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ON GENERALIZED EXTENDED BETA AND HYPERGEOMETRIC FUNCTIONS

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Abstract. In the current study, our aim is to define new generalized extended beta and hypergeometric types of functions. Next, we methodically determine several integral representations, Mellin transforms, summation formulas, and recurrence relations. Moreover, we provide log-convexity, Turán type inequality for the generalized extended beta function and differentiation formulas, transformation formulas, differential and difference relations for the generalized extended hypergeometric type functions. Also, we additionally suggest a generating function. Further, we provide the generalized extended beta distribution by making use of the generalized extended beta function as an application to statistics and obtaining variance, coefficient of variation, moment generating function, characteristic function, cumulative distribution function, and cumulative distribution function's complement.

1. Introduction and Preliminaries

For the sake of brevity, we shall employ the following notation [14] established by Carlson [3, p.33] throughout this study:

$$\mathbb{C}_{>} := \{z \in \mathbb{C} : \Re(z) > 0\},$$
$$\mathbb{C}_{>>} := \{y, z \in \mathbb{C} : \Re(y) > \Re(z) > 0\},$$
$$\mathbb{C}_{>-} := \{z \in \mathbb{C} : \Re(z) > -1\},$$

where the symbol $\mathbb{C}$ denotes the set of complex numbers.

We begin by recalling Euler's beta function $B(\mu,\nu)$ (see [10, 25])

$$B(\mu,\nu) := \begin{cases} \int_0^1 t^{\mu-1}(1-t)^{\nu-1}dt, & (\mu,\nu \in \mathbb{C}_{>}) \\ \frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}, & (\mu,\nu \in \mathbb{C}\setminus \mathbb{Z}_0^-), \end{cases} \tag{1.1}$$

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where $\Gamma(z)$ is the familiar gamma function such that (see [10, 25])

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt \quad (z \in \mathbb{C}_>). \tag{1.2}$$

Here and throughout the paper, $\mathbb{R}_0^+, \mathbb{N}$ and $\mathbb{Z}_0^-$ denote the sets of non-negative real numbers, positive integers and non-positive integers, respectively.

The Gauss' hypergeometric function ${}_2F_1(\delta, \gamma; \xi; z)$ and the Kummer's confluent hypergeometric function ${}_1F_1(\gamma; \xi; z) = \Phi(\gamma; \xi; z)$ are recalled here (see [10, 25, 26]):

$$ {}_2F_1(\delta, \gamma; \xi; z) = \sum_{n=0}^{\infty} \frac{(\delta)_n (\gamma)_n}{(\xi)_n} \frac{z^n}{n!} \tag{1.3}$$
$$(|z| < 1; \delta, \gamma \in \mathbb{C}; \xi \in \mathbb{C} \setminus \mathbb{Z}_0^-),$$

and

$$ {}_1F_1(\gamma; \xi; z) := \Phi(\gamma; \xi; z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\xi)_n} \frac{z^n}{n!} \tag{1.4}$$
$$(z, \gamma \in \mathbb{C}; \xi \in \mathbb{C} \setminus \mathbb{Z}_0^-),$$

where $(\vartheta)_n$ (for $\vartheta \in \mathbb{C}$) is the Pochhammer symbol defined by (see [10])

$$(\vartheta)_n := \begin{cases} 1, & (n = 0), \\ \vartheta(\vartheta + 1)...(\vartheta + n - 1), & (n \in \mathbb{N}). \end{cases} \tag{1.5}$$

Additionally, here we recall the integral representations of ${}_2F_1(\delta, \gamma; \xi; z)$ and ${}_1F_1(\gamma; \xi; z)$ (see [10, 25, 26])

$$ {}_2F_1(\delta, \gamma; \xi; z) = \frac{1}{B(\gamma, \xi - \gamma)} \int_0^1 t^{\gamma-1} (1-t)^{\xi-\gamma-1} (1 - zt)^{-\delta} dt \tag{1.6}$$
$$(\xi, \gamma \in \mathbb{C}_{>>}; |\arg(1 - z)| \leq \pi - \epsilon \ \ (0 < \epsilon < \pi)),$$

and

$$ {}_1F_1(\gamma; \xi; z) := \Phi(\gamma; \xi; z) = \frac{1}{B(\gamma, \xi - \gamma)} \int_0^1 t^{\gamma-1} (1-t)^{\xi-\gamma-1} e^{zt} dt \tag{1.7}$$
$$(\xi, \gamma \in \mathbb{C}_{>>}).$$

The extension of the Beta function was presented by Chaudhry et al. ([4, 5]) in 1997, which is as follows:

$$B(\mu, \nu; p) = \int_0^1 t^{\mu-1} (1-t)^{\nu-1} \exp\left(\frac{-p}{t(1-t)} \right) dt \quad (p \in \mathbb{C}_>). \tag{1.8}$$

By making use of $B(\mu, \nu; p)$, defined in (1.8) Chaudhry et al. [6] extended the classical hypergeometric function and the confluent hypergeometric function as follows:

$$F_p(\delta, \gamma; \xi; z) = \sum_{n=0}^{\infty} \frac{B(\gamma + n, \, \xi - \gamma; p)}{B(\gamma, \, \xi - \gamma)} (\delta)_n \frac{z^n}{n!} \tag{1.9}$$

$$(p \geq 0,\ |z|< 1;\ \xi, \gamma \in \mathbb{C}_{>>}),$$

and

$$\Phi_p(\gamma;\xi;z) = \sum_{n=0}^{\infty} \frac{B(\gamma+n,\, \xi-\gamma;p)}{B(\gamma,\, \xi-\gamma)} \frac{z^n}{n!} \tag{1.10}$$

$$(p \geq 0;\ \xi, \gamma \in \mathbb{C}_{>>}).$$

Further, in 2014, Choi et al. [10] proposed a further extended Beta function as:

$$B(\mu,\nu;p,q) = \int_0^1 t^{\mu-1}(1-t)^{\nu-1} \exp\left(\frac{-p}{t} - \frac{q}{1-t}\right) dt \tag{1.11}$$
$$(\min\{\Re(\mu), \Re(\nu)\} > 0;\ \min\{\Re(p), \Re(q)\} > 0).$$

They have studied various integral formulas, properties, summation formulas, Mellin transforms, recurrence type relations of $B(\mu,\nu;p,q)$ in a systematic manner. They have also presented connections of $B(\mu,\nu;p,q)$ with other special functions.

Choi et al. [10] have proposed an extension of the extended confluent hypergeometric function and the extended Gauss' hypergeometric function through the use of $B(\mu,\nu;p,q)$

$$F_{p,q}(\delta,\gamma;\xi;z) = \sum_{n=0}^{\infty} \frac{B(\gamma+n,\, \xi-\gamma;p,q)}{B(\gamma,\, \xi-\gamma)} (\delta)_n \frac{z^n}{n!} \tag{1.12}$$

$$(p, q \in \mathbb{C}_{>},\ |z|< 1;\ \xi, \gamma \in \mathbb{C}_{>>}),$$

and

$$\Phi_{p,q}(\gamma;\xi;z) = \sum_{n=0}^{\infty} \frac{B(\gamma+n,\, \xi-\gamma;p,q)}{B(\gamma,\, \xi-\gamma)} \frac{z^n}{n!} \tag{1.13}$$

$$(p, q \in \mathbb{C}_{>};\ \xi, \gamma \in \mathbb{C}_{>>}).$$

Extensions and generalizations of several hypergeometric type higher transcendental functions have been studied by various authors, for example, (see [1, 7, 8, 9, 15, 17, 18, 19, 20, 21, 22, 23]).

In this paper, our aim is to systematically study the introduction of a new generalized extended Beta function and establish its various properties. These include integral representations, Mellin transforms, summation formulas, and log convexity. Additionally, as an application, we formulate a generalized extended Beta distribution. Moreover, we present generalized extended hypergeometric functions along with their various properties.

2. Generalized Extended Beta Function and its Properties

Definition 2.1. For $\min\{\Re(\mu), \Re(\nu)\} > 0$; $\min\{\Re(p), \Re(q)\} > 0$ and $\alpha, \eta \in \mathbb{C}_{>-}$, the extended Beta function $B_{\alpha;\eta}(\mu, \nu;\ p; q)$ has a new generalized extension that is defined as follows:

$$B_{\alpha;\eta}(\mu, \nu; p; q) = \int_0^1 t^{\mu-1}(1-t)^{\nu-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dt. \tag{2.1}$$

Here in (2.1), $\mathbf{E}_\alpha(z)$ is Mittag-Leffler function defined by (see [28, p.255, Eq. (3.1)]):

$$\mathbf{E}_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)} \qquad (z \in \mathbb{C},\ \alpha \in \mathbb{C}_{>-}), \tag{2.2}$$

and $\mathbf{S}_\eta(z)$ is Bessel-Struve Kernel function given as follows (see [2, 12, 13, 24]):

$$\mathbf{S}_\eta(z) = \frac{\Gamma(\eta+1)}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma(m/2+\eta+1)}\frac{z^m}{m!} \qquad (z \in \mathbb{C},\ \eta \in \mathbb{C}_{>-}). \tag{2.3}$$

Obviously, for $p = 0 = q$ in (2.2) and (2.3), we have $\mathbf{E}_\alpha(0) = 1 = \mathbf{S}_\eta(0)$; afterward, (2.1) reduces to (1.1), which is the standard Beta function.

