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A NOTE ON MIXED POLYNOMIALS AND NUMBERS

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Abstract. The main object of this article is to propose a unified extension of Bernoulli, Euler and Genocchi polynomials by means of a new family of mixed polynomials whose generating function is given in terms of generalized Bessel function. We also discuss here some fundamental properties of our introduced mixed polynomials by making use of the series arrangement technique. Furthermore, some conclusions of our present study are also pointed out in the last section.

1. Introduction

Throughout in this paper, let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of natural numbers, real numbers and complex numbers, respectively, and let

$$\mathbb{N}_0 := \{0, 1, 2, 3, ...\} = \mathbb{N} \cup \{0\}.$$

The generating functions of the Bernoulli polynomials, Euler polynomials and Genocchi polynomials are defined, respectively, by (see, [24], [26], [15])

$$\frac{t}{e^t - 1} e^{ut} = \sum_{n=0}^{\infty} \mathbb{B}_n(u) \frac{t^n}{n!} \ (|t| < 2\pi), \tag{1}$$

$$\frac{2}{e^t + 1} e^{ut} = \sum_{n=0}^{\infty} \mathbb{E}_n(u) \frac{t^n}{n!} \ (|t| < \pi) \tag{2}$$

and

$$\frac{2t}{e^t + 1} e^{ut} = \sum_{n=0}^{\infty} \mathbb{G}_n(u) \frac{t^n}{n!} \ (|t| < \pi). \tag{3}$$

Obviously, for $u = 0$, we have

$$\mathbb{B}_n(0) = \mathbb{B}_n, \ \mathbb{E}_n(0) = \mathbb{E}_n, \text{ and } \mathbb{G}_n(0) = \mathbb{G}_n,$$

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where $\mathbb{B}_n$, $\mathbb{E}_n$ and $\mathbb{G}_n$ are the well known Bernoulli, Euler and Genocchi numbers, respectively.

These polynomials play a crucial role in several parts of analysis, calculus of finite differences and various other fields of mathematical analysis, for example, in statistics, numerical analysis, combinatorics etc. A useful generalization of the Bernoulli, Euler and Genocchi polynomials are described, respectively, by means of the following generating functions (see, [16]–[21]):

$$\left(\frac{t}{e^t-1}\right)^p e^{ut} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(p)}(u)\frac{t^n}{n!} \ (|t|<2\pi,\ 1^p:=1), \tag{4}$$

$$\left(\frac{2}{e^t+1}\right)^p e^{ut} = \sum_{n=0}^{\infty} \mathbb{E}_n^{(p)}(u)\frac{t^n}{n!} \ (|t|<\pi,\ 1^p:=1) \tag{5}$$

and

$$\left(\frac{2t}{e^t+1}\right)^p e^{ut} = \sum_{n=0}^{\infty} \mathbb{G}_n^{(p)}(u)\frac{t^n}{n!} \ (|t|<\pi,\ 1^p:=1). \tag{6}$$

Due to notable applications in mathematical analysis, a unified treatment of such polynomials introduced from time to time by many researchers (see, for example, [1], [2], [3], [6], [14], [8]–[23], [25] and the references cited therein). The main motive of the present investigation is to propose a new extension of Bernoulli, Euler and Genocchi polynomials by means of a single generating function involving generalized Bessel function of the first kind $w^b_{\nu,c}(t)$.

The generalized Bessel function of the first kind $w^b_{\nu,c}(t)$ is defined as follows (see, [4], see also [7]):

$$w^b_{\nu,c}(t) = \sum_{k=0}^{\infty} \frac{(-c)^k\ (t/2)^{\nu+2k}}{k!\ \Gamma(\nu+k+\frac{1+b}{2})}, \tag{7}$$

where $b, c, \nu \in \mathbb{C}$ with $\mathbb{R}(\nu) > -1$ and $w^b_{\nu,c}(0) = 0$.

Furthermore, for some particular values of the parameters, the generalized Bessel function $w^b_{\nu,c}(t)$ reduces to the under mentioned hyperbolic sine and cosine functions (see, [4], see also [7]):

(i) On setting $\nu = 1 - \frac{b}{2}$ and replacing c by $-c^2$ in (7), we get

$$w^b_{1-\frac{b}{2},\ -c^2}(t) = \left(\frac{2}{t}\right)^{\frac{b}{2}} \frac{\sinh ct}{\sqrt{\pi}}. \tag{8}$$

(ii) Further, on setting $\nu = -\frac{b}{2}$ and replacing c by $-c^2$ in (7), we have

$$w^b_{-\frac{b}{2},\ -c^2}(t) = \left(\frac{2}{t}\right)^{\frac{b}{2}} \frac{\cosh ct}{\sqrt{\pi}}. \tag{9}$$

Here, $\sinh t$ and $\cosh t$ denotes the hyperbolic sine and cosine functions, which can be expressed in terms of exponential functions as follows:

$$\sinh t = \frac{e^t - e^{-t}}{2} \tag{10}$$

and

$$\cosh t = \frac{e^t + e^{-t}}{2}. \tag{11}$$

2. Mixed polynomials and numbers

In this section, we propose a unified extension of Bernoulli, Euler and Genocchi polynomials (numbers) by means of the mixed polynomials whose generating function is given in terms of the generalized Bessel function of the first kind $w^b_{\nu,c}(t)$.

