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CR-PRODUCT OF A HOLOMORPHIC STATISTICAL MANIFOLD

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Abstract. This study inspects the structure of CR-product of a holomorphic statistical manifold. Findings concerning geodesic submanifolds and totally geodesic foliations in the context of dual connections have been demonstrated. The integrability of distributions in CR-statistical submanifolds has been characterized. The statistical version of CR-product in the holomorphic statistical manifold has been researched. Additionally, some assertions for curvature tensor field of the holomorphic statistical manifold have been substantiated.

1. Introduction

The analysis of geometric structures on sets of certain probability distributions led to the emergence of the statistical manifold. Introduced by [13] and investigated thoroughly by [1], [2], [10], [16], [17], these manifolds have applications in the field of statistical inference, neural networks, control system, face recognition and image analysis, etc.

The concept of CR-submanifolds of a Kaehler manifold was first initiated by [3] and further developed by [4], [6], [5], [18]. The researchers explored the geometry of CR-submanifolds in various manifolds such as a Hermitian manifold, a Sasakian manifold, and a Kenmotsu manifold etc. Their statistical version, namely, CR-statistical submanifolds in holomorphic statistical manifolds was investigated intensively by Furuhata et al. [8], [7], [9]. Contemporarily, [11], [12], [14] and [15] et al. obtained several results on CR-statistical submanifolds of the holomorphic statistical manifold.

In the present research work, various results for the geodesicity and totally geodesic foliations in CR-statistical submanifolds of the holomorphic statistical manifold have been developed. The integrability of totally real distributions has been worked upon. The conditions for a CR- statistical submanifold to be a CR-product have been derived. Some expressions for the curvature tensor field

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in the structure of a mixed foliate CR-statistical submanifold and CR-product of the holomorphic statistical manifold have been provided.

2. Preliminaries

This section addresses some key concepts pertaining to the theory of submanifolds of a holomorphic statistical manifold.

Definition 2.1. [8] *Let $\bar{M}$ be a C^∞ manifold of dimension $\bar{m} \geq 2$, $\bar{\nabla}$ be an affine connection on $\bar{M}$, and $\bar{g}$ be a Riemannian metric on $\bar{M}$. Then ($\bar{M}$,$\bar{\nabla}$,$\bar{g}$) is called a statistical manifold if*
(i) $\bar{\nabla}$ is of torsion free, and
(ii) $(\bar{\nabla}_X \bar{g})(Y,Z) = (\bar{\nabla}_Y \bar{g})(X,Z)$ for $X, Y, Z \in \Gamma(T\bar{M})$.

Moreover, an affine connection $\bar{\nabla}^*$ is called the dual connection of $\bar{\nabla}$ with respect to $\bar{g}$ if

$$X\bar{g}(Y,Z) = \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}^*_X Z) \text{ for } X, Y, Z \in \Gamma(T\bar{M}).$$

If ($\bar{M}$,$\bar{\nabla}$,$\bar{g}$) is a statistical manifold, then so is ($\bar{M}$,$\bar{\nabla}^*$,$\bar{g}$). We therefore denote the statistical manifold by $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$.

Let M be a submanifold of a statistical manifold ($\bar{M}$,$\bar{\nabla}$, $\bar{g}$) and g be the induced metric on M. If the normal space of M is denoted by $T_x^\perp M :=\{v \in T_x\bar{M} \mid \bar{g}(v,w) = 0, w \in T_x M\}$, then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y), \quad \bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{1}$$

$$\bar{\nabla}^*_X Y = \nabla^*_X Y + B^*(X,Y), \quad \bar{\nabla}^*_X \xi = -A^*_\xi X + \nabla^{\perp *}_X \xi, \tag{2}$$

for X, $Y \in \Gamma(TM)$, $\xi \in \Gamma(T^\perp M)$.

Further, the following holds for X, $Y \in \Gamma(TM)$, $\xi \in \Gamma(T^\perp M)$:

$$\bar{g}(B(X,Y), \xi) = g(A^*_\xi X, Y), \quad \bar{g}(B^*(X,Y), \xi) = g(A_\xi X, Y). \tag{3}$$

Let $\bar{R}$ and R be the curvature tensor fields with respect to $\bar{\nabla}$ and ∇, respectively. Then the equations of Gauss and Codazzi are respectively given by

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z,W) = {} & \bar{g}(R(X,Y)Z,W) + \bar{g}(B(X,Z), B^*(Y,W)) \\ & - \bar{g}(B^*(X,W), B(Y,Z)), \end{aligned} \tag{4}$$

$$\bar{g}(\bar{R}(X,Y)Z, JZ) = \bar{g}((\bar{\nabla}_X B)(Y,Z) - (\bar{\nabla}_Y B)(X,Z), JZ), \tag{5}$$

where $(\bar{\nabla}_X B)(Y,Z) = \nabla^\perp_X B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$ for X, Y, Z and W tangent to M.