For $\alpha = 1$ and $z = -p/t$ in (2.2), we get (see [28, p.256, Eq.(3.2)])

$$\mathbf{E}_1\left(\frac{-p}{t}\right) = e^{-p/t}. \tag{2.4}$$

Similarly, for $\eta = \frac{-1}{2}$ and $z = \frac{-q}{1-t}$ in (2.3), we have

$$\mathbf{S}_{\frac{-1}{2}}\left(\frac{-q}{1-t}\right) = e^{\frac{-q}{1-t}}. \tag{2.5}$$

Now for $\alpha = 1$ and $\eta = \frac{-1}{2}$; from (2.4) and (2.5), the equation (2.1) reduces to (1.11).

2.1. Integral Representations of $B_{\alpha;\eta}(\mu, \nu; p; q)$

Lemma 2.2. *The following integral formula for Mittag-Leffler function holds true:*

$$\int_0^\infty p^{b-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)dp = \frac{t^b\,\Gamma(b)\,\Gamma(1-b)}{\Gamma(1-\alpha b)}, \tag{2.6}$$

where α, $b \in \mathbb{C}$, $p \geq 0$ and $\alpha \in \mathbb{C}_{>}$.

Proof. The following integral formula we have (see [28, p.273, Eq. (3.119.)]) is

$$\int_0^\infty p^{b-1}\mathbf{E}_\alpha(-\lambda p)dp = \frac{\Gamma(b)\,\Gamma(1-b)}{\lambda^b\,\Gamma(1-\alpha b)}. \tag{2.7}$$

Substituting $\lambda = \frac{1}{t}$, we obtain the result of (2.6). □

Lemma 2.3. *The following integral formula for Bessel-Struve Kernel function holds true:*

$$\int_0^\infty q^{c-1}\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dq = \frac{(1-t)^c}{\sqrt{\pi}}\,\frac{\Gamma(c)\,\Gamma\left(\frac{1-c}{2}\right)\,\Gamma(\eta+1)}{\Gamma(\eta-\frac{c}{2}+1)}, \tag{2.8}$$

where η, $c \in \mathbb{C}$, $q \geq 0$, *and* $\eta \in \mathbb{C}_>$.

Proof. The integral representation of $\mathbf{S}_\eta(-z)$ we have (see [2, p.5, Eq.(2.1)]) is

$$\mathbf{S}_\eta(-z) := \frac{2}{\sqrt{\pi}}\,\frac{\Gamma(\eta+1)}{\Gamma(\eta+\frac{1}{2})}\int_0^1 (1-t^2)^{\eta-\frac{1}{2}}\,e^{-zt}dt. \tag{2.9}$$

Taking transformation of $z \to zq$ and integrating with respect to q having limit from 0 to ∞, we get

$$\begin{aligned}&\int_0^\infty q^{c-1}\mathbf{S}_\eta(-zq)dq\\ &\quad= \int_0^\infty q^{c-1}\cdot\left\{\frac{2}{\sqrt{\pi}}\,\frac{\Gamma(\eta+1)}{\Gamma(\eta+\frac{1}{2})}\int_0^1 (1-t^2)^{\eta-\frac{1}{2}}\,e^{-zqt}dt\right\}dq.\end{aligned}$$

Since the uniform convergence of the above integral ensures that the double integrals' order can be interchanged, we can ascertain

$$\begin{aligned}&\int_0^\infty q^{c-1}\mathbf{S}_\eta(-zq)dq\\ &= \frac{2}{\sqrt{\pi}}\,\frac{\Gamma(\eta+1)}{\Gamma(\eta+\frac{1}{2})}\int_0^1 (1-t^2)^{\eta-\frac{1}{2}}\left\{\int_0^\infty q^{c-1}\,e^{-zqt}dq\right\}dt.\end{aligned}$$

Applying the well-known Gamma function formula (see [11, 16, 24])

$$\Gamma(\beta)\theta^{-\beta} = \int_0^\infty t^{\beta-1}e^{-\theta t}dt \qquad (\theta,\,\beta \in \mathbb{C}_>),$$

to the above integral expression, then solving the integration, we obtain

$$\int_0^\infty q^{c-1}\mathbf{S}_\eta(-zq)dq = \frac{1}{z^c\,\sqrt{\pi}}\,\frac{\Gamma(c)\,\Gamma\left(\frac{1-c}{2}\right)\,\Gamma(\eta+1)}{\Gamma(\eta-\frac{c}{2}+1)}. \tag{2.10}$$

By substituting $z = \frac{1}{1-t}$, our result in (2.8) is obtained. □

Theorem 2.4. *The following integral formula holds true:*

$$\begin{aligned}&\int_0^\infty\int_0^\infty p^{b-1}q^{c-1}B_{\alpha;\eta}(\mu,\nu;p;q)\\ &= \frac{\Gamma(b)\,\Gamma(1-b)\,\Gamma(c)\,\Gamma\left(\frac{1-c}{2}\right)\,\Gamma(\eta+1)}{\sqrt{\pi}\,\Gamma(1-\alpha b)\,\Gamma(\eta-\frac{c}{2}+1)}\,B(\mu+b,\,\nu+c)\end{aligned} \tag{2.11}$$

$$(p,q,b,c \in \mathbb{C}_>;\,(\mu+b),(\nu+c) \in \mathbb{C}_>;\ \alpha,\,\eta \in \mathbb{C}_{>-}).$$

Proof. By multiplying $p^{b-1}\,q^{c-1}$ on each side of (2.1) and integrating the resultant identity with respect to p and q $(0 \le p, q \le \infty)$, we obtain

(2.12)

$$\int_0^\infty \int_0^\infty p^{b-1} q^{c-1} B_{\alpha;\eta}(\mu,\nu;p;q) = \int_0^\infty \int_0^\infty p^{b-1} q^{c-1} \left\{ \int_0^1 t^{\mu-1}(1-t)^{\nu-1} \mathbf{E}_\alpha\left(\frac{-p}{t}\right) \mathbf{S}_\eta\left(\frac{-q}{1-t}\right) dt \right\} dp\, dq.$$

The order of integration is guaranteed to be switched by uniform convergence of the integral. We therefore get

$$\int_0^\infty \int_0^\infty p^{b-1} q^{c-1} B_{\alpha;\eta}(\mu,\nu;p;q) = \int_0^1 t^{\mu-1}(1-t)^{\nu-1} \cdot \left\{ \int_0^\infty p^{b-1} \mathbf{E}_\alpha\left(\frac{-p}{t}\right) dp \cdot \int_0^\infty q^{c-1} \mathbf{S}_\eta\left(\frac{-q}{1-t}\right) dq \right\} dt.$$

By using (2.6) and (2.8) in above integral, we get

$$\begin{aligned} &\int_0^\infty \int_0^\infty p^{b-1} q^{c-1} B_{\alpha;\eta}(\mu,\nu;p;q) \\ &= \frac{\Gamma(b)\,\Gamma(1-b)\,\Gamma(c)\,\Gamma\left(\frac{1-c}{2}\right)\,\Gamma(\eta+1)}{\sqrt{\pi}\,\Gamma(1-\alpha b)\,\Gamma(\eta-\frac{c}{2}+1)} \int_0^1 t^{\mu+b-1}(1-t)^{\nu+c-1} dt \\ &= \frac{\Gamma(b)\,\Gamma(1-b)\,\Gamma(c)\,\Gamma\left(\frac{1-c}{2}\right)\,\Gamma(\eta+1)}{\sqrt{\pi}\,\Gamma(1-\alpha b)\,\Gamma(\eta-\frac{c}{2}+1)} B(\mu+b,\, \nu+c). \end{aligned}$$

□

Remark 2.5. The special case $\alpha = 1$ and $\eta = \frac{-1}{2}$ of (2.11) and then $b = 1 = c$ in the same equation reduce to two corresponding results in Choi et al. [10].