Let

$$\mathbb{G}^{b,c,d}_{\nu,a}(t) = \frac{2^{b-d} t^{a-\frac{b}{2}} e^{-\frac{t}{2}}}{\sqrt{\pi}\ w^b_{\nu,c}(\frac{t}{2})}, \tag{12}$$

where $a, d \in \mathbb{N}_0$, $b, c, \nu \in \mathbb{C}$ and $w^b_{\nu,c}(t)$ is the generalized Bessel function of the first kind defined by (7).

Definition 2.1. The new mixed polynomials $\Phi^{a,b,c,d}_{n,\nu}(u)$ for nonnegative integer n are defined by

$$\mathbb{G}^{b,c,d}_{\nu,a}(t) e^{ut} = \sum_{n=0}^{\infty} \Phi^{a,b,c,d}_{n,\nu}(u) \frac{t^n}{n!}, \tag{13}$$

where $\mathbb{G}^{b,c,d}_{\nu,a}(t)$ is given in (12).

If we assign some particular values to the parameters of our mixed polynomials $\Phi^{a,b,c,d}_{n,\nu}(u)$ then these polynomials easily reduces to the Bernoulli polynomials $\mathbb{B}_n(u)$, Euler polynomials $\mathbb{E}_n(u)$ and Genocchi polynomials $\mathbb{G}_n(u)$. The special cases of (13) are given as follows:

Case-I (Connection with Bernoulli polynomials). Setting $\nu = 1 - \frac{b}{2}$, $c = -1,\ a = d = 1$ in (13) and then by using (12) and (8), we arrive at

$$\frac{t\ e^{-\frac{t}{2}}}{2 \sinh \frac{t}{2}}\ e^{ut} = \sum_{n=0}^{\infty} \Phi^{1,b,-1,1}_{n,1-\frac{b}{2}}(u) \frac{t^n}{n!},$$

or equivalently,

$$\frac{t}{e^t - 1} e^{ut} = \sum_{n=0}^{\infty} \Phi^{1,b,-1,1}_{n,1-\frac{b}{2}}(u) \frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} \mathbb{B}_n(u)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \Phi^{1,b,-1,1}_{n,1-\frac{b}{2}}(u)\frac{t^n}{n!},$$

where $\mathbb{B}_n(u)$ are the Bernoulli polynomials defined by (1).

Hence, we have

$$\Phi^{1,b,-1,1}_{n,1-\frac{b}{2}}(u) = \mathbb{B}_n(u). \tag{14}$$

Case-II (Connection with Euler polynomials). Further, on setting $\nu = -\frac{b}{2}$, $c = -1,\ a = d = 0$ in (13) and then by using (12) and (9), we get

$$\frac{e^{-\frac{t}{2}}}{\cosh\frac{t}{2}}\ e^{ut} = \sum_{n=0}^{\infty} \Phi^{0,b,-1,0}_{n,-\frac{b}{2}}(u)\frac{t^n}{n!},$$

or equivalently,

$$\frac{2}{e^t+1}e^{ut} = \sum_{n=0}^{\infty} \Phi^{0,b,-1,0}_{n,-\frac{b}{2}}(u)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} \mathbb{E}_n(u)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \Phi^{0,b,-1,0}_{n,-\frac{b}{2}}(u)\frac{t^n}{n!},$$

where $\mathbb{E}_n(u)$ are the Euler polynomials given in (2).

Hence, we get

$$\Phi^{0,b,-1,0}_{n,-\frac{b}{2}}(u) = \mathbb{E}_n(u). \tag{15}$$

Case-III (Connection with Genocchi polynomials). Taking $\nu = -\frac{b}{2}$, $c = -1,\ a = 1,\ d = 0$ in (13) and then by using (12) and (9), we obtain

$$\frac{t\ e^{-\frac{t}{2}}}{\cosh\frac{t}{2}}\ e^{ut} = \sum_{n=0}^{\infty} \Phi^{1,b,-1,0}_{n,-\frac{b}{2}}(u)\frac{t^n}{n!},$$

or equivalently,

$$\frac{2t}{e^t+1}e^{ut} = \sum_{n=0}^{\infty} \Phi^{1,b,-1,0}_{n,-\frac{b}{2}}(u)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} \mathbb{G}_n(u)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \Phi^{1,b,-1,0}_{n,-\frac{b}{2}}(u)\frac{t^n}{n!},$$

where $\mathbb{G}_n(u)$ are the Genocchi polynomials given in (3).

