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ON SPATIAL QUATERNIONIC SMARANDACHE RULED SURFACES

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Abstract. In this paper, we investigate the spatial quaternionic expressions of the ruled surfaces whose base curves are formed by the Smarandache curve. Moreover, we formulate the striction curves and dralls of these surfaces. If the quaternionic Smarandache ruled surfaces are closed, the pitches and angle of pitches are interpreted. Finally, we calculate the integral invariants of these surfaces using quaternionic formulas.

1. Introduction

The concept of quaternions was introduced in 1843 by the Irish mathematician William Rowan Hamilton. The quaternions have applications in many disciplines. Some of these are computer graphics, the development of vision devices, robot kinematics, control theory, quantum theory, molecular dynamics, animation representations, and navigation devices [7, 8, 16, 5, 6]. New interpretations have been made using quaternions in the theory of curves and surfaces. Bharathi and Nagaraj expressed the Serenet-Ferret invariants of any curve using quaternions in 1987 [1]. Chen and Lie reached new results by making correlations between quaternionic transformations and minimal surfaces in 2005 [4]. In addition, Çetin and Kocayiğit investigated the Serret-Frenet formulas of Samarandache curves in terms of quaternions [3]. Şenyurt and Eren investigate special Smarandache curves created by the Frenet vectors of spacelike anti-Salkowski curve with a spacelike principal normal [15]. Öztürk et al. introduce Smarandache curves of an affine C^{∞} curve in affine 3-space. Besides, they calculate the relationship between the Frenet frames of the curve couple and the Frenet invariants of each derived curve [11].

Ruled surfaces are surfaces that can be generated by moving a straight line along a curve. There have been numerous studies conducted in various spaces and frames. The authors introduce the concept of partner-ruled surfaces, defined in the Flc frame on a polynomial curve. They investigate the requirements

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for two of these surfaces to be simultaneously developable and minimal. Additionally, they examine the geodesic, asymptotic, and curvature lines of the parameter curves in the partner-ruled surfaces [9]. In [10], they present a new approach to understanding the geometric characteristics and local singularities of time-like surfaces. The method is based on the introduction of the geometric invariant, which allows us to derive necessary and sufficient conditions for a time-like surface to be a time-like developable ruled surface. They then use singularity theory to classify the singularities of this surface, providing a complete characterization of its geometric features. Şenyurt and Çalışkan investigate the ruled surface with the theory of quaternion. They express integral invariants and calculate the ruled surfaces drawn by Frenet vectors belonging to spatial quaternionic curves [14]. In [2], the author analyzes the quaternionic ruled surfaces according to the alternative frame. Ouarab demonstrates a new method for constructing special ruled surfaces and investigates their minimalist and developability properties. The author introduces the concept of Smarandache ruled surfaces, which are defined based on the Darboux frame of a curve on a regular surface. The paper also presents theorems that provide sufficient and necessary conditions for these surfaces to be minimal and developable. Furthermore, the authors examine Smarandache ruled surfaces in terms of alternative frame [12, 13].

In Section 2, we present the geometric preliminaries regarding the basic problem of the paper mentioned in the introduction. In Section 3, we define these surfaces using quaternionic Smarandache curves as base curves and Frenet vectors as their directives. Moreover, we calculate the striction curves, dralls, pitches, and angle of pitches for these surfaces. Finally, we exemplify the findings.

2. Preliminaries

In this section, we show the notions of the quaternions and the spatial quaternionic curves. We demonstrate the definitions of quaternionic Smarandache curves and we investigate the Frenet-Serret invariants of these curves.

2.1. Quaternions and quatenionic ruled surface

The real quaternion q is expressed as the sum of a scalar $S_q = q_0$ and a vector $V_q = q_1e_1 + q_2e_2 + q_3e_3$ such that

$$q = q_0 + q_1e_1 + q_2e_2 + q_3e_3,$$

where the set $\{e_i \mid 1 \leqslant i \leqslant 3\}$ is the standard orthonormal basis set. The components q_0, q_1, q_2 and q_3 are real numbers, and e_i, $(1 \leqslant i \leqslant 3)$ are quaternionic units that satisfy the non-commutative multiplication rules
$e_i \times e_j = e_k = -e_j \times e_i$ and $e_i \times e_i = -1$ for all $1 \leqslant i, j \leqslant 3$. The complex

conjugate $\bar{q}$ is defined by

$$\bar{q} = S_q - V_q = q_0 - q_1 e_1 - q_2 e_2 - q_3 e_3.$$

Let Q denote the set of quaternions. The quaternion inner product is defined by the following real-valued, symmetric, and bilinear form:

$$\begin{aligned} & h : Q \times Q \to \mathbb{R} \\ (1) \qquad & (p, q) \to h(p, q) = \frac{1}{2}(p \times \bar{q} + q \times \bar{p}). \end{aligned}$$

For $p = S_p + V_p$ and $q = S_q + V_q$, the quaternionic product is defined by

$$p \times q = S_p S_q + S_p V_q + S_q V_p - \langle V_p, V_q \rangle + V_p \wedge V_q,$$

where $\langle , \rangle$ and $\wedge$ denote the inner product and cross product in $\mathbb{R}^3$ and thus, the spatial quaternionic cross product obtains as

$$p \times q = -\langle V_p, V_q \rangle + V_p \wedge V_q.$$

The norm of a quaternion q is

$$\rho(q)^2 = h(q, q) = q \times \bar{q} = \bar{q} \times q = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2.$$

Since $\rho(q) = 1$, the quaternion q is called unit quaternion. The inverse of the quaternion q is given by

$$q^{-1} = \frac{\bar{q}}{\rho(q)}.$$

The space of the spatial quaternions is classified by $\{q \in Q | \, q + \bar{q} = 0\}$ where Q denotes quaternion set [1, 6].