Theorem 2.6. *Several integral representations for $B_{\alpha;\eta}(\mu,\nu;p;q)$ are valid as follows:*

(2.13)

$$B_{\alpha;\eta}(\mu,\nu;p;q) = 2\int_0^{\frac{\pi}{2}} (\cos^{2\mu-1}\theta)\,(\sin^{2\nu-1}\theta)\,\mathbf{E}_\alpha(-p\sec^2\theta)\mathbf{S}_\eta(-q\csc^2\theta)\,d\theta;$$

(2.14) $$B_{\alpha;\eta}(\mu,\nu;p;q) = \int_0^\infty \frac{\varphi^{\mu-1}}{(1+\varphi)^{\mu+\nu}}\,\boldsymbol{E}_\alpha\left[\frac{-p(1+\varphi)}{\varphi}\right] \boldsymbol{S}_\eta[-q(1+\varphi)]\,d\varphi;$$

(2.15)

$$B_{\alpha;\eta}(\mu,\nu;p;q) = 2^{1-\mu-\nu}\int_{-1}^1 (1+\varphi)^{\mu-1}\,(1-\varphi)^{\nu-1}\,\boldsymbol{E}_\alpha\left[\frac{-2p}{1+\varphi}\right] \boldsymbol{S}_\eta\left[\frac{-2q}{1-\varphi}\right] d\varphi;$$

$$B_{\alpha;\eta}(\mu,\nu;p;q) = (\mathfrak{g}-\mathfrak{f})^{1-\mu-\nu}\int_{\mathfrak{f}}^{\mathfrak{g}} (\varphi-\mathfrak{f})^{\mu-1}(\mathfrak{g}-\varphi)^{\nu-1} \cdot \left\{ \boldsymbol{E}_\alpha\left[\frac{-p(\mathfrak{g}-\mathfrak{f})}{(\varphi-\mathfrak{f})}\right] \mathbf{S}_\eta\left[\frac{-q(\mathfrak{g}-\mathfrak{f})}{(\mathfrak{g}-\varphi)}\right]\right\} d\varphi \tag{2.16}$$
$$(p,q,\mu,\nu\in\mathbb{C}_{>};\alpha,\eta\in\mathbb{C}_{>-}).$$

Proof. With the transformations $t=\cos^2\theta$, $t=\frac{\varphi}{1+\varphi}$, $t=\frac{1+\varphi}{2}$ and $t=\frac{\varphi-\mathfrak{f}}{\mathfrak{g}-f}$ in equation (2.1), the equations (2.13), (2.14), (2.15), and (2.16) can be obtained, respectively. □

Remark 2.7. The corresponding result in [10] is obtained by reducing the integrals in Theorem(2.6) to the specific cases $\alpha=1$ and $\eta=\frac{-1}{2}$. Next, if we assume $p=q$, the integrals in Theorem(2.6) reduce to the equivalent results in [5] following $\alpha=1$ and $\eta=\frac{-1}{2}$. Additionally, it can be deduced with ease that the integrals in Theorem(2.6) yield certain established formulas for the Beta function in the particular situation $p=0=q$.

2.2. Mellin Transforms of $B_{\alpha;\eta}(\mu,\nu;p;q)$

Theorem 2.8. *With respect to $B_{\alpha;\eta}(\mu,\nu;p;q)$, the following Mellin transformation formula is valid:*

(2.17)
$$B_{\alpha;\eta}(\mu,\nu;p;q) = \frac{1}{(2\pi i)^2}\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty}\int_{\varepsilon_2-i\infty}^{\varepsilon_2+i\infty} \frac{\Gamma(x)\,\Gamma(1-x)\,\Gamma(y)\,\Gamma\left(\frac{1-y}{2}\right)\,\Gamma(\eta+1)}{\sqrt{\pi}\,\Gamma(1-\alpha x)\,\Gamma(\eta-\frac{y}{2}+1)} \cdot \frac{\Gamma(\mu+x)\,\Gamma(\nu+y)}{\Gamma(\mu+x+\nu+y)} p^{-x}q^{-y}\,dx\,dy$$

$$(p,\,q\in\mathbb{C}_{>};\ \varepsilon_1>0,\ \varepsilon_2>0;\ \alpha,\ \eta\in\mathbb{C}_{>-}).$$

Proof. With Mellin transform applied to both sides of (2.1), we obtain

(2.18)
$$M\{B_{\alpha;\eta}(\mu,\nu;p;q);p\to x,q\to y\} = \int_0^\infty\int_0^\infty p^{x-1}q^{y-1}\left\{\int_0^1 t^{\mu-1}(1-t)^{\nu-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dt\right\}dp\,dq.$$

The order of integration is guaranteed to be switched by uniform convergence of the integral. We therefore get

$$M\{B_{\alpha;\eta}(\mu,\nu;p;q);p\to x,q\to y\} = \int_0^1 t^{\mu-1}(1-t)^{\nu-1} \cdot\left\{\int_0^\infty p^{x-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)dp\right\}\cdot\left\{\int_0^\infty q^{y-1}\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dq\right\}dt. \tag{2.19}$$

Now putting the results of (2.6) and (2.8) in (2.19) and then applying the definition of the Beta function (1.1), we obtain

$$M\{B_{\alpha;\eta}(\mu,\nu;p;q);p\to x,q\to y\}$$
$$=\frac{\Gamma(x)\,\Gamma(1-x)\,\Gamma(y)\,\Gamma\left(\frac{1-y}{2}\right)\,\Gamma(\eta+1)}{\sqrt{\pi}\,\Gamma(1-\alpha x)\,\Gamma(\eta-\frac{y}{2}+1)}\frac{\Gamma(\mu+x)\,\Gamma(\nu+y)}{\Gamma(\mu+x+\nu+y)}.$$

Next, the inverse image of the Mellin transform is applied to the first and last sides of the above derived identity to prove the equation (2.17). □

2.3. Properties of $B_{\alpha;\eta}(\mu,\nu;p;q)$

Theorem 2.9. *For $B_{\alpha;\eta}(\mu,\nu;p;q)$, the following relation is valid:*

$$B_{\alpha;\eta}(\mu,\nu;p;q)=B_{\eta;\alpha}(\nu,\mu;q;p) \tag{2.20}$$
$$(p,\,q\in\mathbb{C}_{>};\,\alpha,\,\eta\in\mathbb{C}_{>-}).$$

Proof. Using the transformation $t=(1-t)$ in equation (2.1), one may derive the result in (2.20). □

Remark 2.10. The special case $\alpha=1$ and $\eta=\frac{-1}{2}$ in (2.20) brings about an equivalent result in Choi et al. [10]. After putting $\alpha=1$ and $\eta=\frac{-1}{2}$, taking $p=q$ in (2.20) brings about an equivalent result in Chaudhry et al. [5]. For $p=0=q$, the result in (2.20) reduces to the symmetric property of the Beta function.

Theorem 2.11. *For every extended beta function $B_{\alpha;\eta}(\mu,\nu;p;q)$, the following relation is valid:*

$$B_{\alpha;\eta}(\mu,\nu;p;q)=B_{\alpha;\eta}(\mu+1,\nu;p;q)+B_{\alpha;\eta}(\mu,\nu+1;p;q). \tag{2.21}$$

Proof. From (2.1), we have

$$\begin{aligned}B_{\alpha;\eta}(\mu,\nu;p;q)&=\int_0^1 t^{\mu-1}(1-t)^{\nu-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dt\\&=\int_0^1 t^{\mu-1}(1-t)^{\nu-1}\{t+(1-t)\}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dt\\&=\int_0^1 t^{\mu+1-1}(1-t)^{\nu-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dt\\&\quad+\int_0^1 t^{\mu-1}(1-t)^{\nu+1-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dt\end{aligned}$$
$$\Rightarrow B_{\alpha;\eta}(\mu,\nu;p;q)=B_{\alpha;\eta}(\mu+1,\nu;p;q)+B_{\alpha;\eta}(\mu,\nu+1;p;q),$$

which is our desired result. □

Remark 2.12. The special case $\alpha=1$ and $\eta=\frac{-1}{2}$ of (2.21) brings about an equivalent result in [10]. Then, after putting $\alpha=1$ and $\eta=\frac{-1}{2}$, if we take $p=q$ of (2.21) brings about an equivalent result in [5]. Furthermore, it is seen that the identity in (2.21) yields a known relation for the Beta function in the particular case $p = 0 = q$.

Theorem 2.13. *The following summation formula is satisfied by the extended beta function* $B_{\alpha;\eta}(\mu, \nu; p; q)$*:*

$$B_{\alpha;\eta}(\mu, 1-\nu; p; q) = \sum_{k=0}^{\infty} \frac{(\nu)_k}{k!} B_{\alpha;\eta}(\mu + k, 1; p; q) \tag{2.22}$$
$$(p, q \in \mathbb{C}_>;\ \alpha,\ \eta \in \mathbb{C}_{>-}).$$

Proof. The generalized binomial theorem, we have

$$(1-z)^{-\nu} = \sum_{k=0}^{\infty} \frac{(\nu)_k}{k!} z^k \qquad (|z| < 1),$$

where $(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$ is the Pochhammer Symbol.

Therefore (2.1) can be written as

$$\begin{aligned} B_{\alpha;\eta}(\mu, 1-\nu; p; q) &= \int_0^1 t^{\mu-1}(1-t)^{1-\nu-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right) dt \\ &= \int_0^1 t^{\mu-1}(1-t)^{-\nu}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right) dt \\ &= \int_0^1 t^{\mu-1}\left[\sum_{k=0}^{\infty}\frac{(\nu)_k}{k!}t^k\right]\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right) dt \\ &= \sum_{k=0}^{\infty}\frac{(\nu)_k}{k!}\left\{\int_0^1 t^{\mu+k-1}(1-t)^{1-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right) dt\right\} \end{aligned}$$

$$\Rightarrow B_{\alpha;\eta}(\mu, 1-\nu; p; q) = \sum_{k=0}^{\infty}\frac{(\nu)_k}{k!} B_{\alpha;\eta}(\mu+k, 1; p; q).$$

We obtain the stated result (2.22). □

Remark 2.14. The special case $\alpha = 1$ and $\eta = \frac{-1}{2}$ of (2.22) brings about an equivalent result in [10].