Hence, we have

$$\Phi^{1,b,-1,0}_{n,-\frac{b}{2}}(u) = \mathbb{G}_n(u). \tag{16}$$

Also, it is possible to define the new mixed numbers by taking $u = 0$, i.e.

$$\Phi^{a,b,c,d}_{n,\nu}(0) = \Phi^{a,b,c,d}_{n,\nu}. \tag{17}$$

Obviously, by assigning some particular values to the parameters (which are discussed in case-I, case-II and case-III), these mixed numbers will easily reduces to the Bernoulli numbers $\mathbb{B}_n$, Euler numbers $\mathbb{E}_n$ and Genocchi numbers $\mathbb{G}_n$.

Definition 2.2. For arbitrary real or complex parameter p, the higher order mixed polynomials $\Phi_{n,\nu}^{[p;a,b,c,d]}(u)$ are defined by

$$\left[\mathbb{G}_{\nu,a}^{b,c,d}(t)\right]^p e^{ut} = \sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!}. \tag{18}$$

Clearly, for $p = 1$, (18) reduce to (13). Also, we get
$\Phi_{n,1-\frac{b}{2}}^{[p;1,b,-1,1]}(u) = \mathbb{B}_n^{(p)}(u)$, $\Phi_{n,-\frac{b}{2}}^{[p;0,b,-1,0]}(u) = \mathbb{E}_n^{(p)}(u)$ and $\Phi_{n,-\frac{b}{2}}^{[p;1,b,-1,0]}(u) = \mathbb{G}_n^{(p)}(u)$,

where $\mathbb{B}_n^{(p)}(u)$, $\mathbb{E}_n^{(p)}(u)$ and $\mathbb{G}_n^{(p)}(u)$ are the generalized Bernoulli, Euler and Genocchi polynomials defined by (4), (5) and (6), respectively.

Furthermore for $u = 0$, we get a new family of higher order mixed numbers,
i.e.

$$\Phi_{n,\nu}^{[p;a,b,c,d]}(0) = \Phi_{n,\nu}^{[p;a,b,c,d]}. \tag{19}$$

Some of the higher order mixed polynomials and numbers are listed below:

$$\Phi_{0,a-\frac{b}{2}}^{[p;a,b,c,d]}(u) = \frac{\Gamma(a+\frac{1}{2})}{\pi^{\frac{p}{2}}}\, 2^{(2a-d)p},$$

$$\Phi_{1,a-\frac{b}{2}}^{[p;a,b,c,d]}(u) = \frac{\Gamma(a+\frac{1}{2})}{\pi^{\frac{p}{2}}}\left(u-\frac{p}{2}\right)2^{(2a-d)p},$$

$$\Phi_{2,a-\frac{b}{2}}^{[p;a,b,c,d]}(u) = \frac{\Gamma(a+\frac{1}{2})}{\pi^{\frac{p}{2}}}\left\{\left(u-\frac{p}{2}\right)^2+\frac{pc}{8(a+\frac{1}{2})}\right\}2^{(2a-d)p},$$

$$\Phi_{3,a-\frac{b}{2}}^{[p;a,b,c,d]}(u) = \frac{\Gamma(a+\frac{1}{2})}{\pi^{\frac{p}{2}}}\left\{\left(u-\frac{p}{2}\right)^3+\frac{3pc\left(u-\frac{p}{2}\right)}{8(a+\frac{1}{2})}\right\}2^{(2a-d)p},$$

$$\Phi_{4,a-\frac{b}{2}}^{[p;a,b,c,d]}(u) = \frac{\Gamma(a+\frac{1}{2})}{\pi^{\frac{p}{2}}}\left\{\left(u-\frac{p}{2}\right)^4+\frac{3pc\left(u-\frac{p}{2}\right)^2}{4(a+\frac{1}{2})}-\frac{3pc^2}{64\left(a+\frac{1}{2}\right)\left(a+\frac{3}{2}\right)}\right.$$
$$\left.+\frac{3p(p+1)c^2}{64\left(a+\frac{1}{2}\right)^2}\right\}2^{(2a-d)p},\cdots$$

and

$$\Phi_{0,a-\frac{b}{2}}^{[p;a,b,c,d]} = \frac{\Gamma(a+\frac{1}{2})}{\pi^{\frac{p}{2}}}\ 2^{(2a-d)p},$$

$$\Phi_{1,a-\frac{b}{2}}^{[p;a,b,c,d]} = -\frac{p\ \Gamma(a+\frac{1}{2})}{2\ \pi^{\frac{p}{2}}} 2^{(2a-d)p},$$

$$\Phi_{2,a-\frac{b}{2}}^{[p;a,b,c,d]} = \frac{\Gamma(a+\frac{1}{2})}{\pi^{\frac{p}{2}}}\ \left\{\frac{p^2}{4} + \frac{pc}{8(a+\frac{1}{2})}\right\} 2^{(2a-d)p},$$