Definition 2.2. [12] *Let $(\bar{M}, \bar{J}, \bar{g})$ be a Kaehler manifold and $\bar{\nabla}$ an affine connection of $\bar{M}$. Then, $(\bar{M}, \bar{\nabla}, \bar{J}, \bar{g})$ is called a holomorphic statistical manifold if*
(i) $(\bar{M},\bar{\nabla},\bar{g})$ is a statistical manifold, and
(ii) ω is a $\bar{\nabla}$- parallel 2-form on $\bar{M}$, where ω is defined by $\omega(X, Y) = \bar{g}(X, \bar{J}Y)$, for any $X, Y \in \Gamma(TM)$.

Lemma 2.3. [7] *Let $(\bar{M}, \bar{J}, \bar{\bar{g}})$ be a Kaehler manifold. If we define a connection $\bar{\nabla}$ as $\bar{\nabla} = \bar{\nabla}^{\circ} + K$, where $\bar{\nabla}^{\circ}$ is a Levi-Civita connection on $\bar{M}$ and K is a (1,2)-tensor field satisfying the following conditions:*

$$\begin{aligned} K(X, Y) &= K(Y, X), \\ \bar{g}(K(X, Y), Z) &= \bar{g}(Y, K(X, Z)), \\ K(X, \bar{J}Y) &= -\bar{J}K(X, Y), \end{aligned}$$

for $X, Y, Z \in \Gamma(T\bar{M})$, then $(\bar{M}, \bar{\nabla}, \bar{J}, \bar{g})$ is a holomorphic statistical manifold.

Lemma 2.4. [12] *Let $(\bar{M}, \bar{\nabla}, \bar{J}, \bar{\bar{g}})$ be a holomorphic statistical manifold. Then*

$$\bar{\nabla}_X \bar{J}Y = \bar{J}\bar{\nabla}^*_X Y, \tag{6}$$

for $X, Y, Z \in \Gamma(TM)$, where $\bar{\nabla}^*$ is the dual connection of $\bar{\nabla}$ with respect to $\bar{g}$.

Example 2.5. [12] *For $c \in \mathbb{R}$, let O be an interval in $\{t > 0 | 1 - 2ct^3 > 0\}$ and set a domain $\Omega = O \times \mathbb{R}$ in the (u^1, u^2)-plane $\mathbb{R}^2$. J denotes the standard complex structure on Ω, determined by $J\frac{\partial}{\partial u^1} = \frac{\partial}{\partial u^2}$. Define a Riemannian metric g and an affine connection $\bar{\nabla}$ on Ω by*

$$\begin{aligned} g &= u^1\{(du^1)^2 + (du^2)^2\}, \\ \bar{\nabla}_{\frac{\partial}{\partial u^1}} \frac{\partial}{\partial u^1} &= -\frac{1}{2}\phi(u^1)^{-1}\frac{\partial}{\partial u^1}, \\ \bar{\nabla}_{\frac{\partial}{\partial u^1}} \frac{\partial}{\partial u^2} = \bar{\nabla}_{\frac{\partial}{\partial u^2}} \frac{\partial}{\partial u^1} &= (u^1)^{-1}(1 + \frac{1}{2}\phi(u^1))\frac{\partial}{\partial u^2}, \\ \bar{\nabla}_{\frac{\partial}{\partial u^2}} \frac{\partial}{\partial u^2} &= -\frac{1}{2}\phi(u^1)^{-1}\frac{\partial}{\partial u^2}, \end{aligned}$$

where $\phi(t) = -1 \pm \sqrt{1 - 2ct^3}$. Then $(\Omega, \bar{\nabla}, g, J)$ is a holomorphic statistical manifold of constant holomorphic sectional curvature c.

