbf{c}(-\alpha), \;\; \hat{\mathbf{b}}_1 = \mathbf{c}(-\alpha) + d\mathbf{c}'(-\alpha), \;\; \hat{\mathbf{b}}_2 = \mathbf{c}(0) - d\mathbf{c}'(0), \;\; \hat{\mathbf{b}}_3 = \mathbf{c}(0).$$

The curvature vector κ of $\mathbf{b}$ is

$$\kappa(t) = \frac{(\mathbf{b}'(t) \times \mathbf{b}''(t)) \times \mathbf{b}'(t)}{||\mathbf{b}'(t)||^4},$$

[32] and so we have

$$\begin{aligned} \kappa(0) &= \frac{2a}{3d^2(a^2+c^2)^2} \begin{pmatrix} d\sin(\alpha) + \cos(\alpha) - 1 \\ c^2(d - d\cos(\alpha) + \sin(\alpha) - \alpha) \\ -ac(d - d\cos(\alpha) + \sin(\alpha) - \alpha) \end{pmatrix}, \\ \hat{\kappa}(1) &= \frac{2a}{3d^2(a^2+c^2)^2} \begin{pmatrix} d\sin(\alpha) + \cos(\alpha) - 1 \\ -c^2(d - d\cos(\alpha) + \sin(\alpha) - \alpha) \\ ac(d - d\cos(\alpha) + \sin(\alpha) - \alpha) \end{pmatrix}, \end{aligned}$$

where $\hat{\kappa}$ is the curvatures vector of $\hat{\mathbf{b}}$. The two consecutive cubic approximants meet with G^2 continuity at the junction if and only if $\kappa(0) = \hat{\kappa}(1)$, which is equivalent to

$$d - d\cos(\alpha) + \sin(\alpha) - \alpha = 0.$$

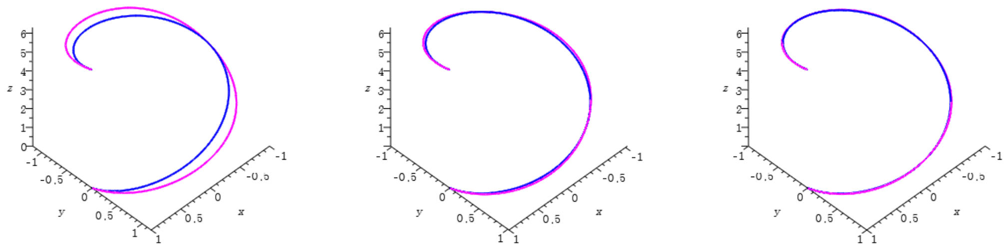

FIGURE 1. Circular helix $\mathbf{c}(t) = (\cos t, \sin t, t)^T$ (blue color), $t \in [0, 2\pi]$, and its approximation curve $\mathbf{b}$ (magenta) with $\alpha = \pi$, $2\pi/3$, $\pi/2$, from left to right.

Thus we have the solution $d = d_1$ where

$$d_1 = \frac{\alpha - \sin\alpha}{1 - \cos\alpha}. \tag{3.2}$$

Proposition 3.1. *The cubic Hermite interpolation of the circular helix with $d = d_1$ is G^2 continuous at both endpoints.*

The asymptotic analysis of the G^2 cubic approximant is as follows. We consider the arc length parametrization of $\mathbf{c}$,

$$\mathbf{c}(s) = \left(\cos\frac{s}{\sqrt{a^2+c^2}}, \sin\frac{s}{\sqrt{a^2+c^2}}, \frac{s}{\sqrt{a^2+c^2}}\right)^T, \; s \in [0, h], \tag{3.3}$$