$$\Phi_{3,a-\frac{b}{2}}^{[p;a,b,c,d]} = \frac{\Gamma(a+\frac{1}{2})}{\pi^{\frac{p}{2}}}\ \left\{-\frac{p^3}{8} - \frac{3p^2c}{16(a+\frac{1}{2})}\right\} 2^{(2a-d)p},$$

$$\Phi_{4,a-\frac{b}{2}}^{[p;a,b,c,d]} = \frac{\Gamma(a+\frac{1}{2})}{\pi^{\frac{p}{2}}}\ \left\{\frac{p^4}{16} + \frac{3cp^3}{16(a+\frac{1}{2})} - \frac{3pc^2}{64\left(a+\frac{1}{2}\right)\left(a+\frac{3}{2}\right)} + \frac{3p(p+1)c^2}{64\left(a+\frac{1}{2}\right)^2}\right\} 2^{(2a-d)p}, \cdots$$

For $p=1$, all the above polynomials and numbers immediately reduce to

$$\Phi_{0,a-\frac{b}{2}}^{a,b,c,d}(u) = \frac{\Gamma(a+\frac{1}{2})}{\sqrt{\pi}}\ 2^{(2a-d)},$$

$$\Phi_{1,a-\frac{b}{2}}^{a,b,c,d}(u) = \frac{\Gamma(a+\frac{1}{2})}{\sqrt{\pi}}\ \left(u-\frac{1}{2}\right) 2^{(2a-d)},$$

$$\Phi_{2,a-\frac{b}{2}}^{a,b,c,d}(u) = \frac{\Gamma(a+\frac{1}{2})}{\sqrt{\pi}}\ \left\{\left(u-\frac{1}{2}\right)^2 + \frac{c}{8(a+\frac{1}{2})}\right\} 2^{(2a-d)},$$

$$\Phi_{3,a-\frac{b}{2}}^{a,b,c,d}(u) = \frac{\Gamma(a+\frac{1}{2})}{\sqrt{\pi}}\ \left\{\left(u-\frac{1}{2}\right)^3 + \frac{3c\left(u-\frac{1}{2}\right)}{8(a+\frac{1}{2})}\right\} 2^{(2a-d)},$$

$$\Phi_{4,a-\frac{b}{2}}^{a,b,c,d}(u) = \frac{\Gamma(a+\frac{1}{2})}{\sqrt{\pi}}\ \left\{\left(u-\frac{1}{2}\right)^4 + \frac{3c\left(u-\frac{1}{2}\right)^2}{4(a+\frac{1}{2})} - \frac{3c^2}{64\left(a+\frac{1}{2}\right)\left(a+\frac{3}{2}\right)} + \frac{3c^2}{32\left(a+\frac{1}{2}\right)^2}\right\} 2^{(2a-d)}, \cdots$$

and

$$\Phi_{0,a-\frac{b}{2}}^{a,b,c,d} = \frac{\Gamma(a+\frac{1}{2})}{\sqrt{\pi}}\ 2^{(2a-d)},$$

$$\Phi_{1,a-\frac{b}{2}}^{a,b,c,d} = -\frac{\Gamma(a+\frac{1}{2})}{2\ \sqrt{\pi}}\ 2^{(2a-d)},$$

$$\Phi_{2,a-\frac{b}{2}}^{a,b,c,d} = \frac{\Gamma(a+\frac{1}{2})}{\sqrt{\pi}} \left\{\frac{1}{4} + \frac{c}{8(a+\frac{1}{2})}\right\} 2^{(2a-d)},$$

$$\Phi_{3,a-\frac{b}{2}}^{a,b,c,d} = \frac{\Gamma(a+\frac{1}{2})}{\sqrt{\pi}} \left\{-\frac{1}{8} - \frac{3c}{16(a+\frac{1}{2})}\right\} 2^{(2a-d)},$$

$$\Phi_{4,a-\frac{b}{2}}^{a,b,c,d} = \frac{\Gamma(a+\frac{1}{2})}{\sqrt{\pi}} \left\{\frac{1}{16} + \frac{3c}{16(a+\frac{1}{2})} - \frac{3c^2}{64\left(a+\frac{1}{2}\right)\left(a+\frac{3}{2}\right)} + \frac{3c^2}{32\left(a+\frac{1}{2}\right)^2}\right\} 2^{(2a-d)}, \cdots,$$

which are our mixed polynomials and numbers defined by (13) and (17), respectively.

Remark 2.1. *If we compare our mixed polynomials* $\Phi_{n,\nu}^{a,b,c,d}(u)$ *and* $\Phi_{n,\nu}^{[p;a,b,c,d]}(u)$ *given in* (13) *and* (18), *respectively, with the polynomials* $B_{n,\alpha}(u)$ *defined by Frappier* [5] *then we see that these polynomials are showing two new extensions (in a slightly different form) of* $B_{n,\alpha}(u)$.