3. CR-statistical submanifolds of a holomorphic statistical manifold

Definition 3.1. $(\bar{M}, \bar{\nabla}, \bar{J}, \bar{g})$ *be a holomorphic statistical manifold. Then a statistical submanifold M is called CR- statistical submanifold of holomorphic* *statistical manifold if it is endowed with the pair of orthogonal distributions $(D, D^{\perp})$ satisfying the following conditions:*

$$TM = D \oplus D^{\perp}.$$

The distribution D is invariant if

$$\bar{J}(D_x) = D_x \quad for \;\; each \;\; x \in M.$$

The distribution $D^{\perp}$ is anti-invariant if

$$\bar{J}(D_x^{\perp}) \subset T_x^{\perp} M \quad for \;\; each \;\; x \in M.$$

The projection morphisms of TM to D and $D^{\perp}$ are denoted by T and R, respectively. Then we have

$$X = TX + RX, \tag{7}$$

$$\bar{J}\xi = t\xi + f\xi, \tag{8}$$

for $X \in \Gamma(TM)$ and $\xi \in \Gamma(T^{\perp}M)$, where $t\xi$ and $f\xi$ denote the tangential and the normal components of $\bar{J}\xi$, respectively.

Applying $\bar{J}$ to equation (7), we obtain

$$\bar{J}X = \bar{J}TX + \bar{J}LX.$$

If we put $\bar{J}TX = PX$ and $\bar{J}LX = FX$, then

$$\bar{J}X = PX + FX, \tag{9}$$

where $PX \in \Gamma(D)$ and $FX \in \Gamma(D^{\perp})$. We denote the orthogonal complementary distribution to $\bar{J}(D^{\perp})$ in $\Gamma(TM^{\perp})$ by N. Then we have

$$TM^{\perp} = \bar{J}(D^{\perp}) \oplus N.$$

Definition 3.2. *A CR-statistical submanifold of a holomorphic statistical manifold is called D-totally geodesic with respect to $\bar{\nabla}$ (resp. $\bar{\nabla}^*$) if $B(X, Y) = 0$ (resp. $B^*(X, Y) = 0$) for all $X, Y \in D$.*

Definition 3.3. *A CR-statistical submanifold of a holomorphic statistical manifold is called mixed totally geodesic with respect to $\bar{\nabla}$ (resp. $\bar{\nabla}^*$) if $B(X, Y) = 0$ (resp. $B^*(X, Y)= 0$) for $X \in D$ and $Y \in D^{\perp}$.*

Theorem 3.4. *Let M be a CR-statistical submanifold of the holomorphic statistical manifold $\bar{M}$. Then, M is D-totally geodesic with respect to $\bar{\nabla}$ (resp. $\bar{\nabla}^*$) if and only if $A^*_{\xi}X$ (resp. $A_{\xi}X$) has no component in D.*

Proof : M is D-totally geodesic with respect to $\bar{\nabla}$ if and only if

$$\bar{g}(B(X, \bar{J}Y), \xi) = 0$$

for all $X, Y \in D$ and $\xi \in \Gamma(TM^{\perp})$.

Since M is a CR-statistical submanifold, therefore

$$\bar{g}(B(X,\bar{J}Y),\xi)=\bar{g}(\bar{\nabla}_X\bar{J}Y,\xi)=-\bar{g}(\bar{J}Y,\bar{\nabla}^*_X\xi)$$

follows using (1). Further, from equation (2),

$$\bar{g}(B(X,\bar{J}Y),\xi)=\bar{g}(A^*_\xi X,\bar{J}Y).$$

Hence the hypothesis leads to the required assertion. Similarly, the corresponding result holds for the dual connection.

Theorem 3.5. *Let M be a CR-statistical submanifold of the holomorphic statistical manifold $\bar{M}$.Then,*

1. *the distribution D defines a totally geodesic foliation with respect to $\bar{\nabla}$ (resp. $\bar{\nabla}^*$) if and only if $A_{\bar{J}Z}X$ (resp. $A^*_{\bar{J}Z}X$) has no component in D.*
2. *the distribution $D^\perp$ defines a totally geodesic foliation with respect to $\bar{\nabla}$ (resp. $\bar{\nabla}^*$) if and only if $B(X,Z)$ (resp. $B^*(X,Z)$) has no component in $D^\perp$.*

Proof : The distribution D defines a totally geodesic foliation if and only if $\bar{g}(\nabla_X Y,Z)=0$ for all $X,Y\in\Gamma(D)$ and $Z\in\Gamma(D^\perp)$.

Using (1), we have

$$\begin{aligned}\bar{g}(\nabla_X\bar{J}Y,Z)&=\bar{g}(\bar{\nabla}_X\bar{J}Y,Z)=\bar{g}(\bar{J}\nabla^*_XY,Z)\\&=\bar{g}(Y,\bar{\nabla}_X\bar{J}Z)=\bar{g}(Y,A_{\bar{J}Z}X).\end{aligned}$$

From the hypothesis with respect to dual connection $\bar{\nabla}^*$, we get ((1). The distribution $D^\perp$ defines a totally geodesic foliation if and only if $\bar{g}(\nabla_XY,Z)=0$ for all $X,Y\in\Gamma(D^\perp)$ and $Z\in\Gamma(D)$.