for sufficiently small $h > 0$. It follows from (3.1) and (3.3) that

$$\begin{aligned}
\Delta\mathbf{b}_0 &= (0, ad, cd)^T, \\
\Delta\mathbf{b}_1 &= (a\cos(h) + ad\sin(h) - a, a\sin(h) - ad\cos(h) - ad, ch - 2cd)^T, \\
\Delta\mathbf{b}_2 &= (-ad\sin(h), ad\cos(h), cd)^T, \\
\Delta^2\mathbf{b}_0 &= (a\cos(h) + ad\sin(h) - a, a\sin(h) - ad\cos(h) - 2ad, ch - 3cd)^T, \\
\Delta^2\mathbf{b}_1 &= (-a\cos(h) - 2ad\sin(h) + a, 2ad\cos(h) - a\sin(h) + ad, 3cd - ch)^T, \\
\Delta^3\mathbf{b}_0 &= (-2a\cos(h) - 3ad\sin(h) + 2a, 3ad\cos(h) - 2a\sin(h) + 3ad, 6cd - 2ch)^T
\end{aligned} \tag{3.4}$$

where $\Delta^k\mathbf{b}_i = \Delta^{k-1}\mathbf{b}_{i+1} - \Delta^{k-1}\mathbf{b}_i$, $k = 1, 2, 3$, and $i = 0, \ldots, 3-k$.

Proposition 3.2. *The cubic polynomial approximant with $d = d_1$ has the approximation order four.*

Proof. First, we aim to show that the approximation order is at least four. Since $\mathbf{b}$ is a G^1 Hermite interpolation of $\mathbf{c}$, applying Newton's remainder formula for polynomial interpolation,

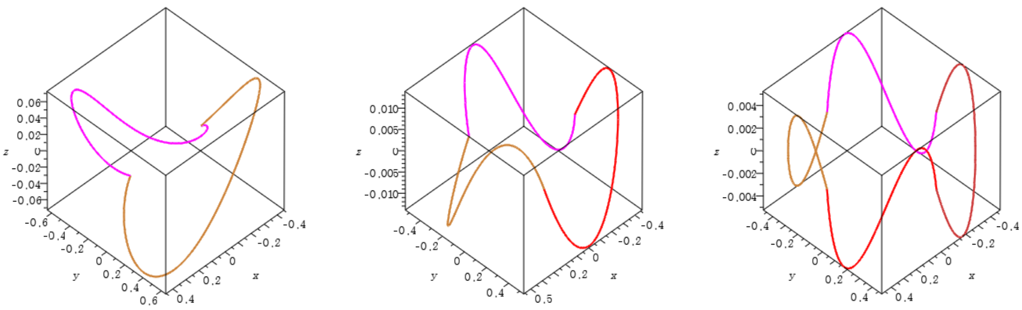

FIGURE 2. Curvature vector curves of the approximation $\mathbf{b}$ with $\alpha = \pi$, $2\pi/3$, $\pi/2$, from left to right, for the circular helix $\mathbf{c}(t) = (\cos t, \sin t, t)^T$, $t \in [0, 2\pi]$. The curvature vector curves are plotted using different colors for each segment.

it is sufficient to show that

$$||\mathbf{b}'(t)|| = \omega h + \mathcal{O}(h^2), \quad \mathbf{b}^{(k)}(t) = \mathcal{O}(h^k) \text{ for } k \geq 2,$$

for some constant $\omega > 0$. Eq. (3.2) yields

$$d_1 = \frac{1}{3}h + \frac{1}{90}h^3 + \mathcal{O}(h^5). \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}
\Delta \mathbf{b}_i &= \frac{ah}{3}(0,1,1)^T + \mathcal{O}(h^2), \ i = 0,1,2, \\
\Delta^2 \mathbf{b}_i &= -\frac{ah^2}{6}(1,0,0)^T + \mathcal{O}(h^3), \ i = 0,1, \\
\Delta^3 \mathbf{b}_0 &= h^3(0, -\frac{a}{10}, \frac{c}{15})^T + \mathcal{O}(h^4).
\end{aligned}$$

Thus $||\mathbf{b}'(t)|| = \sqrt{2}ah/3 + \mathcal{O}(h^2)$, and $\mathbf{b}^{(k)}(t) = \mathcal{O}(h^k)$ for $k = 2, 3$. Note that $\mathbf{b}^{(k)}(t) = \mathbf{0}$ for $k \geq 4$, since $\mathbf{b}$ is a cubic polynomial curve. Hence the accuracy of the G^2 cubic approximant with $d = d_1$ is $\mathcal{O}(h^4)$.