3. Properties of the Mixed polynomials

The main object of this section is to establish some basic properties of our newly introduced polynomials, which are given in the following theorems:

Theorem 3.1. *Let* $r \in \mathbb{N}$. *Then we have the following derivatives for our mixed polynomials:*

$$\frac{d^r}{du^r}\Phi_{n,\nu}^{[p;a,b,c,d]}(u) = \frac{n!}{(n-r)!}\ \Phi_{n-r,\nu}^{[p;a,b,c,d]}(u) \tag{20}$$

and

$$\frac{d^r}{du^r}\Phi_{n,\nu}^{a,b,c,d}(u) = \frac{n!}{(n-r)!}\ \Phi_{n-r,\nu}^{a,b,c,d}(u). \tag{21}$$

Proof. We have

$$\left[\mathbb{G}_{\nu,a}^{b,c,d}(t)\right]^p e^{ut} = \sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!}. \tag{22}$$

Differentiating (22) r-times with respect to u, we get

$$t^r \left[\mathbb{G}_{\nu,a}^{b,c,d}(t)\right]^p e^{ut} = \sum_{n=0}^{\infty} \frac{d^r}{du^r}\Phi_{n,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!} \tag{23}$$

$$t^r \sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{d^r}{du^r}\Phi_{n,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!}. \tag{24}$$

After comparing the coefficients of the powers of t on both sides of (24), we easily get our needed result (20). The result (21) can be established with the help of (20) by setting $p = 1$. □

Theorem 3.2. *For $n \in \mathbb{N}_0$ and $v \in \mathbb{R}$, we have*

$$\Phi_{n,\nu}^{[p;a,b,c,d]}(u+v) = \sum_{k=0}^{n} {}^{n}C_k \; v^k \Phi_{n-k,\nu}^{[p;a,b,c,d]}(u) \tag{25}$$

and

$$\Phi_{n,\nu}^{a,b,c,d}(u+v) = \sum_{k=0}^{n} {}^{n}C_k \; v^k \Phi_{n-k,\nu}^{a,b,c,d}(u). \tag{26}$$

Proof. On replacing u by $u+v$ in (18), we get

$$\left[\mathbb{G}_{\nu,a}^{b,c,d}(t)\right]^p e^{ut} e^{vt} = \sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p;a,b,c,d]}(u+v)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!}\sum_{k=0}^{\infty}\frac{(vt)^k}{k!} = \sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p;a,b,c,d]}(u+v)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \Phi_{n,\nu}^{[p;a,b,c,d]}(u)v^k \frac{t^{n+k}}{n!k!} = \sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p;a,b,c,d]}(u+v)\frac{t^n}{n!}. \tag{27}$$

Now, by applying the Lemma 10 given in [24, p.56], we get our first claimed result (25). For $p = 1$, (25) easily reduces to (26). □

Corollary 3.3. *On setting $v = 1$ in* (25) *and* (26)*, we have*

$$\Phi_{n,\nu}^{[p;a,b,c,d]}(u+1) = \sum_{k=0}^{n} {}^{n}C_k \; \Phi_{n-k,\nu}^{[p;a,b,c,d]}(u) \tag{28}$$

and

$$\Phi_{n,\nu}^{a,b,c,d}(u+1) = \sum_{k=0}^{n} {}^{n}C_k \; \Phi_{n-k,\nu}^{a,b,c,d}(u). \tag{29}$$

Theorem 3.4. *Let p and q are the real or complex parameters, $n \in \mathbb{N}_0$ and $v \in \mathbb{R}$. Then the following identity for $\Phi_{n,\nu}^{[p;a,b,c,d]}(u)$ holds:*

$$\Phi_{n,\nu}^{[p+q;a,b,c,d]}(u+v) = \sum_{k=0}^{n} {}^{n}C_k \; \Phi_{n-k,\nu}^{[p;a,b,c,d]}(u) \; \Phi_{k,\nu}^{[q;a,b,c,d]}(v). \tag{30}$$

Proof. Taking

$$\left[\mathbb{G}_{\nu,a}^{b,c,d}(t)\right]^{p+q} e^{(u+v)t} = \sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p+q;a,b,c,d]}(u+v)\frac{t^n}{n!}$$

$$\left[\mathbb{G}_{\nu,a}^{b,c,d}(t)\right]^p e^{ut} \left[\mathbb{G}_{\nu,a}^{b,c,d}(t)\right]^q e^{vt} = \sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p+q;a,b,c,d]}(u+v)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!} \sum_{k=0}^{\infty} \Phi_{k,\nu}^{[q;a,b,c,d]}(v)\frac{t^k}{k!} = \sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p+q;a,b,c,d]}(u+v)\frac{t^n}{n!}. \tag{31}$$

Further, by applying the Lemma 10 given in [24, p.56], we easily arrive at (30). □