Now from (1),

$$\begin{aligned}\bar{g}(\nabla_XY,\bar{J}Z)&=\bar{g}(\bar{\nabla}_XY,\bar{J}Z)\\&=-\bar{g}(Y,\bar{\nabla}^*_X\bar{J}Z)=\bar{g}(\bar{J}Y,B(X,Z)),\end{aligned}$$

which completes the proof.

Theorem 3.6. *Let M be a CR-statistical submanifold of the holomorphic statistical manifold $\bar{M}$. If M is a totally geodesic submanifold with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$, then $TM^\perp$ is a Killing distribution on M.*

Proof : From equation (1) and using the relationship of dual connections in holomorphic statistical manifold, we get

$$\begin{aligned}\bar{g}(B(X,Y),\xi)&=\bar{g}(\bar{\nabla}_XY,\xi)=-\bar{g}(Y,\bar{\nabla}^*_X\xi)\\&=-\bar{g}(Y,[X,\xi])-\bar{g}(Y,\bar{\nabla}^*_\xi X)\\&=-\bar{g}(Y,[X,\xi])-\xi\bar{g}(Y,X)+\bar{g}(\bar{\nabla}_\xi Y,X)\\&=-\bar{g}(Y,[X,\xi])-\xi\bar{g}(Y,X)+\bar{g}(\bar{\nabla}_Y\xi,X)+\bar{g}([\xi,Y],X)\\&=-(L_\xi\bar{g})(X,Y)-\bar{g}(\bar{\nabla}^*_YX,\xi),\\\bar{g}(B(X,Y),\xi)&=-(L_\xi\bar{g})(X,Y)-\bar{g}(B^*(X,Y),\xi).\end{aligned}$$

The totally geodesicity of M with respect to dual connections in above equation proves the result.

Theorem 3.7. *Let M be a CR-statistical submanifold of the holomorphic statistical manifold $\bar{M}$.Then*

(i) $\bar{g}(JA_{JZ}U, X) = \bar{g}(\nabla^*_U Z, X)$,
(ii) $A_{JW}Z = A_{JZ}W$,
(iii) $A^*_\xi JX = -A_{J\xi}X$,

for any U tangent to N, $X \in \Gamma(D)$, $Z, W \in \Gamma(D^\perp)$ and ξ in N.

Proof : From Lemma 2.4, for U tangent to N, $Z \in \Gamma(D^\perp)$,

$$-A_{JZ}U + D_U JZ = J\nabla^*_U Z + JB^*(U, Z).$$

Taking inner product with X,

$$\bar{g}(-A_{JZ}U, X) = \bar{g}(J\nabla^*_U Z, X).$$

By applying $\bar{J}$ on both sides, we get the identity (i).

For $Z, W \in \Gamma(D^\perp)$,

$$-A_{JW}Z + B(Z, JW) = P\nabla^*_Z W + F\nabla^*_Z W + tB^*(Z, W) + fB^*(Z, W).$$

Taking tangential parts of the above equation, we have

$$-A_{JW}Z = P\nabla^*_Z W + tB^*(Z, W),$$

Therefore,

$$-A_{JZ}W = P\nabla^*_W Z + tB^*(W, Z),$$
$$A_{JW}Z - A_{JZ}W = P[W, Z].$$

Now for any ξ in N

$$\bar{g}(B(JX, Y), \xi) = \bar{g}(\bar{\nabla}_{JX}Y - \nabla_{JX}Y, \xi).$$

From the concept of holomorphic statistical manifold, we infer

$$\bar{g}(B(JX, Y), \xi) = -\bar{g}(Y, \bar{\nabla}^*_{JX}\xi) = \bar{g}(Y, A^*_\xi JX),$$
$$\bar{g}(JB^*(X, Y), \xi) = \bar{g}(J\bar{\nabla}^*_X Y, \xi) = -\bar{g}(\bar{\nabla}^*_X Y, J\xi) = -\bar{g}(Y, A_{J\xi}X).$$

The above equation leads to the other two required identities.

Theorem 3.8. *Let M be a CR-statistical submanifold of the holomorphic statistical manifold $\bar{M}$.Then the totally real distribution $D^\perp$ of a CR-statistical submanifold is integrable if*

$$\nabla^\perp_W JZ = \nabla^\perp_Z JW,$$

for any $Z, W \in \Gamma(D^\perp)$ and ξ in N.