On the other hand, it follows from

$$||\mathcal{P}_{xy}\mathbf{b}(1/2)|| - a = \frac{a}{4}\frac{3h/2 + \sin(h)/2 - 4\sin(h/2)}{\sin(h/2)} > 0, \tag{3.6}$$

that the midpoint $\mathbf{b}(1/2)$ lies outside of the cylinder C containing the circular helix. Here, $\mathcal{P}_{xy}$ is the orthogonal projection from $\mathbb{R}^3$ onto the xy-plane. The inequality in (3.6) holds for the following reason. A function $g : [0, \infty) \to \mathbb{R}$ is defined by $g(h) = 3h/2 + \sin(h)/2 -$

α	$d_H(\mathbf{c}, \mathbf{b})$	decreasing ratio
π	1.781×10^{-1}	
$\pi/2$	9.817×10^{-3}	4.181
$\pi/4$	5.990×10^{-4}	4.035
$\pi/8$	3.723×10^{-5}	4.008
$\pi/16$	2.323×10^{-6}	4.002
$\pi/32$	1.452×10^{-7}	4.000
$\pi/64$	9.072×10^{-9}	4.000
$\pi/128$	5.670×10^{-10}	4.000
$\pi/256$	3.574×10^{-11}	3.988

TABLE 1. Hausdorff distance $d_H(\mathbf{c}, \mathbf{b})$ between the circular helix $\mathbf{c}(t) = (\cos t, \sin t, t)^T$ on the interval $[0, \pi/2^j]$, $j = 0, 1, \ldots, 8$, and its cubic approximant $\mathbf{b}$.

$4\sin(h/2)$. Since

$$g'(h) = \frac{3}{2} + \frac{1}{2}\cos h - 2\cos\frac{h}{2} = (1 - \cos\frac{h}{2})^2,$$

$g(0) = g'(0) = 0$, and $g'(h) > 0$ for all $h > 0$ except for $h = 4n\pi$, $n \in \mathbb{N}$, we have $g(h) > 0$ for all $h > 0$. Let $d_C(\mathbf{x})$ be the shortest distance from the point $\mathbf{x} \in \mathbb{R}^3$ to the cylinder C. Since the Hausdorff distance $d_H(\mathbf{c}, \mathbf{b})$ between $\mathbf{c}$ and $\mathbf{b}$ satisfies

$$d_H(\mathbf{c}, \mathbf{b}) \geq d_C(\mathbf{b}(1/2)) = ||\mathcal{P}_{xy}\mathbf{b}(1/2)|| - a = \frac{a}{640}h^4 + \mathcal{O}(h^6),$$

the approximation order of the G^2 cubic approximant is four. □

As a numerical example, we consider the circular helix $\mathbf{c}(t) = (\cos t, \sin t, t)^T$, $t \in [0, 2\pi]$. The Hausdorff distances $d_H(\mathbf{c}, \mathbf{b})$ between $\mathbf{c}(t)$ on the interval $[0, \pi/2^j]$ and its cubic approximant $\mathbf{b}$ are obtained and listed in Table 1. We can see that all decreasing ratios are closed to four, which is the approximation order.