Definition 2.1. [1] *The spatial quaternionic curve α is defined by*

$$\begin{aligned} \alpha &: I \subset \mathbb{R} \to Q, \\ s &\to \alpha(s) = \sum_{i=1}^{3} \alpha_i(s) e_i \end{aligned}$$

where $I = [0, 1]$ is an interval in real line $\mathbb{R}$ and $s \in [0, 1]$ is the arc-length parameter.

Theorem 2.2. [1] *Let α be a spatial quaternionic curve with the arc-length parameter s and be non-zero curvatures $\{\kappa, \tau\}$, and Frenet frame $\{t, n, b\}$ of the quaternionic curve α. Then the Serret-Frenet formulae of the spatial quaternionic curve α at a point $\alpha(s)$ are*

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix}_s = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

such that

$$t(s) = \alpha'(s), \; n(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \; b(s) = t(s) \times n(s),$$

where the vectors t, n and b are unit tangent, unit principal normal, and unit binormal vectors of the spatial quaternionic curve α, respectively.

Lemma 2.3. [14] *The drall and the striction curve of the quaternionic ruled surface drawn by arbitrary vector X are respectively given by*

$$P=\frac{1}{2}\frac{(X\times X')\times\overline{\alpha'}+\alpha'\times\overline{(X\times X')}}{\rho(X')^{2}} \tag{2}$$

and

$$r(s)=\alpha(s)-\frac{1}{2}\frac{X'\times\bar{t}+t\times\overline{X'}}{\rho(X')^{2}}. \tag{3}$$

Definition 2.4. [14] *For given closed spatial quaternionic ruled surface, the magnitude of $l_x=\oint\limits_{\alpha} h(d\alpha, X)$ is called the pitch of this surface.*

Theorem 2.5. [14] *Let D be Steiner rotation vector and V be Steiner translation vector. The angle of pitch and the pitch of the closed spatial quaternionic ruled surface, λ_x and l_x are equal to $\lambda_x = h(D, X)$ and $l_x = h(V, X)$.*

2.2. Quaternionic Smarandache curves

Definition 2.6. [3] *Let $\alpha = \alpha(s)$ be a spatial quaternion curve and $\{t, n, b\}$ be Frenet-Serret vectors. The spatial quaternionic $tn-$Smarandache curves with the arc-length parameter s^* are defined by*

$$\beta_1(s^*(s))=\frac{1}{\sqrt{2}}(t(s)+n(s)).$$

The Frenet-Serret invariants of the spatial quaternionic $tn-$Smarandache curves are

$$\begin{cases} T^{\beta_1} &= \dfrac{1}{\sqrt{2\kappa^2+\tau^2}}(-\kappa t+\kappa n+\tau b),\\ N^{\beta_1} &= \dfrac{1}{\sqrt{a_1^2+a_2^2+a_3^2}}(a_1t+a_2n+a_3b),\\ B^{\beta_1} &= \dfrac{1}{\sqrt{2\kappa^2+\tau^2}\sqrt{a_1^2+a_2^2+a_3^2}}(b_1t+b_2n+b_3b),\\ k_1^{\beta_1} &= \dfrac{\sqrt{2}\sqrt{a_1^2+a_2^2+a_3^2}}{(2\kappa^2+\tau^2)^2},\\ k_2^{\beta_1} &= \dfrac{\sqrt{2}\left(\left(\kappa'+\kappa^2\right)(\kappa c_3-\tau c_2)+\left(\kappa^2-\kappa'+\tau^2\right)(\kappa c_3+\tau c_1)+\kappa(\kappa\tau+\tau')(c_1+c_2)\right)}{(2\kappa\kappa'+\tau\tau')^2+(2\kappa^2\tau+\kappa\tau'-\kappa'\tau+\tau^3)^2+(\kappa\tau'-\kappa'\tau)^2+(2\kappa^3+\kappa\tau^2)^2}, \end{cases}$$

where

$$\begin{cases} a_1 = -2\kappa^4 - \kappa'\tau^2 - \kappa^2\tau^2 + \kappa\tau\tau', \\ a_2 = -2\kappa^4 - 3\kappa^2\tau^2 + \kappa'\tau^2 - \tau^2 - \kappa\tau\tau', \\ a_3 = 2\kappa^3\tau + 2\kappa^2\tau' + \kappa\tau^3 - 2\kappa\kappa'\tau, \\ b_1 = \kappa a_3 - \tau a_2, \\ b_2 = \kappa a_3 + \tau a_1, \\ b_3 = -\kappa\left(a_1 + a_2\right), \\ c_1 = -\kappa'' - 3\kappa\kappa' + \kappa^3 + \kappa\tau^2, \\ c_2 = -3\kappa\kappa' - \kappa^3 - 3\tau\tau' + \kappa'' - \kappa\tau^2, \\ c_3 = -\kappa^2\tau + 2\kappa'\tau - \tau^3 + \kappa\tau' + \tau''. \end{cases}$$

Definition 2.7. [3] *Let $\alpha = \alpha(s)$ be a spatial quaternion curve and $\{t, n, b\}$ be Frenet-Serret vectors. The spatial quaternionic $tb-$Smarandache curves with the arc-length parameter s^* are defined by*

$$\beta_2\left(s^*(s)\right) = \frac{1}{\sqrt{2}}\left(t(s) + b(s)\right).$$

The Frenet-Serret invariants of the spatial quaternionic $tb-$Smarandache curves are

