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ON ATTRACTORS OF TYPE 1 ITERATED FUNCTION SYSTEMS†

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Abstract. This paper discusses the properties of attractors of Type 1 IFS which construct self similar fractals on product spaces. General results like continuity theorem and Collage theorem for Type 1 IFS are established. An algebraic equivalent condition for the open set condition is studied to characterize the points outside a feasible open set. Connectedness properties of Type 1 IFS are mainly discussed. Equivalence condition for connectedness, arc wise connectedness and locally connectedness of a Type 1 IFS is established. A relation connecting separation properties and topological properties of Type 1 IFS attractors is studied using a generalized address system in product spaces. A construction of 3D fractal images is proposed as an application of the Type 1 IFS theory.

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1. Introduction

Studies on abstract spaces and their approaches to model real life situations are recent research areas today. Even though it is challenging to visualize the structures of functions, scientists use abstract spaces to provide relations between various fields which can suggest solutions to unsolved problems in other related areas. Fractal geometry solves several recent issues in the field of geometry. They are the most accepted approximation of natural structures. This theory introduced by Mandelbrot in 1975 deals with irregularities and chaos[19]. This geometry helps to model the natural phenomena and complex structures in nature. These structures are constructed mathematically using a finite set of

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contraction mappings called iterated function systems. The mathematical construction of fractals using hyperbolic iterated function systems (IFS) is widely studied. Hutchinson proved the existence and uniqueness of a non empty compact set called attractor for an IFS[17]. Further separation properties of IFS were studied using open set condition (OSC). Later Lalley proved that an open set satisfying this separation property is not influenced by the attractor. Strong open set condition (SOSC) was introduced to resolve this problem. The equivalence of OSC and SOSC to the positive Hausdorff measure of the attractor was proved by Schief[30].

Several generalizations and detailed studies on iterated function systems were carried out in the literature. Many authors have contributed in popularising IFS to construct self similar structures in different forms. Following Hutchinson's work, Barnsley, Falconer and Secelean put forward the research on generalized self similar sets with more resemblance to nature. Falconer generalized Hutchinson's mathematical definition of self similarity to sub and super self similarity [10, 11]. Secelean introduced countable and generalized countable IFS [31, 32]. He also considered IFS with generalized contractions and IFS on generalized spaces [34, 33]. Further studies were conducted by Bandt, Schief, Mcclure, Vallin and more others. Christoph Bandt and Siegfried Graf considered non-overlapping attractors with discontinuous action of a family of similitudes, to generalize the separation property open set condition[7]. Extending work by Levy, Mandelbrot, Dekking, Bedford, and others, Christoph Bandt constructed just touching fractals from matrices of integers[6]. M. Mcclure and R. W. Vallin considered the collection of sub-self similar sets and super self similar sets and showed that they are dense subsets in the collection of subsets of the Euclidean space[21]. A. Schief obtained an equivalent condition for both strong open set condition and open set condition of a self similar set[30]. He also investigated this result in complete metric spaces and shown that the open set condition no longer implies equality of Hausdorff and similarity dimension of self-similar sets[29]. The product of two hyperbolic IFSs and their properties were studied by Duvall and Husch[9]. Balu and Mathew introduced an IFS which constructs self similar sets in product spaces[4]. Aswathy and Mathew proposed a new form of product IFS which allows finding corresponding projection IFS which construct cross-sections of the higher dimensional fractal[2]. There are hyperbolic IFSs in product spaces that are not constructed as product of two IFSs. A class of such IFSs, namely Type 1 IFS were proposed by Aswathy and Mathew[2]. This collection contains the set of product IFSs. Since every IFS in the product space are not a product IFSs, the study on these generalized IFSs is relevant in higher dimensional spaces.

The study on these special attractors in the product spaces is the primary objective of this work. An algebraic equivalence of the separation property open set condition is to be studied using the address system. Thus a characterisation of separation properties in product spaces is another objective of the work.

In this paper, the analytical and topological properties of attractors of Type 1 IFS are studied. The separation property and open set condition are studied in product spaces and a feasible open set for Type 1 IFS is proposed. An equivalent condition on connected properties of attractors is also discussed. A relation between separation properties and topological properties is studied using the address system in product spaces. An application of imitating 3D fractal images using Type 1 IFS is proposed in the final section.