For given error tolerance, if the approximation error is lager than the tolerance, then subdivisions are required. Our method subdivides the circular helix by the same angle so that all segments of the circular helix are congruent. Thus it is easy to find the smallest number of subdivision to obtain the cubic approximant having the error less than the tolerance. For the given tolerance TOL, the smallest number of subdivision is the positive integer n satisfying

$$d_H(\mathbf{c}_{\frac{\alpha}{n+1}}, \mathbf{b}^{n+1}) < TOL \leq d_H(\mathbf{c}_{\frac{\alpha}{n}}, \mathbf{b}^n), \tag{3.7}$$

where $\mathbf{c}_{\frac{\alpha}{n}}$ is the circular helix defined on the interval $[0, \alpha/n]$ and $\mathbf{b}^n$ is its cubic approximant. In this case, the number of subdivided segments is $n + 1$. The smallest numbers of subdivided segments of the circular helix $\mathbf{c}$ having the approximation error less than the tolerances $TOL = 10^{-1}, 10^{-2}, \ldots, 10^{-5}$, are obtained in Table 2.

In Figure 1, the circular helix $\mathbf{c}$ (blue color) is subdivided into two, three, and four segments, respectively, and then each segment is approximated by cubic polynomial curve (magenta).

TOL	number of subdivided segments
10^{-1}	3
10^{-2}	4
10^{-3}	8
10^{-4}	13
10^{-5}	23

TABLE 2. Smallest number of subdivided segments of the circular helix $\mathbf{c}(t) = (\cos t, \sin t, t)^T$ of angle $\alpha = 2\pi$ whose approximation error is less than the tolerance.

The composite cubic polynomial curves approximating $\mathbf{c}$ are G^2 continuous. The curvature vectors κ of cubic polynomial curves are plotted by different colors, as shown in Figure 2. It is observed that the curvature vectors are continuous, which illustrates Proposition 3.1. Note that all curvature vectors κ of cubic polynomial curves are congruent, since all subdivided circular helix segments with the same angle are congruent. Since the circular helix $\mathbf{c}$ is obtained by a translation of the circular helix $\mathbf{c}(t), t \in [2\pi, 4\pi]$, the curvature vector κ of the composite cubic polynomial curves approximating $\mathbf{c}$ is a closed curve, as shown in Figure 2.

4. APPROXIMATION OF ELLIPTICAL HELIX BY G^2 CUBIC POLYNOMIAL CURVE

In this section the elliptical helix is approximated by the G^2 cubic Bézier curve which is G^1 Hermite interpolation of the elliptical helix. The elliptical helix $\mathbf{e}$ is represented [33, 34] by

$$\mathbf{e}(t) = (a\cos(t_0 + t), b\sin(t_0 + t), ct)^T, t \in [0, \alpha],$$

for some real number t_0. The curve $\mathbf{e}$ can be obtained by an affine transformation A of the circular helix such as $\mathbf{e}(t) = A\mathbf{c}(t)$, where

$$A = \begin{pmatrix} \cos t_0 & -\frac{b}{a}\sin t_0 & 0 \\ \sin t_0 & \frac{b}{a}\cos t_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A cubic Bézier curve $\mathbf{p}$ is G^1 Hermite interpolation of $\mathbf{e}$ if its control points satisfy

$$\mathbf{p}_0 = \mathbf{e}(0),\ \mathbf{p}_1 = \mathbf{e}(0) + d_1\mathbf{e}'(0),\ \mathbf{p}_2 = \mathbf{e}(\alpha) - d_1\mathbf{e}'(\alpha),\ \mathbf{p}_3 = \mathbf{e}(\alpha), \tag{4.1}$$

where d_1 is defined in (3.2). Then, $\mathbf{p}$ can be also obtained from the affine transformation A of the approximant $\mathbf{b}$, i.e.,

$$\mathbf{p}(t) = A\mathbf{b}(t).$$

It is well known that the parametric and geometric continuity are invariant under affine transformations [12]. Thus the following proposition holds.

Proposition 4.1. *The cubic Bézier curve* $\mathbf{p}$ *with the control points in (4.1) has* G^2 *continuity.*

Using properties of affine transformation, we can see that the approximant $\mathbf{p}$ has the approximation order four.