Theorem 2.15. *The following summation formula is satisfied by the extended beta function* $B_{\alpha;\eta}(\mu, \nu; p; q)$*:*

$$B_{\alpha;\eta}(\mu, \nu; p; q) = \sum_{k=0}^{\infty} B_{\alpha;\eta}(\mu + k, \nu + 1; p; q) \tag{2.23}$$
$$(p, q \in \mathbb{C}_>; \alpha,\ \eta \in \mathbb{C}_{>-}).$$

Proof. By using the result, we write

$$(1-z)^{\nu-1} = (1-z)^{\nu}\sum_{k=0}^{\infty} z^k \qquad (|z| < 1). \tag{2.24}$$

Using (2.24) in (2.1), we get

$$\begin{aligned} B_{\alpha;\eta}(\mu,\nu;p;q) &= \int_0^1 t^{\mu-1}\left\{(1-t)^{\nu}\sum_{k=0}^{\infty} t^k\right\}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dt \\ &= \sum_{k=0}^{\infty}\left\{\int_0^1 t^{\mu+k-1}(1-t)^{\nu+1-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dt\right\} \\ \Rightarrow B_{\alpha;\eta}(\mu,\nu;p;q) &= \sum_{k=0}^{\infty} B_{\alpha;\eta}(\mu+k,\nu+1;p;q). \end{aligned}$$

We arrive at the provided outcome in (2.23). □

Theorem 2.16. *For every extended beta function* $B_{\alpha;\eta}(\mu,\nu;p;q)$*, the following relation is valid:*

$$B_{\alpha;\eta}(\mu,-\mu-m;p;q) = \sum_{k=0}^{m}\binom{m}{k}B_{\alpha;\eta}(\mu+k,-\mu-k;p;q), \quad (m\in\mathbb{N}_0). \tag{2.25}$$

Proof. Substituting $\nu=-\mu-m$ in the known result given in (2.21)

$$B_{\alpha;\eta}(\mu,\nu;p;q) = B_{\alpha;\eta}(\mu+1,\nu;p;q) + B_{\alpha;\eta}(\mu,\nu+1;p;q),$$

we arrive at

$$B_{\alpha;\eta}(\mu,-\mu-m;p;q) = B_{\alpha;\eta}(\mu+1,-\mu-m;p;q) + B_{\alpha;\eta}(\mu,-\mu-m+1;p;q).$$

Writing this formula recursively with $m=1,2,3,...$, we obtain

$$B_{\alpha;\eta}(\mu,-\mu-1;p;q) = B_{\alpha;\eta}(\mu,-\mu;p;q) + B_{\alpha;\eta}(\mu+1,-\mu-1;p;q),$$

$$\begin{aligned} B_{\alpha;\eta}(\mu,-\mu-2;p;q) &= B_{\alpha;\eta}(\mu,-\mu;p;q) + 2B_{\alpha;\eta}(\mu+1,-\mu-1;p;q) \\ &\quad + B_{\alpha;\eta}(\mu+2,-\mu-2;p;q), \end{aligned}$$

and so on. Taking another step further leads us to (2.25). □

2.4. Log-Convexity and Turán type Inequality of $B_{\alpha;\eta}(\mu,\nu;p;q)$

Theorem 2.17. {*Log-Convexity*} *The following inequalities hold true:*

$$B_{\alpha;\eta}(\lambda\mu_1+(1-\lambda)\mu_2,\nu;p;q) \le B_{\alpha;\eta}(\mu_1,\nu;p;q)^{\lambda}\cdot B_{\alpha;\eta}(\mu_2,\nu;p;q)^{1-\lambda} \tag{2.26}$$

$$(\lambda\in(0,1);\ \mu_1<\mu_2;\ \nu\in\mathbb{C},\ p,q\in\mathbb{C}_{>};\ \alpha,\ \eta\in\mathbb{C}_{>-}),$$

and

$$B_{\alpha;\eta}(\mu,\ \lambda\nu_1+(1-\lambda)\nu_2;p;q) \le B_{\alpha;\eta}(\mu,\nu_1;p;q)^{\lambda}\cdot B_{\alpha;\eta}(\mu,\ \nu_2;p;q)^{1-\lambda} \tag{2.27}$$

$$(\lambda\in(0,1);\ \nu_1<\nu_2;\ \nu\in\mathbb{C},\ p,q\in\mathbb{C}_{>};\ \alpha,\ \eta\in\mathbb{C}_{>-}).$$

Proof. For integrals, the Hölder's inequality [27, Eq. (2)] asserts that

$$\int_a^b |f(t)\,g(t)|\,dt \le \left[\int_a^b |f(t)|^j dt\right]^{\frac{1}{j}} \left[\int_a^b |g(t)|^k dt\right]^{\frac{1}{k}}, \tag{2.28}$$

where, $\frac{1}{j}+\frac{1}{k}=1$ with $j,k>1$. Applying Hölder's inequality of (2.28) and substituting $\mu=\lambda\mu_1+(1-\lambda)\mu_2$ in (2.1), we obtain

$$\begin{aligned}
&B_{\alpha;\eta}(\lambda\mu_1+(1-\lambda)\mu_2,\nu;p;q)\\
&=\int_0^1 t^{(\lambda\mu_1+(1-\lambda)\mu_2)-1}(1-t)^{\nu-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dt\\
&=\int_0^1 \left[t^{\mu_1-1}(1-t)^{\nu-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)\right]^\lambda\\
&\quad\cdot\left[t^{\mu_2-1}(1-t)^{\nu-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)\right]^{1-\lambda} dt\\
&\le\left[\int_0^1 t^{\mu_1-1}(1-t)^{\nu-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dt\right]^\lambda\\
&\quad\cdot\left[\int_0^1 t^{\mu_2-1}(1-t)^{\nu-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dt\right]^{1-\lambda}
\end{aligned}$$

$\Rightarrow B_{\alpha;\eta}(\lambda\mu_1+(1-\lambda)\mu_2,\nu;p;q)\le B_{\alpha;\eta}(\mu_1,\nu;p;q)^\lambda\cdot B_{\alpha;\eta}(\mu_2,\nu;p;q)^{1-\lambda}$.
Hence the result (2.26) is obtained. A similar argument off substituting $\nu=\lambda\nu_1+(1-\lambda)\nu_2$ in (2.1) proves the result in (2.27). □

Corollary 2.18. {*Turán Type Inequality*} *The following inequalities hold true:*

$$B_{\alpha;\eta}\left(\frac{\mu_1+\mu_2}{2},\nu;p;q\right)^2 - B_{\alpha;\eta}(\mu_1,\nu;p;q)\cdot B_{\alpha;\eta}(\mu_2,\nu;p;q)\le 0 \tag{2.29}$$

$$(\mu_1<\mu_2;\ \nu\in\mathbb{C},\ p,q\in\mathbb{C}_>;\ \alpha,\ \eta\in\mathbb{C}_{>-}),$$

and

$$B_{\alpha;\eta}\left(\mu,\frac{\nu_1+\nu_2}{2};p;q\right)^2 - B_{\alpha;\eta}(\mu,\nu_1;p;q)\cdot B_{\alpha;\eta}(\mu,\nu_2;p;q)\le 0 \tag{2.30}$$

$$(\nu_1<\nu_2;\ \nu\in\mathbb{C},\ p,q\in\mathbb{C}_>;\ \alpha,\ \eta\in\mathbb{C}_{>-}).$$

Proof. For $\lambda=\frac{1}{2}$ in (2.26) and (2.27), above equations (2.29) and (2.30) can be obtained, respectively. □

Remark 2.19. The special case $\alpha=1$ and $\eta=\frac{-1}{2}$ of (2.26) brings about an equivalent result in Luo et al. [14, Eq. (15)]. By setting $p=0=q$, the well-known result that we get

$$B(\lambda\mu_1+(1-\lambda)\mu_2,\,\nu)\ \le\ B(\mu_1,\nu)^\lambda\cdot B(\mu_2,\nu)^{1-\lambda}. \tag{2.31}$$

3. A Generalized Extended Beta Distribution

We define an extended beta distribution in statistical distribution theory as follows:

$$f(\nu) = \begin{cases} \frac{1}{B_{\alpha;\eta}(\mu,\nu;p;q)} t^{\mu-1}(1-t)^{\nu-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right), & \text{if } (0<t<1) \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

$$(p,\, q>0;\, -\infty<\mu,\nu<\infty;\, \alpha,\, \eta\in\mathbb{C}_{>-}).$$

We next go through a few basic characteristics of the extended beta distribution (3.1).