Theorem 3.5. *For $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, we have*

$$\Phi_{n,\nu}^{[p;a,b,c,d]}(\alpha u) = \sum_{k=0}^{n} {}^nC_k \ u^k(\alpha-1)^k \ \Phi_{n-k,\nu}^{[p;a,b,c,d]}(u) \tag{32}$$

and

$$\Phi_{n,\nu}^{a,b,c,d}(\alpha u) = \sum_{k=0}^{n} {}^nC_k \ u^k(\alpha-1)^k \ \Phi_{n-k,\nu}^{a,b,c,d}(u). \tag{33}$$

Proof. Taking

$$\begin{aligned}\sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p;a,b,c,d]}(\alpha u)\frac{t^n}{n!} &= \left[\mathbb{G}_{\nu,a}^{b,c,d}(t)\right]^p e^{\alpha ut} \\ &= \left[\mathbb{G}_{\nu,a}^{b,c,d}(t)\right]^p e^{ut}e^{(\alpha-1)ut} \\ &= \sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{\{(\alpha-1)ut\}^k}{k!}\end{aligned}$$

$$\sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p;a,b,c,d]}(\alpha u)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}^nC_k \ \{(\alpha-1)u\}^k \ \Phi_{n-k,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!}. \tag{34}$$

On equating the coefficients of the powers of t in both sides of (34), we get our needed result (32). The result (33) can be established with the help of (32) by putting $p = 1$. □

Theorem 3.6. *For $n \in \mathbb{N}$, each of the following identities holds:*

$$\Phi_{n,\nu}^{[p;a,b,c,d]}(u+1) - \Phi_{n,\nu}^{[p;a,b,c,d]}(u) = \sum_{k=0}^{n-1} \frac{n!}{k!(n-1-k)!} \ \Phi_{n-1-k,\nu}^{[p;a,b,c,d]}(u) \ \mathbb{B}_k^{(-1)} \tag{35}$$

and

$$\Phi_{n,\nu}^{a,b,c,d}(u+1) - \Phi_{n,\nu}^{a,b,c,d}(u) = \sum_{k=0}^{n-1} \frac{n!}{k!(n-1-k)!} \ \Phi_{n-1-k,\nu}^{a,b,c,d}(u) \ \mathbb{B}_k^{(-1)}, \tag{36}$$

where $\mathbb{B}_k^{(-1)}$ are the generalized Bernoulli numbers obtained from (4) by taking $u = 0$.

Proof. Taking

$$\left[\mathbb{G}_{\nu,a}^{b,c,d}(t)\right]^p e^{ut}(e^t-1) = t\left[\mathbb{G}_{\nu,a}^{b,c,d}(t)\right]^p e^{ut}\left(\frac{t}{e^t-1}\right)^{-1}$$

$$= t\sum_{n=0}^{\infty}\Phi_{n,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!}\sum_{k=0}^{\infty}\mathbb{B}_k^{(-1)}\frac{t^k}{k!}$$

$$\sum_{n=0}^{\infty}\Phi_{n,\nu}^{[p;a,b,c,d]}(u+1)\frac{t^n}{n!}-\sum_{n=0}^{\infty}\Phi_{n,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!} = t\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\Phi_{n,\nu}^{[p;a,b,c,d]}(u)\mathbb{B}_k^{(-1)}\frac{t^{n+k}}{n!\;k!}$$

(37)

$$\sum_{n=0}^{\infty}\left[\Phi_{n,\nu}^{[p;a,b,c,d]}(u+1)-\Phi_{n,\nu}^{[p;a,b,c,d]}(u)\right]\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\Phi_{n-k,\nu}^{[p;a,b,c,d]}(u)\mathbb{B}_k^{(-1)}\frac{t^{n+1}}{(n-k)!\;k!}.$$

On equating the coefficients of the powers of t in both sides of (37), we get our required result (35). The result (36) can be established with the help of (35) by taking $p=1$. □

Theorem 3.7. *For $n,k\in\mathbb{N}_0$, each of the following relations holds:*

$$\Phi_{n,\nu}^{[p;a,b,c,d]}(u+1)+\Phi_{n,\nu}^{[p;a,b,c,d]}(u) = 2\sum_{k=0}^{n}{}^nC_k\;\Phi_{n-k,\nu}^{[p;a,b,c,d]}(u)\;\mathbb{E}_k^{(-1)} \tag{38}$$

and

$$\Phi_{n,\nu}^{a,b,c,d}(u+1)+\Phi_{n,\nu}^{a,b,c,d}(u) = 2\sum_{k=0}^{n}{}^nC_k\;\Phi_{n-k,\nu}^{a,b,c,d}(u)\;\mathbb{E}_k^{(-1)}, \tag{39}$$

where $\mathbb{E}_k^{(-1)}$ are the generalized Euler numbers obtained from (5) by setting $u=0$.