$$\begin{cases} T^{\beta_2} = n, \\ N^{\beta_2} = \dfrac{1}{\sqrt{\kappa^2+\tau^2}}\left(-\kappa t + \tau b\right), \\ B^{\beta_2} = \dfrac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau t + \kappa b\right), \\ k_2^{\beta_2} = \dfrac{\sqrt{2}\left(\tau-\kappa\right)\left(d_3\left(-\kappa^2+\kappa\tau\right) - d_1\left(\kappa\tau-\tau^2\right)\right)}{(\kappa-\tau)^2(\kappa'-\tau')^2 + (\kappa^2\tau - 2\kappa\tau^2+\tau^3)^2 + (-2\kappa^2\tau+\kappa\tau^2+\kappa^3)^2}, \\ k_1^{\beta_2} = \dfrac{\sqrt{2}\sqrt{\kappa^2+\tau^2}}{\kappa-\tau}, \end{cases}$$

where

$$\begin{cases} d_1 = -3\kappa\kappa' + \kappa'\tau + 2\kappa\tau', \\ d_2 = -\kappa^3 + \kappa^2\tau + \kappa'' - \tau'' - \kappa\tau^2 + \tau^3, \\ d_3 = 2\kappa'\tau - 3\tau\tau' + \kappa\tau'. \end{cases}$$

Definition 2.8. [3] *Let $\alpha = \alpha(s)$ be a spatial quaternion curve and $\{t, n, b\}$ be Frenet-Serret vectors. The spatial quaternionic $nb-$Smarandache curves with the arc-length parameter s^* are defined by*

$$\beta_3\left(s^*(s)\right) = \frac{1}{\sqrt{2}}\left(n(s) + b(s)\right).$$

The Frenet-Serret invariants of the spatial quaternionic $nb-$Smarandache curves are

$$\begin{cases} T^{\beta_3} = \dfrac{1}{\sqrt{\kappa^2+2\tau^2}}\left(-\kappa t-\tau n+\tau b\right), \\ N^{\beta_3} = \dfrac{1}{\sqrt{f_1^2+f_2^2+f_3^2}}\left(f_1 t+f_2 n+f_3 b\right), \\ B^{\beta_3} = \dfrac{1}{\sqrt{\kappa^2+2\tau^2}\sqrt{f_1^2+f_2^2+f_3^2}}\left(g_1 t+g_2 n+g_3 b\right), \\ k_1^{\beta_3} = \dfrac{\sqrt{2}\sqrt{f_1^2+f_2^2+f_3^2}}{\left(\kappa^2+2\tau^2\right)^2}, \\ k_2^{\beta_3} = \dfrac{\sqrt{2}\left(\left(-\kappa'\tau+\kappa\tau^2\right)\left(h_2+h_3\right)+\left(\kappa^2+\tau'+\tau^2\right)\left(\kappa h_3+\tau h_1\right)-\left(\tau^2-\tau'\right)\left(\kappa h_2-\tau h_1\right)\right)}{\left(\kappa\kappa'+2\tau\tau'\right)^2+\left(\kappa^2\tau+2\tau^3\right)^2+\left(\kappa\tau'-\kappa'\tau\right)^2+\left(\kappa^3+\kappa\tau'+2\kappa\tau^2-\kappa'\tau\right)^2}, \end{cases}$$

where

$$\begin{cases} f_1 = -\kappa^3\tau-2\kappa'\tau^2+2\kappa\tau^3+2\kappa\tau\tau', \\ f_2 = -\kappa^4-\kappa^2\tau'-3\kappa^2\tau^2-2\tau^4+\kappa\kappa'\tau, \\ f_3 = -\kappa^2\tau^2+\kappa^2\tau'-2\tau^4-\kappa\kappa'\tau, \\ g_1 = -\tau\left(f_3+f_2\right), \\ g_2 = \kappa f_3+\tau f_1, \\ g_3 = -\kappa f_2+\tau f_1, \\ h_1 = -\kappa''+\kappa'\tau+2\kappa\tau'+\kappa^3+\kappa\tau^2, \\ h_2 = -3\kappa\kappa'+\kappa^2\tau-\tau''-3\tau\tau'+\tau^3, \\ h_3 = -\kappa^2\tau-3\tau\tau'-\tau^3+\tau''. \end{cases}$$

3. Spatial Quaternionic Smarandache Ruled Surfaces

In this section, we define ruled surfaces whose base curves are called Smarandache curves. Moreover, we express the notions of the drall, striction curve, the pitch and angle of pitch. Finally, we calculate integral invariants, and we find some interesting results.

Definition 3.1. *Let $\alpha = \alpha(s)$ be a unit speed spatial quaternionic curve and $\{t, n, b\}$ be Frenet-Serret vectors. The ruled surfaces generated by the spatial quaternionic Smarandache curves are defined as follows:*

$$\begin{cases} \Phi_1(s,v) = \dfrac{1}{\sqrt{2}}\left(t(s)+n(s)\right)+vb(s), \\ \Phi_2(s,v) = \dfrac{1}{\sqrt{2}}\left(t(s)+b(s)\right)+vn(s), \\ \Phi_3(s,v) = \dfrac{1}{\sqrt{2}}\left(n(s)+b(s)\right)+vt(s). \end{cases}$$

These ruled surfaces are called spatial quaternionic $tn-$Smarandache ruled surface, spatial quaternionic $tb-$Smarandache ruled surface and spatial quaternionic $nb-$Smarandache *ruled surface, respectively.*

Theorem 3.2. *Let Φ_1, Φ_2, and Φ_3 be spatial quaternionic Smarandache ruled surfaces, then the striction curves r_{β_1}, r_{β_2} and r_{β_3} of Φ_1, Φ_2 and Φ_3 are*

$$\begin{cases} r_{\beta_1} = \beta_1 + \dfrac{\kappa}{\tau\sqrt{2\kappa^2+\tau^2}} b, \\ r_{\beta_2} = \beta_2, \\ r_{\beta_3} = \beta_3 + \dfrac{\tau}{\kappa\sqrt{\kappa^2+2\tau^2}} t, \end{cases}$$

respectively, where $\kappa \neq 0$ and $\tau \neq 0$.