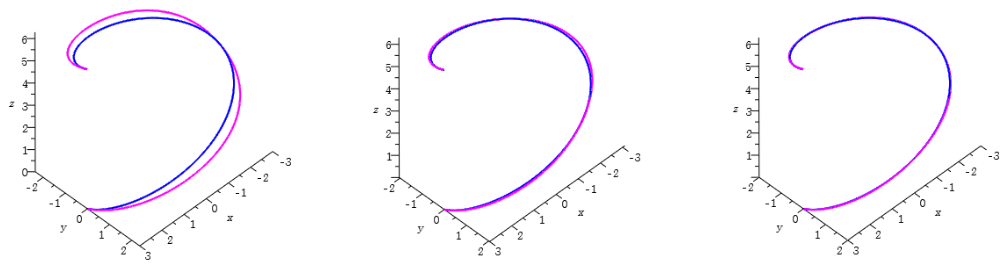

FIGURE 3. Elliptical helix $\mathbf{e}(t) = (3\cos t, 2\sin t, t)^T$ (blue color), $t \in [0, 2\pi]$, and its approximation curve $\mathbf{p}$ (magenta) with $\alpha = \pi$, $2\pi/3$, $\pi/2$, from left to right.

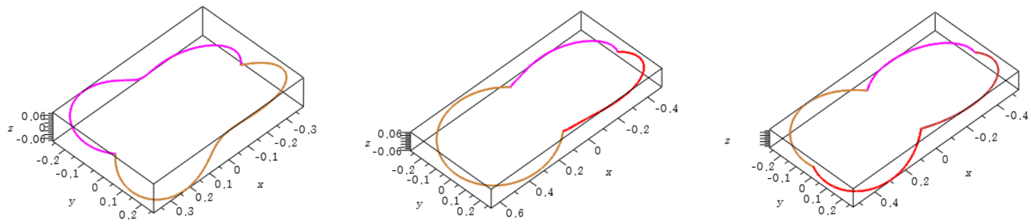

FIGURE 4. Curvature vector curves of the approximation $\mathbf{p}$ with $\alpha = \pi$, $2\pi/3, \pi/2$, from left to right, for the elliptical helix $\mathbf{e}(t) = (3\cos t, 2\sin t, t)^T$, $t \in [0, 2\pi]$. The curvature vector curves are plotted using different colors for each segments.

Proposition 4.2. *The cubic polynomial approximant* $\mathbf{p}$ *of the elliptical helix* $\mathbf{e}$ *has the approximation order four.*

Proof. Since A is an affine transformation satisfying $\mathbf{e} = A\mathbf{c}$ and $\mathbf{p} = A\mathbf{b}$,

$$d_H(\mathbf{e}, \mathbf{p}) \le ||A|| d_H(\mathbf{c}, \mathbf{b}), \tag{4.2}$$

where

$$||A|| = \sup\{||A\mathbf{x}|| : ||\mathbf{x}|| = 1, \mathbf{x} \in \mathbb{R}^3\}.$$

It follows from $||A|| = \max\{1, b/a\}$ that $d_H(\mathbf{e}, \mathbf{p}) = \mathcal{O}(h^4)$ for sufficiently small $h > 0$.

On the other hand, by the affine transformation A, the midpoint $\mathbf{p}(1/2)$ also lies outside of the cylinder E containing the elliptical helix $\mathbf{e}$. Let $d_E(\mathbf{x})$ be the shortest distance from the

α	$d_H(\mathbf{e},\mathbf{p})$	decreasing ratio
π	3.562×10^{-1}	
$\pi/2$	2.367×10^{-2}	3.911
$\pi/4$	1.668×10^{-3}	3.827
$\pi/8$	1.094×10^{-4}	3.931
$\pi/16$	6.933×10^{-6}	3.980
$\pi/32$	4.349×10^{-7}	3.995
$\pi/64$	2.721×10^{-8}	3.999
$\pi/128$	1.701×10^{-9}	4.000
$\pi/256$	1.063×10^{-10}	4.000