The k^{th} moment of X, if k is any real number, is provided by

$$E(X^k) = \frac{B_{\alpha;\eta}(\mu+k,\nu;p;q)}{B_{\alpha;\eta}(\mu,\nu;p;q)} \tag{3.2}$$

$$(\mu,\nu\in\mathbb{R},\ p,q\in\mathbb{R}^+,\ \alpha,\ \eta\in\mathbb{C}_{>-}).$$

For $k=1$, the specific instance of (3.2) produces the mean of our suggested extended beta distribution, which is:

$$E(X) = \frac{B_{\alpha;\eta}(\mu+1,\nu;p;q)}{B_{\alpha;\eta}(\mu,\nu;p;q)}. \tag{3.3}$$

The variance of the distribution we proposed may be represented as follows:

$$\begin{aligned} Var(X) &= E(X^2)-[E(X)]^2 = E[(X-E(X))^2] \\ &= \frac{B_{\alpha;\eta}(\mu+2,\nu;p;q)\,B_{\alpha;\eta}(\mu,\nu;p;q)\ -[B_{\alpha;\eta}(\mu+1,\nu;p;q)]^2}{[B_{\alpha;\eta}(\mu,\nu;p;q)]^2}. \end{aligned} \tag{3.4}$$

The ratio of the standard deviation to the mean termed the coefficient of variation, may be represented as follows for this distribution:

$$C.V = \sqrt{\frac{B_{\alpha;\eta}(\mu+2,\nu;p;q)\,B_{\alpha;\eta}(\mu,\nu;p;q)}{B_{\alpha;\eta}(\mu+1,\nu;p;q)}-1}. \tag{3.5}$$

About the inception of this distribution, the moment generating function (m.g.f.), is provided by

$$M_X(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}E(X^k),$$

whence

$$M_X(t) = \frac{1}{B_{\alpha;\eta}(\mu,\nu;p;q)}\sum_{k=0}^{\infty}B_{\alpha;\eta}(\mu+k,\nu;p;q)\,\frac{t^k}{k!}. \tag{3.6}$$

It is possible to compute the proposed distribution's characteristic function in the following way:

$$E(e^{it\mu}) = \sum_{k=0}^{\infty}\frac{i^k t^k}{k!}\,E(X^k),$$

whence

$$E(e^{it\mu}) = \frac{1}{B_{\alpha;\eta}(\mu,\nu;p;q)} \sum_{k=0}^{\infty} B_{\alpha;\eta}(\mu+k,\nu;p;q)\frac{i^k t^k}{k!}. \tag{3.7}$$

Our suggested extended beta distribution (3.1) has the following cumulative distribution function:

$$F(a) = p[X < a] = \int_0^a f(a)da,$$

so as to

$$F(a) = \frac{B_{\alpha;\eta;a}(\mu,\nu;p;q)}{B_{\alpha;\eta}(\mu,\nu;p;q)}, \tag{3.8}$$

wherein $B_{\alpha;\eta;a}(\mu,\nu;p;q)$ represents an incomplete extended beta function as described by

$$B_{\alpha;\eta;a}(\mu,\nu;p;q) = \int_0^a t^{\mu-1}(1-t)^{\nu-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dt \tag{3.9}$$
$$(p,\ q \in \mathbb{C}_>;\ -\infty < \mu,\nu < \infty;\ \alpha,\ \eta \in \mathbb{C}_{>-}).$$

Our suggested distribution's reliability function, which is just the cumulative distribution function's complement, is provided by

$$R(a) = p[X \geq a] = 1 - F(a) = \int_a^\infty f(a)da,$$

so that

$$R(a) = \frac{\hat{B}_{\alpha;\eta;a}(\mu,\nu;p;q)}{B_{\alpha;\eta}(\mu,\nu;p;q)}. \tag{3.10}$$

However, the (upper) incomplete extended beta function $\hat{B}_{\alpha;\eta;a}(\mu,\nu;p;q)$ is defined by

$$\hat{B}_{\alpha;\eta;a}(\mu,\nu;p;q) = \int_a^\infty t^{\mu-1}(1-t)^{\nu-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)dt \tag{3.11}$$
$$(p,\ q > 0;\ -\infty < \mu,\nu < \infty;\ \alpha,\ \eta \in \mathbb{C}_{>-}).$$

4. Generalized Extension of Gauss' and Confluent Hypergeometric Functions

Using $B_{\alpha;\eta}(\mu,\nu;p;q)$, we extend Gauss' and Confluent hypergeometric functions as outlined in this section:

$$F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q) = \sum_{k=0}^{\infty} \frac{B_{\alpha;\eta}(\gamma+k,\xi-\gamma;p;q)}{B(\gamma,\xi-\gamma)}(\delta)_k \frac{x^k}{k!} \tag{4.1}$$
$$(p,q \geq 0,\ |x| < 1;\ \xi,\ \gamma \in \mathbb{C}_{>>};\ \alpha,\ \eta \in \mathbb{C}_{>-}),$$

and

$$\Phi_{\alpha;\eta}(\gamma;\xi;x;p;q) = \sum_{k=0}^{\infty} \frac{B_{\alpha;\eta}(\gamma+k,\xi-\gamma;p;q)}{B(\gamma,\xi-\gamma)} \frac{x^k}{k!} \tag{4.2}$$
$$(p,q \geq 0;\ \xi,\ \gamma \in \mathbb{C}_{>>};\ \alpha,\ \eta \in \mathbb{C}_{>-}).$$

The functions $F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q)$ and $\Phi_{\alpha;\eta}(\gamma;\xi;x;p;q)$ are used in this context represents a further generalization of the extended Gauss' hypergeometric function and a further generalization of the extended Confluent hypergeometric function, respectively.

4.1. Integral Representations

Theorem 4.1. *For $p,q \geq 0$, the integral formulae listed below are valid:*

$$F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q) = \frac{1}{B(\gamma,\ \xi-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{\xi-\gamma-1}(1-xt)^{-\delta} \cdot \boldsymbol{E}_\alpha\left(\frac{-p}{t}\right) \boldsymbol{S}_\eta\left(\frac{-q}{1-t}\right) dt \tag{4.3}$$
$$(|\arg(1-x)| < \pi;\ \xi,\ \gamma \in \mathbb{C}_{>>};\ \alpha,\ \eta \in \mathbb{C}_{>-}).$$

Proof. By using the definition of (2.1) in place of the $B_{\alpha;\eta}(\gamma+k,\ \xi-\gamma;p;q)$ in (4.1), it is simple to derive the integral formula

$$F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q) = \frac{1}{B(\gamma,\ \xi-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{\xi-\gamma-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right) \cdot \mathbf{S}_\eta\left(\frac{-q}{1-t}\right)\left\{\sum_{k=o}^{\infty}(\delta)_k \frac{(xt)^k}{k!}\right\} dt \tag{4.4}$$
$$(|x| < 1;\ \xi,\ \gamma \in \mathbb{C}_{>>};\ \alpha,\ \eta \in \mathbb{C}_{>-}).$$

Making use of the extended binomial expansion, $(1-xt)^{-\delta} = \sum_{k=0}^{\infty}(\delta)_k \frac{(xt)^k}{k!}$ in (4.4), we find the integral in (4.3). □

Theorem 4.2. *For $p,q \geq 0$, the integral formulae listed below are valid:*

$$F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q) = \frac{1}{B(\gamma,\ \xi-\gamma)} \int_0^\infty \varpi^{\gamma-1}(1+\varpi)^{\delta-\xi}[1+\varpi(1-x)]^{-\delta} \cdot \boldsymbol{E}_\alpha\left[\frac{-p(1+\varpi)}{\varpi}\right] \boldsymbol{S}_\eta\left[-q(1+\varpi)\right] d\varpi \tag{4.5}$$
$$(|\arg(1-x)| < \pi;\ \xi,\ \gamma \in \mathbb{C}_{>>};\ \alpha,\ \eta \in \mathbb{C}_{>-});$$

$$F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q) = \frac{2}{B(\gamma,\ \xi-\gamma)} \int_0^{\frac{\pi}{2}} \frac{\sin^{2\gamma-1}\varpi\ \cos^{2\xi-2\gamma-1}\varpi}{(1-x\sin^2\varpi)^\delta} \cdot \boldsymbol{E}_\alpha(-p\csc^2\varpi)\, \boldsymbol{S}_\eta(-p\sec^2\varpi)\, d\varpi \tag{4.6}$$

$$(|\arg(1-x)|<\pi;\ \xi,\ \gamma \in \mathbb{C}_{>>};\ \alpha,\ \eta \in \mathbb{C}_{>-});$$

(4.7)

$$F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q) = \frac{2^{1-\delta-\xi}}{B(\gamma,\,\xi-\gamma)} \int_{-1}^{1} (1+\varpi)^{\gamma-1}(1-\varpi)^{\xi-\gamma-1}[2-x(1+\varpi)]^{-\delta} \cdot \boldsymbol{E}_\alpha\left[\frac{-2p}{1+\varpi}\right] \boldsymbol{S}_\eta\left[\frac{-2q}{1-\varpi}\right] d\varpi$$

$$(|\arg(1-x)|<\pi;\ \xi,\ \gamma \in \mathbb{C}_{>>};\ \alpha,\ \eta \in \mathbb{C}_{>-}).$$

Proof. Setting $t = \frac{\varpi}{1+\varpi}$, $t = \sin^2 \varpi$ and $t = \frac{1+\varpi}{2}$ in (4.3) yields the integrals in (4.5), (4.6), and (4.7), respectively. □

Note 4.3. A similar argument in (4.2) will establish the integral (4.8). The integral is,

$$\Phi_{\alpha;\eta}(\gamma;\xi;x;p;q) = \frac{1}{B(\gamma,\,\xi-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{\xi-\gamma-1}\, e^{xt} \cdot \mathbf{E}_\alpha\left(\frac{-p}{t}\right) \mathbf{S}_\eta\left(\frac{-q}{1-t}\right) dt \tag{4.8}$$

$$(p,q \geq 0;\ \xi,\ \gamma \in \mathbb{C}_{>>};\ \alpha,\ \eta \in \mathbb{C}_{>-}).$$

Remark 4.4. The integrals in (4.3)-(4.8) have special cases $\alpha = 1$ and $\eta = \frac{-1}{2}$, which result in the corresponding integral representations provided in Choi et al.[10]. Upon setting $\alpha = 1$ and $\eta = \frac{-1}{2}$, the corresponding integral representations found in Chaudhry et al.[6] can potentially be obtained by taking $p = q$ of the integrals in (4.3)-(4.8). It is clear that the integral representations in (4.3)–(4.8) reduce to those integrals in the special case of $p = 0 = q$ [26].