Proof. Taking

$$\left[\mathbb{G}_{\nu,a}^{b,c,d}(t)\right]^p e^{ut}(e^t+1) = 2\left[\mathbb{G}_{\nu,a}^{b,c,d}(t)\right]^p e^{ut}\left(\frac{2}{e^t+1}\right)^{-1}$$

$$= 2\sum_{n=0}^{\infty}\Phi_{n,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!}\sum_{k=0}^{\infty}\mathbb{E}_k^{(-1)}\frac{t^k}{k!}$$

$$\sum_{n=0}^{\infty}\Phi_{n,\nu}^{[p;a,b,c,d]}(u+1)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\Phi_{n,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!} = 2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\Phi_{n,\nu}^{[p;a,b,c,d]}(u)\mathbb{E}_k^{(-1)}\frac{t^{n+k}}{n!\;k!}$$

(40)

$$\sum_{n=0}^{\infty}\left[\Phi_{n,\nu}^{[p;a,b,c,d]}(u+1)+\Phi_{n,\nu}^{[p;a,b,c,d]}(u)\right]\frac{t^n}{n!} = 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\Phi_{n-k,\nu}^{[p;a,b,c,d]}(u)\mathbb{E}_k^{(-1)}\frac{t^n}{(n-k)!\;k!}.$$

On equating the coefficients of the powers of t in both sides of (40), we arrive at our needed result (38). The result (39) can be established with the help of (38) by setting $p = 1$. $\square$

Theorem 3.8. *Let $n \in \mathbb{N}$. Then each of the following relations holds:*

$$\Phi_{n,\nu}^{[p;a,b,c,d]}(u+1)+\Phi_{n,\nu}^{[p;a,b,c,d]}(u) = 2\sum_{k=0}^{n-1} \frac{n!}{k!(n-1-k)!}\,\Phi_{n-1-k,\nu}^{[p;a,b,c,d]}(u)\,\mathbb{G}_k^{(-1)} \tag{41}$$

and

$$\Phi_{n,\nu}^{a,b,c,d}(u+1)+\Phi_{n,\nu}^{a,b,c,d}(u) = 2\sum_{k=0}^{n-1} \frac{n!}{k!(n-1-k)!}\,\Phi_{n-1-k,\nu}^{a,b,c,d}(u)\,\mathbb{G}_k^{(-1)}, \tag{42}$$

where $\mathbb{G}_k^{(-1)}$ are the generalized Genocchi numbers obtained from (6) *by putting* $u = 0$.

Proof. Proof of this theorem is similar to that of Theorem 3.6 and Theorem 3.7. $\square$

4. Concluding remarks

In the present article, we have studied a new class of extended Bernoulli, Euler and Genocchi polynomials by means of a single generating function involving generalized Bessel function of the first kind $w_{\nu,c}^{b}(t)$. We have also presented their various interesting properties by applying the series manipulation technique. In this section, we briefly discuss the variations in the generating functions of our newly introduced polynomials.

The generalized Bessel function of the first kind $w_{\nu,c}^{b}(t)$ has the following connections with the Wright hypergeometric function ${}_p\Psi_q$ and Fox H-function $H_{c,d}^{a,b}$ (see for details [26]):

$$w_{\nu,c}^{b}(t) = \left(\frac{t}{2}\right)^{\nu} {}_0\Psi_1\left[\begin{matrix} - \\ (\nu+\frac{1+b}{2},\ 1)\end{matrix}\ \Big|\ -\frac{ct^2}{4}\right] \tag{43}$$

and

$$w_{\nu,c}^{b}(t) = \left(\frac{t}{2}\right)^{\nu} H_{0,2}^{1,0}\left[\frac{ct^2}{4}\ \Big|\ \begin{matrix} - \\ (0,1),\ (-\nu+\frac{1-b}{2},\ 1)\end{matrix}\right]. \tag{44}$$

Therefore, the generating function of our mixed polynomials given in (18), is easily converted in terms of Wright hypergeometric function and Fox H-function as follows:

$$\left[\frac{\Omega_{\nu,a}^{b,d}(t)}{{}_0\Psi_1\left[\begin{array}{c} - \\ (\nu+\frac{1+b}{2},\ 1) \end{array}\Big| -\frac{ct^2}{4}\right]}\right]^p e^{ut} = \sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!} \tag{45}$$

and

$$\left[\frac{\Omega_{\nu,a}^{b,d}(t)}{H_{0,2}^{1,0}\left[\frac{ct^2}{4}\Big| \begin{array}{c} - \\ (0,1),\ (-\nu+\frac{1-b}{2},\ 1) \end{array}\right]}\right]^p e^{ut} = \sum_{n=0}^{\infty} \Phi_{n,\nu}^{[p;a,b,c,d]}(u)\frac{t^n}{n!}, \tag{46}$$

where $\Omega_{\nu,a}^{b,d}(t) = \frac{2^{b-d+2\nu}t^{a-\frac{b}{2}-\nu}e^{-\frac{t}{2}}}{\sqrt{\pi}}$.