Proof. Let $\alpha = \alpha(s)$ be a unit speed spatial quaternion curve with the Frenet vectors $\{t, n, b\}$ and the Smarandache curves $\beta_1, \beta_2, \beta_3$ of the spatial quaternion curve α. By using the quaternionic inner product, the striction curve of the spatial quaternionic ruled surface Φ_1 can be written by

$$r_{\beta_1} = \beta_1 - \frac{h\left(b', T^{\beta_1}\right)}{\rho(b')^2} b = \beta_1 - \frac{\frac{1}{2}\left(b' \times \bar{T}^{\beta_1} + T^{\beta_1} \times \bar{b}'\right)}{\sqrt{h\left(b', b'\right)}} b.$$

Substituting $b' = \tau n$ and using the complex conjugate of a quaternion, we arrive at

$$r_{\beta_1} = \beta_1 - \frac{\frac{1}{2}\left(\tau\left(n \times T^{\beta_1}\right) + \tau\left(T^{\beta_1} \times n\right)\right)}{\tau^2} b.$$

Considering the spatial quaternion, the striction curve is

$$r_{\beta_1} = \beta_1 + \frac{\kappa}{\tau\sqrt{2\kappa^2+\tau^2}} b.$$

If the equation (1), the Frenet invariants and spatial quaternions are used, the striction curves of the ruled surfaces Φ_2, Φ_3 are found

$$\begin{aligned} r_{\beta_2} &= \beta_2 - \frac{h\left(n', T^{\beta_2}\right)}{\rho(n')^2} n = \beta_2 - \frac{\frac{1}{2}\left(n' \times \bar{T}^{\beta_2} + T^{\beta_2} \times \bar{n}'\right)}{\sqrt{h\left(n', n'\right)}} n \\ &= \beta_2 + \frac{\frac{1}{2}\left(-\kappa\left(t \times T^{\beta_2}\right) + \tau\left(b \times T^{\beta_2}\right) - \kappa\left(T^{\beta_2} \times t\right) + \tau\left(T^{\beta_2} \times b\right)\right)}{\kappa^2+\tau^2} n \\ &= \beta_2 \end{aligned}$$

and

$$\begin{aligned} r_{\beta_3} &= \beta_3 - \frac{h\left(t', T^{\beta_3}\right)}{\rho(t')^2} t = \beta_3 - \frac{\frac{1}{2}\left(t' \times \bar{T}^{\beta_3} + T^{\beta_3} \times \bar{t}'\right)}{\sqrt{h\left(t', t'\right)}} t \\ &= \beta_3 - \frac{\frac{1}{2}\left(\kappa\left(n \times T^{\beta_3}\right) + \kappa\left(T^{\beta_3} \times t\right)\right)}{\kappa^2} t \\ &= \beta_3 + \frac{\tau}{\kappa\sqrt{\kappa^2+2\tau^2}} t. \end{aligned}$$

□

Theorem 3.3. *Let Φ_1, Φ_2, and Φ_3 be spatial quaternionic Smarandache ruled surfaces, then the dralls of the closed spatial quaternionic Smarandache ruled surfaces are*

$$\begin{cases} P_{\beta_1} = \dfrac{-\kappa}{\tau\sqrt{2\kappa^2+\tau^2}}, \\ P_{\beta_2} = 0, \\ P_{\beta_3} = \dfrac{\tau}{\kappa\sqrt{\kappa^2+2\tau^2}}, \end{cases}$$

respectively, where $\kappa \neq 0$ and $\tau \neq 0$.

Proof. Let $\alpha = \alpha(s)$ be a unit speed spatial quaternion curve with the Frenet vectors $\{t, n, b\}$ and the Smarandache curves $\beta_1, \beta_2, \beta_3$ of the spatial quaternion curve α. According to Lemma 2.3. and quaternionic inner product, the dralls of the closed spatial quaternionic Smarandache ruled surface Φ_1 drawn by the motion of the binormal vector b is given by

$$P_{\beta_1} = \frac{1}{2}\frac{(b \times b') \times \overline{\beta'_1} + \beta'_1 \times \overline{(b \times b')}}{\rho(b')^2}.$$

Using the Frenet invariants and the spatial quaternions, we recompute for drall as follows:

$$\begin{aligned} P_{\beta_1} &= -\frac{1}{2}\frac{(b \times b') \times T^{\beta_1} + T^{\beta_1} \times (b \times b')}{h(b', b')} \\ &= -\frac{1}{2}\frac{(b \times (-\tau n)) \times T^{\beta_1} + T^{\beta_1} \times (b \times (-\tau n))}{\langle b', b' \rangle} \\ &= \frac{1}{2}\frac{\begin{array}{c}\tau\left(\frac{-\kappa}{\sqrt{2\kappa^2+\tau^2}}(t \times t) + \frac{\kappa}{\sqrt{2\kappa^2+\tau^2}}(t \times n) + \frac{\tau}{\sqrt{2\kappa^2+\tau^2}}(t \times b)\right) \\ +\tau\left(\frac{-\kappa}{\sqrt{2\kappa^2+\tau^2}}(t \times t) + \frac{\kappa}{\sqrt{2\kappa^2+\tau^2}}(n \times t) + \frac{\tau}{\sqrt{2\kappa^2+\tau^2}}(b \times t)\right)\end{array}}{\tau^2} \\ &= \frac{-\kappa}{\tau\sqrt{2\kappa^2+\tau^2}}. \end{aligned}$$