Theorem 4.5. *The following relation for $F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q)$ holds true:*

$$F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q) = F_{\alpha;\eta}(\delta,\gamma+1;\xi;x;p;q) + F_{\alpha;\eta}(\delta,\gamma;\xi+1;x;p;q) \tag{4.9}$$

$$(p,q \geq 0;\ |x|<1;\ \xi,\ \gamma \in \mathbb{C}_{>>};\ \alpha,\ \eta \in \mathbb{C}_{>-}).$$

Proof. We have from (4.3)

$$F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q) = \frac{1}{B(\gamma,\,\xi-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{\xi-\gamma-1}(1-xt)^{-\delta} \cdot \mathbf{E}_\alpha\left(\frac{-p}{t}\right) \mathbf{S}_\eta\left(\frac{-q}{1-t}\right) dt$$

$$= \frac{1}{B(\gamma,\,\xi-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{\xi-\gamma-1}[t+(1-t)](1-xt)^{-\delta} \cdot \mathbf{E}_\alpha\left(\frac{-p}{t}\right) \mathbf{S}_\eta\left(\frac{-q}{1-t}\right) dt$$

$$=\frac{1}{B(\gamma,\,\xi-\gamma)}\int_0^1 t^{\gamma+1-1}(1-t)^{\xi-\gamma-1}(1-xt)^{-\delta}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)\,dt$$
$$+\frac{1}{B(\gamma,\,\xi-\gamma)}\int_0^1 t^{\gamma-1}(1-t)^{\xi+1-\gamma-1}(1-xt)^{-\delta}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)\,dt$$
$$\Rightarrow F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q)=F_{\alpha;\eta}(\delta,\gamma+1;\xi;x;p;q)+F_{\alpha;\eta}(\delta,\gamma;\xi+1;x;p;q).$$

□

Note 4.6. An analogous argument will prove the relation for $\Phi_{\alpha;\eta}(\gamma;\xi;x;p;q)$ as:

(4.10)
$$\Phi_{\alpha;\eta}(\gamma;\xi;x;p;q)=\Phi_{\alpha;\eta}(\gamma+1;\xi;x;p;q)+\Phi_{\alpha;\eta}(\gamma;\xi+1;x;p;q)$$
$$(p,q\geq 0\,;\,\xi,\,\gamma\in\mathbb{C}_{>>};\,\alpha,\,\eta\in\mathbb{C}_{>-}).$$

4.2. Mellin Transforms

Theorem 4.7. *For* $p,q\in\mathbb{C}_{>}$; $\epsilon_1>0$, $\epsilon_2>0$; $\xi,\,\gamma\in\mathbb{C}_{>>}$ *and* $\alpha,\,\eta\in\mathbb{C}_{>-}$ *the Mellin-Barnes integral formulas listed below are valid:*

(4.11)
$$F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q)=\frac{1}{(2\pi i)^2\,B(\gamma,\,\xi-\gamma)}\int_{\epsilon_1-i\infty}^{\epsilon_1+i\infty}\int_{\epsilon_2-i\infty}^{\epsilon_2+i\infty}$$
$$\cdot\left\{\frac{\Gamma(\phi)\,\Gamma(1-\phi)\,\Gamma(\psi)\,\Gamma\left(\frac{1-\psi}{2}\right)\,\Gamma(\eta+1)}{\sqrt{\pi}\,\Gamma(1-\alpha\,\psi)\,\Gamma(\eta-\frac{\psi}{2}+1)}\right\}\cdot B(\gamma+\phi,\,\xi+\psi-\gamma)$$
$$\cdot F(\delta,\,\gamma+\phi;\,\xi+\phi+\psi;\,x)\,p^{-\phi}q^{-\psi}d\phi\,d\psi,$$

and

(4.12)
$$\Phi_{\alpha;\eta}(\gamma;\xi;x;p;q)=\frac{1}{(2\pi i)^2\,B(\gamma,\,\xi-\gamma)}\int_{\epsilon_1-i\infty}^{\epsilon_1+i\infty}\int_{\epsilon_2-i\infty}^{\epsilon_2+i\infty}$$
$$\cdot\left\{\frac{\Gamma(\phi)\,\Gamma(1-\phi)\,\Gamma(\psi)\,\Gamma\left(\frac{1-\psi}{2}\right)\,\Gamma(\eta+1)}{\sqrt{\pi}\,\Gamma(1-\alpha\,\phi)\,\Gamma(\eta-\frac{\psi}{2}+1)}\right\}\cdot B(\gamma+\phi,\,\xi+\psi-\gamma)$$
$$\cdot\Phi(\gamma+\phi;\,\xi+\phi+\psi;\,x)\,p^{-\phi}q^{-\psi}\,d\phi\,d\psi.$$

Proof. Multiplying both sides in (4.3) by $p^{\phi-1}\,q^{\psi-1}$, then integrate the resultant identity from 0 to ∞ with regard to p and q, we obtain

$$M\left\{F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q);\,\phi,\,\psi\right\}=\int_0^\infty\int_0^\infty p^{\phi-1}\,q^{\psi-1}\,F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q)\,dp\,dq$$

(4.13)
$$=\frac{1}{B(\gamma,\,\xi-\gamma)}\int_0^1 t^{\gamma-1}(1-t)^{\xi-\gamma-1}(1-xt)^{-\delta}$$
$$\cdot\left[\int_0^\infty p^{\phi-1}\mathbf{E}_\alpha\left(\frac{-p}{t}\right)\,dp\cdot\int_0^\infty q^{\psi-1}\mathbf{S}_\eta\left(\frac{-q}{1-t}\right)\,dq\right]dt.$$

By using of (2.6) and (2.8) in (4.13), we obtain

$$
\begin{aligned}
&M\left\{F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q);\ \phi,\ \psi\right\}\\
&=\left[\frac{\Gamma(\phi)\,\Gamma(1-\phi)\,\Gamma(\psi)\,\Gamma\left(\frac{1-\psi}{2}\right)\,\Gamma(\eta+1)}{\sqrt{\pi}\,\Gamma(1-\alpha\,\phi)\,\Gamma(\eta-\frac{\psi}{2}+1)}\right]\\
&\cdot\frac{1}{B(\gamma,\,\xi-\gamma)}\int_0^1 t^{\gamma+\phi-1}(1-t)^{\xi+\psi-\gamma-1}(1-xt)^{-\delta}dt\\
&=\left[\frac{\Gamma(\phi)\,\Gamma(1-\phi)\,\Gamma(\psi)\,\Gamma\left(\frac{1-\psi}{2}\right)\,\Gamma(\eta+1)}{\sqrt{\pi}\,\Gamma(1-\alpha\,\phi)\,\Gamma(\eta-\frac{\psi}{2}+1)}\right]\\
&\cdot\frac{B(\gamma+\phi,\,\xi+\psi-\gamma)}{B(\gamma,\,\xi-\gamma)}\,F(\delta,\,\gamma+\phi;\,\xi+\phi+\psi;\,x).
\end{aligned}
\tag{4.14}
$$

Now, (4.11) is proved by obtaining the inverse Mellin transforms of each side of (4.14). An analogous argument will prove (4.12). □

Remark 4.8. The special case $\alpha=1$ and $\eta=\frac{-1}{2}$ of (4.11) and (4.12) lead to corresponding results in Choi et al. (see[10, Eq. (10.1) and (10.2)]).