TABLE 3. Hausdorff distance $d_H(\mathbf{e},\mathbf{p})$ between the elliptical helix $\mathbf{e}(t) = (3\cos t, 2\sin t, t)^T$ on the interval $[0, \pi/2^j]$, $j = 0, 1, \ldots, 8$, and its cubic approximant $\mathbf{p}$.

point $\mathbf{x} \in \mathbb{R}^3$ to E. If $b \geq a$, then

$$d_E(\mathbf{p}(1/2)) \geq d_C(\mathbf{b}(1/2)),$$

and if $b < a$, then

$$d_E(\mathbf{p}(1/2)) \geq \frac{b}{a} d_C(\mathbf{b}(1/2)).$$

Thus we have

$$d_E(\mathbf{p}(1/2)) \geq \min\{1, \frac{b}{a}\} d_C(\mathbf{b}(1/2)).$$

Since

$$d_H(\mathbf{e},\mathbf{p}) \geq d_E(\mathbf{p}(1/2)) \geq \min\{1, \frac{b}{a}\} \frac{a}{640} h^4 + \mathcal{O}(h^6),$$

the approximant $\mathbf{p}$ has the approximation order four. □

We consider the elliptical helix $\mathbf{e}(t) = (3\cos t, 2\sin t, t)^T$, $t \in [0, 2\pi]$, as a numerical example. The Hausdorff distances $d_H(\mathbf{e},\mathbf{p})$ between $\mathbf{e}(t)$ on the interval $[0, \pi/2^j]$ and its cubic approximant $\mathbf{p}$ are obtained and listed in Table 3, and thus it illustrates Proposition 4.2.

For given tolerance, our method subdivides the elliptical helix by the same angle when the subdivision is needed. The subdivided segments of the elliptical helix are not congruent in general. For the fast calculation, we use the upper bound of the Hausdorff distance $d_H(\mathbf{e},\mathbf{p})$ in (4.2). Hence it is easy to find the smallest number of subdivision to obtain the cubic approximant having the error bound less than the tolerance using (3.7). If $a < b$, we can use the upper bound in (4.2) by the interchange of a and b. Using this method, the smallest numbers of subdivided segments of the elliptical helix $\mathbf{e}$ having the approximation error bound less than the tolerances can be obtained, as shown in Table 4.

In Figure 3, the elliptical helix $\mathbf{e}$ (blue color) is subdivided into two, three, and four segments, respectively, and then each segment is approximated by cubic polynomial curve (magenta). The composite cubic polynomial curves approximating $\mathbf{e}$ are G^2 continuous. In Figure

TOL	number of subdivided segments
10^{-1}	3
10^{-2}	6
10^{-3}	10
10^{-4}	17
10^{-5}	30

TABLE 4. Smallest number of subdivided segments of the elliptical helix $\mathbf{e}(t) = (3\cos t, 2\sin t, t)^T$ of angle $\alpha = 2\pi$ whose approximation error bound is less than the tolerance.

4, the curvature vectors κ of cubic polynomial curves are plotted by different colors. It is observed that the curvature vectors are continuous, which illustrates Proposition 4.1. Note that the curvature vectors of approximant segments are not congruent in general, since the subdivided segments of the elliptical helix are not congruent. By the same reason as in the circular helix case, the curvature vector κ of the composite cubic polynomial curves approximating $\mathbf{e}$ is a closed curve, as shown in Figure 4.

5. CONCLUSION

In this paper we presented the approximation method of the circular helix by G^2 cubic polynomial curves. The cubic polynomial curves are also G^1 Hermite interpolation of the circular helix. We showed that there exists uniquely one cubic polynomial curve satisfying G^2 continuity and G^1 Hermite interpolation of the circular helix. We also showed that the approximant has the approximation order four. Our method can be extended to approximating the elliptical helix. Using the affine transformation property, we proved that the approximant for the elliptical helix has the G^2 continuity and the approximation order four.

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