In similar way, we determine the dralls P_{β_2} and P_{β_3} of the surfaces Φ_2 and Φ_3 as

$$\begin{aligned} P_{\beta_2} &= \frac{1}{2}\frac{(n \times n') \times \overline{\beta'_2} + \beta'_2 \times \overline{(n \times n')}}{\rho(b')^2} \\ &= -\frac{1}{2}\frac{(n \times (-\kappa t + \tau b)) \times T^{\beta_2} + T^{\beta_2} \times (n \times (-\kappa t + \tau b))}{\langle b', b' \rangle} \\ &= -\frac{1}{2}\frac{\kappa\left(b \times T^{\beta_2}\right) + \tau\left(t \times T^{\beta_2}\right) + \kappa\left(T^{\beta_2} \times b\right) + \tau\left(T^{\beta_2} \times t\right)}{\kappa^2+\tau^2} \\ &= -\frac{1}{2}\frac{\kappa(b \times n) + \tau(t \times n) + \kappa(n \times b) + \tau(n \times t)}{\kappa^2+\tau^2} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}
P_{\beta_3} &= \frac{1}{2}\frac{(t\times t')\times\overline{\beta'_3}+\beta'_3\times\overline{(t\times t')}}{\rho(t')^2}\\
&= -\frac{1}{2}\frac{(t\times(\kappa n))\times T^{\beta_3}+T^{\beta_3}\times(t\times(\kappa n))}{\langle t',t'\rangle}\\
&= -\frac{1}{2}\frac{\kappa\begin{pmatrix}\frac{-\kappa}{\sqrt{\kappa^2+2\tau^2}}(b\times t)-\frac{\tau}{\sqrt{\kappa^2+2\tau^2}}(b\times n)+\frac{\tau}{\sqrt{\kappa^2+2\tau^2}}(b\times b)\\ -\frac{\kappa}{\sqrt{\kappa^2+2\tau^2}}(t\times b)-\frac{\tau}{\sqrt{\kappa^2+2\tau^2}}(n\times b)+\frac{\tau}{\sqrt{\kappa^2+2\tau^2}}(b\times b)\end{pmatrix}}{\kappa^2}\\
&= \frac{\tau}{\kappa\sqrt{\kappa^2+2\tau^2}}.
\end{aligned}$$

□

Corollary 3.4. *Let Φ_1, Φ_2, and Φ_3 be spatial quaternionic Smarandache ruled surfaces, then the following expressions exist:*

(i) *The quaternionic Smarandache ruled surface Φ_1 is not developable,*
(ii) *Since the drall of Φ_2 is zero, the quaternionic Smarandache ruled surface Φ_2 is developable,*
(iii) *The quaternionic Smarandache ruled surface Φ_3 is developable if and only if the base curve of Φ_3 is planar.*

Theorem 3.5. *Let Φ_1, Φ_2, and Φ_3 be spatial quaternionic Smarandache ruled surfaces, then the angles of pitch of the closed spatial quaternionic Smarandache ruled surfaces are*

$$\begin{cases}\lambda_{\beta_1}=\oint\frac{k_2^{\beta_1}\tau}{\sqrt{2\kappa^2+\tau^2}}+\oint\frac{k_1^{\beta_1}b_3}{\sqrt{2\kappa^2+\tau^2}\sqrt{a_1^2+a_2^2+a_3^2}},\\ \lambda_{\beta_2}=\oint k_2^{\beta_2},\\ \lambda_{\beta_3}=-\oint\frac{k_2^{\beta_3}\kappa}{\sqrt{\kappa^2+2\tau^2}}+\oint\frac{k_1^{\beta_3}g_1}{\sqrt{\kappa^2+2\tau^2}\sqrt{f_1^2+f_2^2+f_3^2}},\end{cases}$$

respectively.

Proof. Let $\alpha=\alpha(s)$ be a unit speed spatial quaternion curve with the Frenet vectors $\{t,n,b\}$ and the Smarandache curves β_1,β_2,β_3 of the spatial quaternion curve α. Taking into consideration Steiner vector $d^{\beta_1}=\oint w^{\beta_1}=\oint T^{\beta_1}k_2^{\beta_1}+B^{\beta_1}k_1^{\beta_1}$, the angle of pitch of the closed spatial quaternionic ruled surface λ_{β_1} of the surfaces Φ_1 is written by

$$\lambda_{\beta_1}=h\left(d^{\beta_1},b\right)=\frac{1}{2}\left(d^{\beta_1}\times\bar{b}+b\times\bar{d}^{\beta_1}\right)=-\frac{1}{2}\left(d^{\beta_1}\times b+b\times d^{\beta_1}\right).$$

Using the Frenet invariants and the spatial quaternions, we calculate the angle of the pitch as follows:

$$\begin{aligned}
\lambda_{\beta_1} &= -\frac{1}{2}\left(\oint k_2^{\beta_1}\left(T^{\beta_1}\times b\right)+\oint k_1^{\beta_1}\left(B^{\beta_1}\times b\right)+\oint k_2^{\beta_1}\left(b\times T^{\beta_1}\right)+\oint k_1^{\beta_1}\left(b\times B^{\beta_1}\right)\right)\\
&= -\frac{1}{2}\begin{pmatrix}
\oint \frac{k_2^{\beta_1}}{\sqrt{2\kappa^2+\tau^2}}\left(-\kappa\left(t\times b\right)+\kappa\left(n\times b\right)+\tau\left(b\times b\right)\right)\\
+\oint \frac{k_1^{\beta_1}}{\sqrt{2\kappa^2+\tau^2}\sqrt{a_1^2+a_2^2+a_3^2}}\left(b_1\left(t\times b\right)+b_2\left(n\times b\right)+b_3\left(b\times b\right)\right)\\
+\oint \frac{k_2^{\beta_1}}{\sqrt{2\kappa^2+\tau^2}}\left(-\kappa\left(b\times t\right)+\kappa\left(b\times n\right)+\tau\left(b\times b\right)\right)\\
+\oint \frac{k_1^{\beta_1}}{\sqrt{2\kappa^2+\tau^2}\sqrt{a_1^2+a_2^2+a_3^2}}\left(b_1\left(b\times t\right)+b_2\left(b\times n\right)+b_3\left(b\times b\right)\right)
\end{pmatrix}\\
&= \oint \frac{k_2^{\beta_1}\tau}{\sqrt{2\kappa^2+\tau^2}}+\oint \frac{k_1^{\beta_1}b_3}{\sqrt{2\kappa^2+\tau^2}\sqrt{a_1^2+a_2^2+a_3^2}}.
\end{aligned}$$