Proof : Since M is a holomorphic statistical manifold, therefore from (1) and (2)

$$-A_{JW}Z + \nabla^{\perp}_{Z}JW = J\nabla^{*}_{Z}W + JB^{*}(Z,W),$$

for $Z, W \in \Gamma(D^{\perp})$ and ξ in N. Then,

$$-A_{JZ}W + \nabla^{\perp}_{W}JZ = J\nabla^{*}_{W}Z + JB^{*}(W,Z),$$

$$J[Z,W] = \nabla^{\perp}_{W}JZ - \nabla^{\perp}_{Z}JW.$$

Since $D^{\perp}$ is a totally real distribution, the desired result follows.

Theorem 3.9. *Let M be a CR-statistical submanifold of the holomorphic statistical manifold $\bar{M}$. Then, the distribution D is integrable if and only if the second fundamental form of M satisfies*

$$B(X,\bar{J}Y) = B(Y,\bar{J}X),$$

for all $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Proof : For $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, using (6), we get

$$\begin{aligned}\bar{g}([X,Y],Z) &= \bar{g}(\bar{J}\bar{\nabla}^{*}_{X}Y,\bar{J}Z) - \bar{g}(\bar{J}\bar{\nabla}^{*}_{Y}X,\bar{J}Z)\\ &= \bar{g}(\bar{\nabla}_{X}\bar{J}Y,\bar{J}Z) - \bar{g}(\bar{\nabla}_{Y}\bar{J}X,\bar{J}Z).\end{aligned}$$

Therefore, we infer

$$\bar{g}([X,Y],Z) = \bar{g}(B(X,\bar{J}Y) - B(Y,\bar{J}X),\bar{J}Z).$$

Hence the result.

Definition 3.10. *A CR-statistical submanifold of a holomorphic statistical manifold is called mixed foliate if the distribution is integrable and M is mixed totally geodesic with respect to $\bar{\nabla}$ (resp. $\bar{\nabla}^{*}$).*

Theorem 3.11. *Let M be a mixed foliate CR-statistical submanifold of the holomorphic statistical manifold $\bar{M}$. Then,*

$$\bar{g}(R(X,\bar{J}X)Z,\bar{J}Z) = -2\bar{g}(A^{*}_{\bar{J}Z}\bar{J}X,\bar{J}A^{*}_{\bar{J}Z}X),$$

for all $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Proof : For $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$ in a mixed foliate submanifold M of $\bar{M}$, we get

$$\bar{g}(R(X,\bar{J}X)Z,\bar{J}Z) = -\bar{g}(B(\bar{J}X,\nabla_{X}Z),\bar{J}Z) + \bar{g}(B(X,\nabla_{\bar{J}X}Z,\bar{J}Z).$$

From equation (3), we obtain

$$\bar{g}(R(X,\bar{J}X)Z,\bar{J}Z) = -\bar{g}(A^{*}_{\bar{J}Z}\bar{J}X,\nabla_{X}Z) + \bar{g}(A^{*}_{\bar{J}Z}X,\nabla_{\bar{J}X}Z).$$

Now from Theorem (3.7), we derive

$$\bar{g}(R(X,\bar{J}X)Z,\bar{J}Z) = -\bar{g}(A^{*}_{\bar{J}Z}\bar{J}X,\bar{J}A^{*}_{\bar{J}Z}X) - \bar{g}(\bar{J}A^{*}_{\bar{J}Z}X,A^{*}_{\bar{J}Z}\bar{J}X).$$

Thus, the result follows.

4. CR-product in the holomorphic statistical manifold

In this section, we study the statistical version of CR-product in the holomorphic statistical manifold. We also derive conditions for a CR-statistical manifold to be a CR-product.

Definition 4.1. [6] *A CR-statistical submanifold M of holomorphic statistical manifold is called a CR-product if both the distribution D and $D^{\perp}$ define totally geodesic foliations on M.*

Lemma 4.2. *Let M be a CR-statistical submanifold of the holomorphic statistical manifold $\bar{M}$. Then the distribution D defines a totally geodesic foliation with respect to $\bar{\nabla}$ (resp. $\bar{\nabla}^*$) if and only if*

$$B^*(X, \bar{J}Y) = 0 \; (resp. \; B(X, \bar{J}Y) = 0),$$

for any $X, Y \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$.

Proof : For $X, Y \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$,

$$\bar{g}(\nabla_X Y, V) = (\bar{J}\bar{\nabla}_X Y, \bar{J}V) = \bar{g}(\bar{\nabla}^*_X \bar{J}Y, \bar{J}V),$$
$$\bar{g}(\nabla_X Y, V) = \bar{g}(B^*(X, \bar{J}Y), \bar{J}V).$$

Further, since D defines a totally geodesic foliation, therefore the required outcome ensues from the concept of holomorphic statistical manifold.