4.3. Differentiation Formulas

Differentiating (4.1) and (4.2) with regard to x and using the following formulas will provide the differentiation formulas for generalized extended hypergeometric functions:

$$B(\gamma,\,\xi-\gamma)=\frac{\xi}{\gamma}B(\gamma+1,\,\xi-\gamma)\quad \textit{and}\quad (\chi)_{k+1}=\chi(\chi+1)_k.$$

Theorem 4.9. *The differentiation formulas listed below are valid:*

$$\frac{d}{dx}F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q)=\frac{\delta\,\gamma}{\xi}F_{\alpha;\eta}(\delta+1,\gamma+1;\xi+1;x;p;q), \tag{4.15}$$

$$\frac{d^k}{dx^k}F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q)=\frac{(\delta)_k\,(\gamma)_k}{(\xi)_k}F_{\alpha;\eta}(\delta+k,\gamma+k;\xi+k;x;p;q)\qquad (k\in\mathbb{N}_0), \tag{4.16}$$

$$\frac{d^k}{dx^k}\Phi_{\alpha;\eta}(\gamma;\xi;x;p;q)=\frac{(\gamma)_k}{(\xi)_k}\Phi_{\alpha;\eta}(\gamma+k;\xi+k;x;p;q)\qquad (k\in\mathbb{N}_0). \tag{4.17}$$

Proof. When we differentiate (4.1) with regard to x, we obtain

$$\frac{d}{dx}F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q)=\sum_{k=1}^{\infty}\frac{B_{\alpha;\eta}(\gamma+k,\,\xi-\gamma;p;q)}{B(\gamma,\,\xi-\gamma)}\,(\delta)_k\,\frac{x^{k-1}}{(k-1)!}.$$

This demonstrates (4.15) after substituting k by $k+1$ along with (4.3). Applying this procedure repeatedly results in the generic form (4.16). An analogous argument demonstrates (4.17) for $\Phi_{\alpha;\eta}(\gamma;\xi;x;p;q)$. □

Remark 4.10. The corresponding results of (4.16) and (4.17) are obtained for the particular cases $\alpha = 1$ and $\eta = \frac{-1}{2}$. These are reported in Choi et al.[10]. Chaudhry et al. [6] provided the corresponding outcomes for the particular case $p = q$ of (4.16) and (4.17) (after setting $\alpha = 1$ and $\eta = \frac{-1}{2}$). It is clear that the equivalent formulas for hypergeometric functions[26] are obtained in the special case $p = 0 = q$ of (4.16) and (4.17).

4.4. Transformation Formulas

Theorem 4.11. *The transformation formulas shown below are valid:*

$$F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q) = (1-x)^{-\delta} F_{\alpha;\eta}\left(\delta,\xi-\gamma;\xi;\frac{-x}{1-x};q;p\right) \tag{4.18}$$

$$(p,q \geq 0;\ |\arg(1-x)| < \pi;\ \xi,\ \gamma \in \mathbb{C}_{>>};\ \alpha,\ \eta \in \mathbb{C}_{>-}),$$

and

$$\Phi_{\alpha;\eta}(\gamma;\xi;x;p;q) = e^x \Phi_{\alpha;\eta}(\xi-\gamma;\xi;-x;q;p) \tag{4.19}$$

$$(p,q \geq 0;\ \xi,\ \gamma \in \mathbb{C}_{>>};\ \alpha,\ \eta \in \mathbb{C}_{>-}).$$

Proof. After substituting t by $(1-t)$ in (4.3) and using

$$[1-x(1-t)]^{-\delta} = (1-x)^{-\delta}\left[1+\frac{x}{1-x}t\right]^{-\delta},$$

we have

$$F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q) = \frac{(1-x)^{-\delta}}{B(\gamma,\,\xi-\gamma)}\int_0^1 t^{\xi-\gamma-1}(1-t)^{\gamma-1}\left(1+\frac{x}{1-x}t\right)^{-\delta} \cdot \mathbf{E}_\alpha\left(\frac{-p}{1-t}\right)\mathbf{S}_\eta\left(\frac{-q}{t}\right)dt$$

$$\Rightarrow F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q) = (1-x)^{-\delta}F_{\alpha;\eta}\left(\delta,\xi-\gamma;\xi;\frac{-x}{1-x};q;p\right).$$

We may also establish (4.19) in a same manner. □

Theorem 4.12. *For $F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q)$, the following generating function is valid:*

$$\sum_{m=0}^{\infty}(\delta)_m\, F_{\alpha;\eta}(\delta+m,\gamma;\xi;x;p;q)\frac{z^m}{m!} = (1-z)^{-\delta}\, F_{\alpha;\eta}\left(\delta,\gamma;\xi;\frac{x}{1-z};p;q\right) \tag{4.20}$$

$$(p,\ q \geq 0;\ \xi,\ \gamma \in \mathbb{C}_{>>};\ \alpha,\ \eta \in \mathbb{C}_{>-};\ |z| < 1).$$

Proof. Assume that the left-hand side of (4.20) is Ψ. Because of (4.1), we have

$$\Psi = \sum_{m=0}^{\infty}(\delta)_m\left[\sum_{k=0}^{\infty}\frac{(\delta+m)_k\, B_{\alpha;\eta}(\gamma+k,\,\xi-\gamma;p;q)}{B(\gamma,\,\xi-\gamma)}\,\frac{x^k}{k!}\right]\frac{z^m}{m!}.$$

In the foregoing formula, we can now obtain by using the identity $(a)_m(a+m)_k = (a)_k(a+k)_m$ as:

$$\Psi = \sum_{k=0}^{\infty} \frac{(\delta)_k\, B_{\alpha;\eta}(\gamma+k,\, \xi-\gamma; p; q)}{B(\gamma,\, \xi-\gamma)} \left[\sum_{m=0}^{\infty} (\delta+k)_m \frac{z^m}{m!}\right] \frac{x^k}{k!}.$$

On employing the binomial theorem to the inner summation, we obtain

$$\begin{aligned} \Psi &= \sum_{k=0}^{\infty} \frac{(\delta)_k\, B_{\alpha;\eta}(\gamma+k,\, \xi-\gamma; p; q)}{B(\gamma,\, \xi-\gamma)} (1-z)^{-\delta-k} \frac{x^k}{k!} \\ &= (1-z)^{-\delta} \sum_{k=0}^{\infty} \frac{(\delta)_k\, B_{\alpha;\eta}(\gamma+k,\, \xi-\gamma; p; q)}{B(\gamma,\, \xi-\gamma)\, k!} \left(\frac{x}{1-z}\right)^k \\ &= (1-z)^{-\delta} F_{\alpha;\eta}\left(\delta, \gamma; \xi; \frac{x}{1-z}; p; q\right) \\ \Rightarrow \Psi &= (1-z)^{-\delta}\, F_{\alpha;\eta}\left(\delta, \gamma; \xi; \frac{x}{1-z}; p; q\right), \end{aligned}$$

which is the right-hand side of (4.20). □

Theorem 4.13. *The transformation formulas shown below are valid:*

$$F_{\alpha;\eta}\left(\delta, \gamma; \xi; 1-\frac{1}{x}; p; q\right) = x^{\delta} F_{\alpha;\eta}(\delta, \xi-\gamma; \xi; 1-x; q; p) \tag{4.21}$$

$$(p, q \geq 0;\ |\arg(1-x)| < \pi;\ \xi,\ \gamma \in \mathbb{C}_{>>};\ \alpha,\ \eta \in \mathbb{C}_{>-}),$$

and

$$F_{\alpha;\eta}\left(\delta, \gamma; \xi; \frac{x}{1+x}; p; q\right) = (1+x)^{\delta}\, F_{\alpha;\eta}(\delta, \xi-\gamma; \xi; -x; q; p) \tag{4.22}$$

$$(p, q \geq 0;\ |\arg(1+x)| < \pi;\ \xi,\ \gamma \in \mathbb{C}_{>>};\ \alpha,\ \eta \in \mathbb{C}_{>-}).$$

Proof. Replacing x by $\left(1-\frac{1}{x}\right)$ and $\left(\frac{x}{1+x}\right)$ in (4.18) yields (4.21) and (4.22), respectively. □

4.5. Differential and difference relations

Theorem 4.14. *The subsequent relations are valid:*

$$\Delta_a\, F_{\alpha;\eta}(\delta, \gamma; \xi; x; p; q) = \frac{\gamma\, x}{\xi}\, F_{\alpha;\eta}(\delta+1, \gamma+1; \xi+1; x; p; q), \tag{4.23}$$

and

$$\frac{d}{dx} F_{\alpha;\eta}(\delta, \gamma; \xi; x; p; q) = \frac{\delta}{x}\, \Delta_{\delta} F_{\alpha;\eta}(\delta, \gamma; \xi; x; p; q). \tag{4.24}$$

The difference operator defined by Δ_θ *is indicated here as*

$$\Delta_\theta\, f(\theta, ...) = f(\theta+1, ...) - f(\theta, ...).$$

Proof. The difference operator that is derived from (4.3) yields

$$\Delta_a F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q) = F_{\alpha;\eta}(\xi+1,\gamma;\xi;x;p;q) - F_{\alpha;\eta}(\delta,\gamma;\xi;x;p;q)$$

$$= \frac{x}{B(\gamma,\,\xi-\gamma)} \int_0^1 t^{\gamma}(1-t)^{\xi-\gamma-1}(1-xt)^{-\delta-1} \cdot \mathbf{E}_{\alpha}\left(\frac{-p}{t}\right)\mathbf{S}_{\eta}\left(\frac{-q}{1-t}\right)dt. \tag{4.25}$$