Considering $d^{\beta_2}=\oint w^{\beta_2}=\oint T^{\beta_2}k_2^{\beta_2}+B^{\beta_2}k_1^{\beta_2}$ and $d^{\beta_3}=\oint w^{\beta_3}=\oint T^{\beta_3}k_2^{\beta_3}+B^{\beta_3}k_1^{\beta_3}$, we can write the angles of the pitch $\lambda_{\beta_2},\ \lambda_{\beta_3}$ the ruled surfaces $\Phi_2,\ \Phi_3$ respectively,

$$\begin{aligned}
\lambda_{\beta_2} &= h\left(d^{\beta_2},n\right)=\frac{1}{2}\left(d^{\beta_2}\times\bar{n}+n\times\bar{d}^{\beta_2}\right)=-\frac{1}{2}\left(d^{\beta_2}\times n+n\times d^{\beta_2}\right)\\
&= -\frac{1}{2}\left(\oint k_2^{\beta_2}\left(T^{\beta_2}\times n\right)+\oint k_1^{\beta_2}\left(B^{\beta_2}\times n\right)+\oint k_2^{\beta_2}\left(n\times T^{\beta_2}\right)+\oint k_1^{\beta_2}\left(n\times B^{\beta_2}\right)\right)\\
&= -\frac{1}{2}\begin{pmatrix}
\oint k_2^{\beta_2}\left(n\times n\right)+\oint \frac{k_1^{\beta_2}}{\sqrt{\kappa^2+\tau^2}}\left(\tau\left(t\times n\right)+\kappa\left(b\times n\right)\right)\\
+\oint k_2^{\beta_2}\left(n\times n\right)+\oint \frac{k_1^{\beta_2}}{\sqrt{\kappa^2+\tau^2}}\left(\tau\left(n\times t\right)+\kappa\left(n\times b\right)\right)
\end{pmatrix}\\
&= \oint k_2^{\beta_2}
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{\beta_3} &= h\left(d^{\beta_3},t\right)=\frac{1}{2}\left(d^{\beta_3}\times\bar{t}+t\times\bar{d}^{\beta_3}\right)=-\frac{1}{2}\left(d^{\beta_3}\times t+t\times d^{\beta_3}\right)\\
&= -\frac{1}{2}\left(\oint k_2^{\beta_3}\left(T^{\beta_3}\times t\right)+\oint k_1^{\beta_3}\left(B^{\beta_3}\times t\right)+\oint k_2^{\beta_3}\left(t\times T^{\beta_3}\right)+\oint k_1^{\beta_3}\left(t\times B^{\beta_3}\right)\right)\\
&= -\frac{1}{2}\begin{pmatrix}
\oint \frac{k_2^{\beta_3}}{\sqrt{\kappa^2+2\tau^2}}\left(-\kappa\left(t\times t\right)-\tau\left(n\times t\right)+\tau\left(b\times t\right)\right)\\
+\oint \frac{k_1^{\beta_3}}{\sqrt{\kappa^2+2\tau^2}\sqrt{f_1^2+f_2^2+f_3^2}}\left(g_1\left(t\times t\right)+g_2\left(n\times t\right)+g_3\left(b\times t\right)\right)\\
+\oint \frac{k_2^{\beta_3}}{\sqrt{\kappa^2+2\tau^2}}\left(-\kappa\left(t\times t\right)-\tau\left(t\times n\right)+\tau\left(t\times b\right)\right)\\
+\oint \frac{k_1^{\beta_3}}{\sqrt{\kappa^2+2\tau^2}\sqrt{f_1^2+f_2^2+f_3^2}}\left(g_1\left(t\times t\right)+g_2\left(t\times n\right)+g_3\left(t\times b\right)\right)
\end{pmatrix}\\
&= \oint \frac{-k_2^{\beta_3}\kappa}{\sqrt{\kappa^2+2\tau^2}}+\oint \frac{k_1^{\beta_3}g_1}{\sqrt{\kappa^2+2\tau^2}\sqrt{f_1^2+f_2^2+f_3^2}}.
\end{aligned}$$

□

Theorem 3.6. *Let Φ_1, Φ_2, and Φ_3 be spatial quaternion Smarandache ruled surfaces, then the pitches of the closed spatial quaternionic Smarandache ruled surfaces are*

$$\begin{cases} l_{\beta_1} = \oint \frac{\tau}{\sqrt{2\kappa^2+\tau^2}} ds, \\ l_{\beta_2} = \oint ds, \\ l_{\beta_3} = -\oint \frac{\kappa}{\sqrt{\kappa^2+2\tau^2}} ds, \end{cases}$$

respectively.