Lemma 4.3. *For a CR-statistical submanifold M of the holomorphic statistical manifold $\bar{M}$, the distribution $D^{\perp}$ defines a totally geodesic foliation with respect to $\bar{\nabla}$ (resp. $\bar{\nabla}^*$) in M if and only if*

$$\bar{g}(B(D, D^{\perp}), JD^{\perp}) = 0 \; (resp. \; \bar{g}(B^*(D, D^{\perp}), JD^{\perp}) = 0),$$

for any $X,\ Y \in \Gamma(D)$ and $V,\ W \in \Gamma(D^{\perp})$.

Proof : From (2), (3), we have

$$\bar{g}(B(X, V), \bar{J}W) = \bar{g}(A^*_{\bar{J}W} V, X) = -\bar{g}(\bar{\nabla}^*_V \bar{J}W, X) = \bar{g}(\bar{J}\bar{\nabla}_V W, X),$$

for $X, Y \in \Gamma(D)$ and $V, W \in \Gamma(D^{\perp})$.

Further from equation (6), we derive

$$\bar{g}(B(X, V), \bar{J}W) = \bar{g}(\bar{\nabla}_V W, \bar{J}X) = \bar{g}(\nabla_V W, \bar{J}X).$$

Therefore, the hypothesis leads to the desired result.

Lemma 4.4. *A CR-statistical submanifold of a holomorphic statistical manifold $\bar{M}$ is a CR product if*

$$A_{JD^{\perp}} D = 0 \; (resp. \; A^*_{JD^{\perp}} D = 0).$$

Proof : For $X \in \Gamma(D)$ and $Y, Z \in \Gamma(D^{\perp})$, using equation (3), we have

$$\bar{g}(A_{\bar{J}Z}X, Y) = \bar{g}(B^{*}(X, Y), \bar{J}Z).$$

This implies that $D^{\perp}$ defines a totally geodesic foliation with respect to $\bar{\nabla}$. Similarly,

$$\bar{g}(A_{\bar{J}Z}X, Y) = \bar{g}(B^{*}(X, Y), \bar{J}Z),$$

for $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, which shows that D defines a totally geodesic foliation in M. Hence M is a CR-product.

Conversely, if M is a CR product, then both the distribution D and $D^{\perp}$ define totally geodesic foliations on M. Therefore, using (3), we obtain $A_{JD^{\perp}}D = 0$ for $X \in \Gamma(D)$ and $Y, Z \in \Gamma(D^{\perp})$. Also, for $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, $A^{*}_{JD^{\perp}}D = 0$. Thus the assertion.

Lemma 4.5. *Let M be a CR statistical submanifold of the holomorphic statistical manifold $\bar{M}$. If the leaf $M^{\perp}$ of $D^{\perp}$ is totally geodesic with respect to $\bar{\nabla}$ (resp. $\bar{\nabla}^{*}$) and D is integrable, then for any X in D and ξ in $JD^{\perp}$, we have*

$$JA_{\xi}X = -A_{\xi}JX \ (resp.\ JA^{*}_{\xi}X = -A^{*}_{\xi}JX).$$

*Proof :*For any $X, Y \in \Gamma(D)$ and $\xi \in JD^{\perp}$, we have

$$\bar{g}(JA_{\xi}X, Y) = -\bar{g}(A_{\xi}X, JY) = -\bar{g}(B^{*}(X, JY), \xi).$$

Now from Theorem (3.9), we obtain

$$\bar{g}(JA_{\xi}X, Y) = -\bar{g}(B^{*}(JX, Y), \xi) = -\bar{g}(A_{\xi}JX, Y).$$

The similar approach holds for the dual part. Hence proved.

Let P and f be the endomorphisms of the tangent bundle TM and the normal bundle $TM^{\perp}$ respectively. Let F and t be the normal-valued 1-form on TM and tangent valued 1-form on $TM^{\perp}$ as defined in (8) and (9). Then

$$\nabla_X PY - P\nabla^{*}_X Y = A_{FY}X + tB^{*}(X, Y), \tag{10}$$

$$\nabla^{\perp}_X FY - F\nabla^{*}_X Y = fB^{*}(X, Y) - B(X, PY), \tag{11}$$

$$\nabla_X t\xi - t\nabla^{\perp *}_X \xi = A_{f\xi}X - PA^{*}_{\xi}X, \tag{12}$$