Furthermore, on setting $p = 1$ in (45) and (46), we easily get the variations in the generating function of our extended polynomials $\Phi_{n,\nu}^{a,b,c,d}(u)$ defined by (13).

References

[1] L. C. Andrews, *Special functions for Engineer and Mathematician,* Macmillan Company, New York, 1985.

[2] T. M. Apostol, *On the Lerch Zeta function,* Pacific. J. Math. **1** (1951), 161–167.

[3] E. T. Bell, *Exponential polynomials,* Ann. of Math. **35** (1934), no. 2, 258–277.

[4] J. Choi, P. Agarwal, S. Mathur, and S. D. Purohit, *Certain new integral formulas involving the generalized Bessel functions,* Bull. Korean Math. Soc. **51** (2014), no. 4, 995–1003.

[5] C. Frappier, *Representation formulas for entire functions of exponential type and generalized Bernoulli polynomials,* J. Austral. Math. Soc. (Series A) **64** (1998), 307–316.

[6] M. Ghayasuddin and N. U. Khan, *Certain new presentation of the generalized polynomials and numbers,* Rend. Circ. Mat. Palermo, II. Ser **70** (2021), 327–339. https://doi.org/10.1007/s12215-020-00502-9

[7] N. U. Khan and M. Ghayasuddin, *A new class of integral formulas associated with generalized Bessel functions,* Sohag J. Math. **3** (2016), no. 2, 1–4.

[8] N. U. Khan, T. Usman and J. Choi, *A new class of generalized polynomials associated with Laguerre and Bernoulli polynomials,* Turkish J. Math. **43** (2019), 486–497.

[9] N. U. Khan, T. Usman and J. Choi, *A new class of generalized polynomials,* Turkish J. Math. **42** (2018), 1366–1379.

[10] N. U. Khan, T. Usman and J. Choi, *A new generalization of Apostol type Laguerre-Genocchi polynomials,* C. R. Acad. Sci. Paris Ser. I **355** (2017), 607–617.

[11] N. U. Khan, T. Usman and J. Choi, *Certain generating function of Hermite-Bernoulli-Laguerre polynomials,* Far East J. Math. Sci. **101** (2017), 893–908.

[12] W. A. Khan, S. Araci and M. Acikgoz, *A new class of Laguerre-based Apostol type polynomials,* Cogent Math. **3**:1243839 (2016), 1–17.

[13] W. A. Khan, M. Ghayasuddin and M. Shadab, *Multiple-Poly-Bernoulli polynomials of the second kind associated with Hermite polynomials,* Fasciculi Math. **58** (2017), 97–112.

[14] B. Kurt, *A further generalization of the Bernoulli polynomials and on the 2D-Bernoulli polynomials $B_n^2(x,y)$,* App. Math. Sci. **4** (2010), no. 47, 2315–2322.

[15] Q. M. Luo, *q-Extensions for the Apostal-Genocchi polynomials,* Gen. Math. **17** (2009), no. 2, 113–125.

[16] Q. M. Luo, *Apostal-Euler polynomials of higher order and gaussian hypergeometric functions,* Taiwanese J. Math. **10** (2006), 917–925.

[17] Q. M. Luo, *Extension for the Genocchi polynomials and its fourier expansions and integral representations,* Osaka J. Math. **48** (2011), 291–310.

[18] Q. M. Luo and H. M. Srivastava, *Some generalizations of the Apostal-Bernoulli and Apostal-Euler polynomials,* J. Math. Anal. App. **308** (2005), 290–302.

[19] Q. M. Luo and H. M. Srivastava, *Some relationships between the Apostal-Bernoulli and Apostal-Euler polynomials,* Comp. Math. App. **51** (2006), 631–642.

[20] Q. M. Luo and H. M. Srivastava, *q-Extensions of some relationships between the Bernoulli and Euler polynomials,* Taiwanese J. Math. **15** (2011), 631–642.

[21] Q. M. Luo and H. M. Srivastava, *Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind,* Appl. Math. Comp. **217** (2011), 5702–5728.

[22] M. A. Pathan and W. A. Khan, *Some implicit summation formulas and symmetric identities for the generalized Hermite-Bernoulli polynomials,* Mediterr. J. Math. **12** (2015), 679–695.

[23] M. A. Pathan and W. A. Khan, *Some implicit summation formulas and symmetric identities for the generalized Hermite-Euler polynomials,* East-West J. Math. **16** (2014), no. 1, 92–109.

[24] E. D. Rainville, *Special functions,* Macmillan Company, New York, 1960. Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[25] H. M. Srivastava and J. Choi, *Zeta and q-Zeta functions and associated series and integrals,* Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[26] H. M. Srivastava and H. L. Manocha, *A treatise on generating functions,* Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

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