Proof. Let $\alpha = \alpha(s)$ be a unit speed spatial quaternion curve with the Frenet vectors $\{t, n, b\}$ and the Smarandache curves $\beta_1, \beta_2, \beta_3$ of the spatial quaternion curve α. For $V^{\beta_1} = \oint d\beta_1 = \oint T^{\beta_1} ds$, the pitch l_{β_1} of the closed spatial quaternionic Smarandache ruled surface Φ_1 is written by

$$l_{\beta_1} = h\left(V^{\beta_1}, b\right) = h\left(\oint T^{\beta_1} ds, b\right).$$

If we substitute the values in the above equation, the pitch l_{β_1} of the closed spatial quaternionic ruled surface Φ_1 is

$$\begin{aligned} l_{\beta_1} &= \frac{1}{2}\left(\oint \left(T^{\beta_1} \times \bar{b}\right) ds + \oint \left(b \times \bar{T}^{\beta_1}\right) ds\right) \\ &= -\frac{1}{2}\left(\oint \left(T^{\beta_1} \times b\right) ds + \oint \left(b \times T^{\beta_1}\right) ds\right) \\ &= -\frac{1}{2}\begin{pmatrix} \oint \frac{1}{\sqrt{2\kappa^2+\tau^2}} \left(-\kappa\left(t \times b\right) + \kappa\left(n \times b\right) + \tau\left(b \times b\right)\right) ds \\ + \oint \frac{1}{\sqrt{2\kappa^2+\tau^2}} \left(-\kappa\left(b \times t\right) + \kappa\left(b \times n\right) + \tau\left(b \times b\right)\right) ds \end{pmatrix} \\ &= \oint \frac{\tau}{\sqrt{2\kappa^2+\tau^2}} ds. \end{aligned}$$

For $V^{\beta_2} = \oint d\beta_2 = \oint T^{\beta_2} ds$ and $V^{\beta_3} = \oint d\beta_3 = \oint T^{\beta_3} ds$, we can write the pitches l_{β_2} and l_{β_3} of the closed spatial quaternionic ruled surfaces Φ_2 and Φ_2 as follows:

$$\begin{aligned} l_{\beta_2} &= h\left(V^{\beta_2}, b\right) = h\left(\oint T^{\beta_2} ds, n\right) \\ &= \frac{1}{2}\left(\oint \left(T^{\beta_2} \times \bar{n}\right) ds + \oint \left(n \times \bar{T}^{\beta_2}\right) ds\right) \\ &= -\frac{1}{2}\left(\oint \left(T^{\beta_2} \times n\right) ds + \oint \left(n \times T^{\beta_2}\right) ds\right) \\ &= -\frac{1}{2}\left(\oint (n \times n) ds + \oint (n \times n)\, ds\right) \\ &= \oint ds \end{aligned}$$

and

$$
\begin{aligned}
l_{\beta_3} &= h\left(V^{\beta_3}, t\right) = h\left(\oint T^{\beta_3} ds, t\right) \\
&= \frac{1}{2}\left(\oint \left(T^{\beta_3} \times \bar{t}\right) ds + \oint \left(t \times \bar{T}^{\beta_3}\right) ds\right) \\
&= -\frac{1}{2}\left(\oint \left(T^{\beta_3} \times t\right) ds + \oint \left(t \times T^{\beta_3}\right) ds\right) \\
&= -\frac{1}{2}\left(\begin{array}{l} \oint \frac{1}{\sqrt{\kappa^2 + 2\tau^2}} \left(-\kappa \left(t \times t\right) - \tau \left(n \times t\right) + \tau \left(b \times t\right)\right) ds \\ + \oint \frac{1}{\sqrt{\kappa^2 + 2\tau^2}} \left(-\kappa \left(t \times t\right) - \tau \left(t \times n\right) + \tau \left(t \times b\right)\right) ds \end{array}\right) \\
&= -\oint \frac{\kappa}{\sqrt{\kappa^2 + 2\tau^2}} ds.
\end{aligned}
$$

□

Example 3.7. *Let us consider a spatial quaternionic curve given by the parametric equation*

$$
\alpha(s) = \frac{3}{4}\left(\cos(s) + \frac{\cos(3s)}{9}, \sin(s) + \frac{\sin(3s)}{9}, \frac{-2\cos(s)}{\sqrt{3}}\right).
$$

The quaternionic Smarandache curves constructed by the Frenet vectors of the spatial quaternionic curve α are

$$
\beta_1\left(s^*(s)\right) = \left(\begin{array}{l} -\frac{\sqrt{3}\left(\cos(s) + \cos(3s)\right) + \cos(s)\left(3\sin(s) + \sin(3s)\right)}{4\sqrt{2}\cos(s)}, \\ \frac{\cos(s)^3 - \sqrt{3}\cos(s)\sin(s)}{\sqrt{2}}, \frac{\sqrt{1+\cos(2s)}\sec(s) + \sqrt{6}\sin(s)}{4} \end{array}\right),
$$

$$
\beta_2\left(s^*(s)\right) = \left(\begin{array}{l} \frac{2\cos(s)\cos(2s) - 3\sin(s) - \sin(3s)}{4\sqrt{2}}, \frac{\cos(s)^3 + \sin(s)^3}{\sqrt{2}}, \\ \frac{\sqrt{6}\left(\cos(s) + \sin(s)\right)}{4} \end{array}\right),
$$

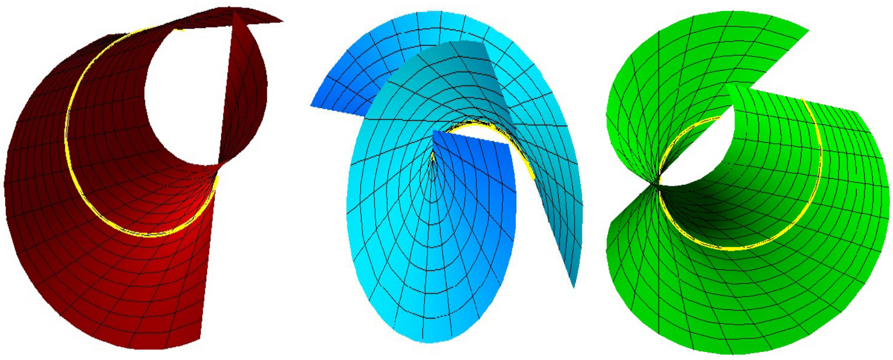
$$
\beta_3\left(s^*(s)\right) = \left(\begin{array}{l} -\frac{\cos(s)\left(-3\cos(s) + 2\sqrt{3}\cos(2s) + \cos(3s)\right)}{4\sqrt{1+\cos(2s)}}, \\ \frac{\cos(s)\sin(s)\left(-\sqrt{3} + \sin(s)\tan(s)\right)}{\sqrt{2}}, \\ \frac{\sqrt{1+\cos(2s)}\left(\sqrt{3} + \sec(s)\right)}{4} \end{array}\right),
$$