$$\nabla^{\perp}_X f\xi - f\nabla^{\perp}_X \xi = -FA^{*}_{\xi}X - B(X, t\xi).$$

Theorem 4.6. *A CR-statistical submanifold of a holomorphic statistical manifold M is a CR product if and only if*

$$\nabla_X PY = P\nabla^{*}_X Y.$$

Proof : For any vectors X, Y tangent to M,

$$A_{FY}X = -tB^{*}(X, Y),$$

follows using equation (10) and the hypothesis. For $U \in \Gamma(D)$, we get $tB^*(X,U) = 0$ which implies that $A_{JZ}U = 0$ for any Z in $D^\perp$ and U in D. Thus,

$$\bar{g}(A_{\bar{J}Z}U, W) = \bar{g}(B^*(U,W), \bar{J}Z).$$

From Lemma (4.3), we conclude that $D^\perp$ defines a totally geodesic foliation in M. Now, for any X, Y in D and Z in $D^\perp$, Theorem (3.7) leads to

$$\begin{aligned} 0 = \bar{g}(A_{\bar{J}Z}Y, X) &= \bar{g}(\bar{J}A_{\bar{J}Z}Y, \bar{J}X) = \bar{g}(\nabla^*_Y Z, \bar{J}X) \\ &= -\bar{g}(Z, \bar{\nabla}_Y \bar{J}X) = -\bar{g}(Z, \nabla_Y \bar{J}X). \end{aligned}$$

This implies that D defines a totally geodesic foliation in M. Hence $\bar{M}$ is a CR-product in $\bar{M}$.

Conversely, if M is a CR product, then both the distributions D and $D^\perp$ define a totally geodesic foliations on M. For $Y \in \Gamma(D)$ and $X \in \Gamma(TM)$, $\nabla_X Y \in \Gamma(D)$. Applying Gauss formula to equation (6) and on comparing normal components, we have $B(X, \bar{J}Y) = \bar{J}B^*(X,Y)$. Hence, for $Y \in \Gamma(D)$, we get $\nabla_X PY = P\nabla^*_X Y$. Similarly, for $Y \in \Gamma(D^\perp)$ and $X \in \Gamma(TM)$, $\nabla_X Y \in \Gamma(D^\perp)$ which proves the result.

Theorem 4.7. *Let M be a CR-product of a holomorphic statistical manifold $\bar{M}$. Then for any unit vectors X in D and Z in $D^\perp$, we have*

$$\bar{g}(R(X, \bar{J}X)Z, \bar{J}Z) = -2\bar{g}(B(X,Z), B^*(X,Z)).$$

Proof : Let M be a CR-product in $\bar{M}$. Then for any unit vectors X in D and Z in $D^\perp$ and using equations (4) and (5), we derive

$$\begin{aligned} \bar{g}(R(X, \bar{J}X)Z, \bar{J}Z) &= \bar{g}(\nabla^\perp_X B(\bar{J}X, Z) - \nabla^\perp_{\bar{J}X} B(X,Z), \bar{J}Z) \\ &= -\bar{g}(B(\bar{J}X, Z), \nabla^{\perp *}_X \bar{J}Z) + \bar{g}(B(X,Z), \nabla^{\perp *}_{\bar{J}X} \bar{J}Z) \\ &= -\bar{g}(B(\bar{J}X, Z), \bar{J}\bar{\nabla}_X Z) + \bar{g}(B(X,Z), \bar{J}\bar{\nabla}_{\bar{J}X} Z). \end{aligned}$$

Further, using Lemma (4.4), we obtain

$$\bar{g}(R(X, \bar{J}X)Z, \bar{J}Z) = -\bar{g}(B(\bar{J}X, Z), \bar{J}B(X,Z)) + \bar{g}(B(X,Z), \bar{J}B(\bar{J}X, Z)),$$
$$\bar{g}(R(X, \bar{J}X)Z, \bar{J}Z) = -2\bar{g}(B^*(X,Z), B(X,Z)).$$

Hence the result follows.

Remark: Let $\bar{M}$ be a holomorphic statistical manifold with negative holomorphic sectional curvature. Then every CR-product in $\bar{M}$ is either a holomorphic submanifold or a totally real submanifold.