By substituting $\delta+1, \gamma+1,$ and $\xi+1$ for the parameters $\delta, \gamma,$ and ξ in (4.3), respectively, we obtain

$$F_{\alpha;\eta}(\delta+1,\gamma+1;\xi+1;x;p;q) = \frac{\xi}{\gamma}\,\frac{1}{B(\gamma,\,\xi-\gamma)} \int_0^1 t^{\gamma}(1-t)^{\xi-\gamma-1}(1-xt)^{-\delta-1} \cdot \mathbf{E}_{\alpha}\left(\frac{-p}{t}\right)\mathbf{S}_{\eta}\left(\frac{-q}{1-t}\right)dt. \tag{4.26}$$

Now using (4.26) in (4.25) seems to yield (4.23). Afterwards, (4.24) is proved by applying the differentiation formula (4.15) in (4.23). □

Note 4.15. By using the similar argument of the Theorem 4.14, we obtain the following relations:

$$\gamma\Delta_{\gamma}\,\Phi_{\alpha;\eta}(\gamma;\xi+1;x;p;q) + \xi\Delta_{\xi}\,\Phi_{\alpha;\eta}(\gamma;\xi;x;p;q) = 0; \tag{4.27}$$

and

$$\frac{d}{dx}\Phi_{\alpha;\eta}(\gamma;\xi;x;p;q) = \frac{\gamma}{\xi}\,\Phi_{\alpha;\eta}(\gamma;\xi+1;x;p;q) - \Delta_{\xi}\Phi_{\alpha;\eta}(\gamma;\xi;x;p;q). \tag{4.28}$$

Remark 4.16. The special case $\alpha = 1$ and $\eta = \frac{-1}{2}$ of equations in Theorem 4.14 and Note 4.15 lead to corresponding outcomes provided in Choi et al.[10]. After solving the preceding equations for $\alpha = 1$ and $\eta = \frac{-1}{2}$, the specific case $p = q$ yields the findings that are correspondingly reported in Chaudhry et al.[6].

4.6. Summation formula

Gauss formulated the following summation formula(see[10, Eq.13.1]):

$${}_2F_1(\delta,\gamma;\xi;1) = \frac{\Gamma(\xi)\,\Gamma(\xi-\delta-\gamma)}{\Gamma(\xi-\delta)\,\Gamma(\xi-\delta)} = \frac{B(\gamma,\,\xi-\delta-\gamma)}{B(\gamma,\,\xi-\gamma)} \qquad ((\xi-\delta-\gamma)\in\mathbb{C}_{>}). \tag{4.29}$$

In 2014, Choi et al. (see[10, Eq.13.2]) have obtained the following summation formula:

$$F_{p,q}(\delta,\gamma;\xi;1) = \frac{B(\gamma,\,\xi-\delta-\gamma;p,q)}{B(\gamma,\,\xi-\gamma)} \qquad (p,q\geq 0;\ (\xi-\delta-\gamma)\in\mathbb{C}_{>}). \tag{4.30}$$

Theorem 4.17. *The summation formula shown below is valid:*

$$F_{\alpha;\eta}(\delta,\gamma;\xi;1;p;q) = \frac{B_{\alpha;\eta}(\gamma,\,\xi-\delta-\gamma;p;q)}{B(\gamma,\,\xi-\gamma)} \tag{4.31}$$

$$(p,q \geq 0;\ (\xi-\delta-\gamma) \in \mathbb{C}_{>>},\ \alpha,\ \eta \in \mathbb{C}_{>>}).$$

Proof. The summation formula (4.31) is produced by taking $x = 1$ in (4.3) and applying (2.1). □

Remark 4.18. To get at (4.30), consider the specific cases $\alpha = 1$ and $\eta = \frac{-1}{2}$ of (4.31). After setting $\alpha = 1$ and $\eta = \frac{-1}{2}$, the particular case of $p = q$ of (4.31) yields the analogous result found in Chaudhry et al. [6]. It is clear that (4.31) for $p = 0 = q$ reduces to (4.29), the Gauss' summation formula.

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References

[1] R. P. Agarwal, A. Kılıçman, R. K. Parmar, and A. K. Rathie, *Certain generalized fractional calculus formulas and integral transforms involving* (p,q)*–Mathieu-type series,* Adv. Differ. Equ. **221** (2019), 1–11.

[2] Á. Baricz, S. R. Mondal, and A. Swaminathan, *Monotonicity properties of the Bessel-Struve kernel,* Bull. Korean Math. Soc. **53** (2016), 1845–1856.

[3] B. C. Carlson, *Special Functions of Applied Mathematics,* Academic Press, New York, 1977.

[4] M. A. Chaudhry and S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, CRC Press (Chapman and Hall), Boca Raton, FL, 2002.

[5] M. A. Chaudhry, A. Qadir, M. Rafique and S. M. Zubair, *Extension of Euler's beta function,* J. Comput. Appl. Math. **78** (1997), 19–32.

[6] M. A. Chaudhry, A. Qadir, H. M. Srivastava, and R. B. Paris, *Extended hypergeometric and confluent hypergeometric functions,* Appl. Math. Comput. **159** (2004), 589–602.

[7] J. Choi and R. K. Parmar, *Fractional Integration And Differentiation of the* (p,q)*–extended Bessel function,* Bull. Korean Math. Soc. **55** (2018), no. 2, 599–610.

[8] J. Choi and R. K. Parmar, *Fractional calculus of the* (p,q)*–extended Struve function,* Far East Journal of Mathematical Sciences **103** (2018), no. 2, 541–559.

[9] J. Choi, R. K. Parmar, and T. K. Pogány, *Mathieu-type series built by* (p,q)*–extended Gaussian hypergeometric function,* Bull. Korean Math. Soc. **54** (2017), no. 3, 789–797.

[10] J. Choi, A. K. Rathie, and R. K. Parmar, *Extension of extended betahypergeometric and confluent hypergeometric functions,* Honam Mathematical J. **36** (2014), no. 2, 339–367.

[11] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications*, Third edition, Chapman and Hall (CRC Press), Taylor and Francis Group, London and New York, 2014.

[12] A. Gasmi and M. Sifi, *The Bessel–Struve intertwining operator on* $\mathbb{C}$ *and mean-periodic functions,* Int. J. Math. Math. Sci. **57-60** (2004), 3171–3185.

[13] A. Gasmi and F. Soltani, *Fock spaces for the Bessel–Struve kernel,* J. Anal. Appl. **3** (2005), 91–106.

[14] M. J. Luo, R. K. Parmar, and R. K. Raina, *On extended Hurwitz–Lerch zeta function,* J. Math. Anal. Appl. **448** (2017), 1281–1304.

[15] D. J. Maširević, R. K. Parmar, and T. K. Pogány, (p,q)–*extended Bessel and modified Bessel functions of the first kind,* Results in Mathematics, **72** (2017), 617–632.
[16] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C.W.C (eds.), *NIST Handbook of Mathematical Functions,* Cambridge: Cambridge University Press, 2010.
[17] R. K. Parmar and T. K. Pogány, *On Mathieu–type series for the unified Gaussian hypergeometric functions,* Appl. Anal. Discrete Math. **14** (2020), 138–149.
[18] R. K. Parmar and R. K. Raina, *On a certain extension of the Hurwitz-Lerch Zeta function,* Ann. West Univ. Timisoara-Math. Computer Sci. **52** (2014), 157–170.
[19] R. K. Parmar, J. Choi, and S. D. Purohit, *Further Generalization of the Extended Hurwitz-Lerch Zeta Function,* Bol. Soc. Paran. Mat. **37** (2019), no. 1, 177-190.
[20] R. K. Parmar, P. Chopra, and R. B. Paris, *On an extension of extended beta and hypergeometric functions,* Journal of Classical Analysis, **11** (2017), no. 2, 91–106.
[21] R. K. Parmar, V.G. Milovanović, and T. K. Pogány, *Multi-parameter Mathieu, and alternating Mathieu series,* Appl. Math. Comput. **400** (2021), 126099.
[22] R. K. Parmar, T. K. Pogány, and R. K. Saxena, *On properties and applications of* (p,q)-*extended* τ-*hypergeometric functions,* Comptes Rendus. Mathématique. **356** (2018), no. 3, 278–282.
[23] R. K. Parmar and T. K. Pogány, *Extended Srivastava's triple hypergeometric* $H_{A,p,q}$ *function and related bounding inequalities,* J. Contemp. Math. Anal. **52** (2017), no. 6, 261–272.
[24] R. K. Parmar and T. K. Pogány, *On* (p,q)–*Extension of Further Members of Bessel-Struve Functions Class,* Miskolc Mathematical Notes **20** (2019), 451–463.
[25] E. D. Rainville, *Special Functions,* Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.
[26] L. J. Slater, *Generalized Hypergeometric Functions,* Cambridge University Press, Cambridge, London, New York, 1966.
[27] E. W. Weisstein, *Hölder's Inequalities,* MathWorld- A Wolfarm Web Resource. https://mathworld.wolfarm.com/HölderInequalities.html .
[28] X. Yang, *Theory and Applications of Special Functions for Scientists and Engineers,* 272–273, Springer Singapore, 2022.

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