and the spatial quaternionic Smarandache ruled surfaces are found as

$$
\Phi_1(s,v)=\begin{pmatrix}
-\dfrac{\sqrt{3}\left(\cos(s)+\cos(3s)\right)+\cos(s)\left(3\sin(s)+\sin(3s)\right)}{4\sqrt{2}\cos(s)} \\
-\dfrac{v\cos(s)\left(-2+\cos(2s)\right)}{2},\ \dfrac{2\cos(s)^3-\sqrt{3}\sin(2s)}{2\sqrt{2}}+v\sin(s)^3, \\
\dfrac{\sqrt{2}+\sqrt{6}\sin(s)+2\sqrt{3}v\cos(s)}{4}
\end{pmatrix},
$$

$$
\Phi_2(s,v)=\begin{pmatrix}
\dfrac{2\cos(s)\cos(2s)-3\sin(s)-\sin(3s)}{4\sqrt{2}}-\dfrac{\sqrt{3}v\left(\cos(s)+\cos(3s)\right)}{4\cos(s)}, \\
\dfrac{\sqrt{2}\left(\cos(s)^3+\sin(s)^3\right)-\sqrt{3}v\sin(2s)}{4},\ \dfrac{\sqrt{6}\left(\cos(s)+\sin(s)\right)+2v}{4}
\end{pmatrix},
$$

$$
\Phi_3(s,v)=\begin{pmatrix}
-\dfrac{\left(-3\cos(s)+2\sqrt{3}\cos(2s)+\cos(3s)\right)}{4\sqrt{2}}+\dfrac{v\left(-3\sin(s)-\sin(3s)\right)}{4}, \\
\dfrac{-\sin(2s)\left(\sqrt{3}-\sin(s)\tan(s)\right)}{2\sqrt{2}}+v\cos(s)^3, \\
\dfrac{\sqrt{2}\left(\sqrt{3}\cos(s)+1\right)+2\sqrt{3}v\sin(s)}{4}
\end{pmatrix}.
$$



(A) The surface $\Phi_1(s,v)$ and the curve β_1

(B) The surface $\Phi_2(s,v)$ and the curve β_2

(C) The surface $\Phi_3(s,v)$ and the curve β_3

FIGURE 1. The spatial quaternionic Smarandache ruled surfaces (top view) and the quaternionic Smarandache curve (yellow) with $s \in [-0.5, 1.5]$ and $v \in [-1, 1]$

4. Conclusion

In this paper, we derive the spatial quaternionic Smarandache ruled surfaces and the striction curves and dralls of these surfaces are calculated. The conditions of these ruled surfaces to be developable are investigated. If the quaternionic Smarandache ruled surfaces are closed, the pitches and angle of pitches are constructed. The integral invariants of these surfaces, whose base curves are formed by the quaternionic Smarandache curve, are calculated using quaternionic formulas.

References

[1] K. Bharathi and M. Nagaraj, *Quaternion valued function of a real Serret-Frenet formulae*, Indian J. Pure Appl. Math. **16** (1985), 741–756.

[2] A. Çalışkan, *The quaternionic ruled surfaces in terms of alternative frame*, Palest. J. Math **11** (2022), 406–412.

[3] M. Çetin and H. Kocayiğit, *On the quaternionic Smarandache curves in Euclidean 3-space*, Int. J. Contemp. Math. Sciences **8** (2013), 139–150.

[4] J. Chen and J. Li, *Quaternionic maps and minimal surfaces*, Ann. Sc. norm. super. Pisa - Cl. sci. **4** (2005), 375–388.

[5] P. R. Girard, *The quaternion group and modern physics*, Eur. J. Phys. **5** (1984), 25–32.

[6] W. R. Hamilton, *Elements of Quaternions*, Chelsea, New York, 1899.

[7] J. A. Hanson and H. Ma, *Quaternion frame approach to streamline visualization*, IEEE Trans. Vis. Comput. Graph. **1** (1995), 164–173.

[8] K. I. Kou and Y. H. Xia, *Linear quaternion differential equations: Basic theory and fundamental results*, Stud. Appl. Math. **141** (2018), 3–45.

[9] Y. Li, K. Eren, K. H. Ayvacı, and S. Ersoy, *Simultaneous characterizations of partner ruled surfaces using Flc frame*, AIMS Math. **7** (2022), 20213–20229.

[10] Y. Li, S. H. Nazra, and R. A. Abdel-Baky, *Singularity properties of timelike sweeping surface in Minkowski 3-space*, Symmetry **14** (2022), 1996.

[11] U. Öztürk, B. Sarıkaya, P. Haskul, and A. Emir, *On Smarandache curves in affine 3-space*, J. New Theory **38** (2022), 61–69.

[12] S. Ouarab, *NC-Smarandache ruled surface and NW-Smarandache ruled surface according to alternative moving frame in* E^3, J. Math. **2021** (2021), 6.

[13] S. Ouarab, *Smarandache ruled surfaces according to Darboux frame in* E^3, J. Math. **2021** (2021), 10.

[14] S. Şenyurt and A. Çalışkan, *The quaternionic expression of ruled surfaces*, Filomat **32** (2018), 5753–5766.

[15] S. Şenyurt and K. Eren, *Smarandache curves of spacelike anti-Salkowski curve with a spacelike principal normal according to Frenet frame*, Gümüşhane University Journal of Science and Technology **10** (2020), 251–260.

[16] K. Shoemake, *On the quaternionic Smarandache curves in Euclidean 3-space*, Comput Graph (ACM) **9** (1985), 245–253.

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