Theorem 4.8. *For a CR-statistical submanifold M of a holomorphic statistical manifold $\bar{M}$, $\nabla_X t\xi = t\nabla^{\perp *}_X \xi$ if and only if $\nabla^\perp_X FY = F\nabla^*_X Y$.*

Proof : For any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(T^\perp M)$, the hypothesis alongwith equation (12) leads to

$$\bar{g}(A_{f\xi}X, Y) = \bar{g}(PA^*_\xi X, Y).$$

From (3), we obtain

$$\bar{g}(A_{f\xi}X,Y) = -\bar{g}(\bar{J}B^*(X,Y),\xi).$$

Also,

$$\bar{g}(PA^*_{\xi}X,Y) = -\bar{g}(B(X,PY),\xi).$$

Therefore, we get

$$\bar{g}(fB^*(X,Y),\xi) = \bar{g}(B(X,PY)),\xi).$$

The result follows from (11).

Theorem 4.9. *Let M be a CR-statistical submanifold of a holomorphic statistical manifold $\bar{M}$. If $\nabla^{\perp}_X FY = F\nabla^*_X Y$, then M is a CR- product.*

Proof : From equation (11) and using the given condition, we get

$$fB^*(X,Y) = B(X,PY).$$

For any Y in $D^{\perp}$, we have $fB^*(X,Y) = 0$. Therefore $\bar{g}(fB^*(X,Y),Y) = 0$. It follows from equation (8) that $\bar{g}(B^*(X,Y),\bar{J}Y) = 0$ and hence $\bar{g}(A_{\bar{J}Y}X,Y) = 0$. Thus $A_{JD^{\perp}}D = 0$. So by Lemma (4.4), M is a CR-product.

Following [12], we present an example of a CR-product in the holomorphic statistical manifold.

Example 4.10. *Let $\mathbb{C}^2 = (\mathbb{R}^4, g, J)$ be the complex Euclidean space, that is $g = \sum_{i=1}^4 dx^i \otimes dx^i$ and $J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^{i+2}}, i = 1,2,\ J\frac{\partial}{\partial x^i} = -\frac{\partial}{\partial x^{i-2}}, i = 3,4$. For functions α_j on $\mathbb{R}^4$, $j = 1,2....8$, define a $(1,2)$-tensor field $K = \sum_{i,j,l=1}^4 k^l_{ij}\frac{\partial}{\partial x^i} \otimes dx^i \otimes dx^j$ on $\mathbb{C}^2$ as follows:*

$$k^1_{11} = \alpha_1,\quad k^3_{13} = k^3_{31} = k^1_{33} = -\alpha_1,\quad k^2_{11} = k^1_{12} = k^1_{21} = \alpha_2,$$

$$k^4_{13} = k^4_{31} = k^3_{14} = k^3_{41} = k^3_{23} = k^3_{32} = k^1_{34} = k^1_{43} = k^2_{33} = -\alpha_2,$$

$$k^3_{11} = k^1_{13} = k^1_{31} = \alpha_3,\quad k^3_{33} = -\alpha_3,$$

$$k^4_{11} = k^1_{14} = k^1_{41} = k^2_{13} = k^2_{31} = k^3_{12} = k^3_{21} = k^1_{23} = k^1_{32} = \alpha_4,$$

$$k^4_{33} = k^3_{34} = k^3_{43} = -\alpha_4,\quad k^2_{12} = k^2_{21} = k^1_{22} = \alpha_5,$$

$$k^4_{14} = k^4_{41} = k^1_{44} = k^4_{23} = k^4_{32} = k^2_{34} = k^2_{43} = k^3_{24} = k^3_{42} = -\alpha_5,$$

$$k^2_{22} = \alpha_6,\quad k^4_{24} = k^4_{42} = k^2_{44} = -\alpha_6,\quad k^4_{22} = k^2_{24} = k^2_{42} = \alpha_7,\ k^4_{44} = -\alpha_7,$$

$$k^4_{12} = k^4_{21} = k^2_{14} = k^2_{41} = k^1_{24} = k^1_{42} = k^2_{23} = k^2_{32} = k^3_{22} = \alpha_8,$$

$$k^4_{34} = k^4_{43} = k^3_{44} = -\alpha_8.$$

Here, K satisfies the conditions of Lemma (2.3). Therefore $\bar{M} = (\mathbb{R}^4, \bar{\nabla} = \nabla^g + K, g, J)$ becomes a holomorphic statistical manifold.

Consider a statistical immersion $f : \mathbb{C} \otimes \mathbb{R} \to \mathbb{C}^2$ and a CR-submanifold $M = \mathbb{C} \otimes \mathbb{R}$ in $\mathbb{C}^2$. Now, (∇, g) is the induced statistical structure on M from $(\bar{\nabla}, \bar{g})$ by the immersion f. Then, (M, ∇, g) becomes a CR-product in the holomorphic statistical manifold $\mathbb{C}^2$.

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