2. Preliminaries

Some definitions and results needed for developing the concept of Type 1 IFS are recalled in this section. Major results on this section are taken from [8]. We work on the metric space (X, d). $H(X)$ denotes the collection of non empty compact subsets of X. For $x \in X$ and $A, B, K \in H(X)$, $d(x, K) = \min\{d(x, y); y \in K\}$ and $d(A, B) = \max\{d(x, B); x \in A\}$. The Hausdorff metric on $H(X)$ is the map $h : H(X) \times H(X) \to \mathbb{R}$ defined by $h(A, B) = \max\{d(A, B), d(B, A)\}$. Then the metric satisfies, $h(A \cup B, C \cup D) \leq \max\{h(A, C), h(B, D)\}$ for every $A, B, C, D \in H(X)$. This metric space is complete with the completeness of the metric space (X, d). A contraction mapping is a transformation S on X such that $d(S(x), S(y)) \leq s.d(x, y)$, for every $x, y \in X$ for a constant contractivity factor $0 \leq s < 1$. Contracting similitude is the case where equality holds. By the Banach contraction theorem, for every contraction map on a complete metric space, there exist a unique fixed point and all points in the space are limited to this fixed point in repeated iterations. An hyperbolic iterated function system is a complete metric space (X, d) and a finite number of contraction mappings $f_i : X \to X$, for $i = 1, 2, \cdots, n$. We denote IFS for hyperbolic IFS throughout this paper. Corresponding to an IFS the transformation W on $H(X)$ defined by $W(B) = \overset{n}{\underset{i=1}{\cup}}(f_i(B))$ for all $B \in H(X)$, is a contraction mapping on the complete metric space $(H(X), h)$. The unique fixed point, $K \in H(X)$ is given by $K = \lim_{n\to\infty} W^{(n)}(B)$ for any $B \in H(X)$ [8]. The fixed point $K \in H(X)$ described is the attractor of the IFS. Associated with an IFS, the code space (Σ, d_c) is defined as the space of m symbols $\{1, 2, 3, \cdots, m\}$, say $\Sigma = \{\sigma_1\sigma_2\sigma_3 \cdots ; \sigma_i \in \{1, 2, 3, \cdots, m\}$ for all $i\}$ with the metric $d_c(\sigma, \omega) = \sum_{i=1}^{m} \frac{|\omega_i - \sigma_i|}{m+1}$, for all $\sigma, \omega \in \Sigma$. The map ϕ from $\Sigma \times N \times X$ defined by $\phi(\sigma, n, x) = S_{\sigma_n} \circ S_{\sigma_{n-1}} \circ S_{\sigma_{n-2}} \circ \cdots \circ S_{\sigma_1}(x)$ has a limiting value in the attractor independent of x as $n \to \infty$. This continuous and onto function $\phi : \Sigma \to K$ defines the addressing of points in the attractor. For any point in the attractor, an address is defined as any element from the collection $\phi^{-1}(a) = \{\sigma \in \Sigma; \phi(\sigma) = a\}$. The separation properties of IFS give an idea of how the self similar copies of the attractor are topologically arranged. Hyperbolic IFSs are classified as totally disconnected, just touching and overlapping based on the separation properties defined using the addressing of points in the attractor. If every point in the attractor has a unique address, the IFS is totally disconnected. Hyperbolic IFSs which are not totally disconnected but satisfy the OSC, are called just touching. A collection of contractive maps $f_i's$ is said to satisfy the OSC if there exists a non empty open set V such that $\overset{m}{\underset{i=1}{\cup}} f_i(V) \subseteq V$ and $f_i(V) \cap f_j(V) = \phi$ for $i \neq j$. Study on OSC was initiated by Moran in 1946 which shows the overlap of the copies A_i. The open set V is called the feasible open set of the contractions or of the attractor.

Topological properties of attractors of IFS were studied in the literature [27, 24]. An attractor, K of an IFS, is said to be connected if there does not exist two disjoint closed subsets K_1 and K_2 of K such that $K = K_1 \cup K_2$. An attractor which is not connected is called a disconnected attractor. If the only non empty connected subsets of K are singletons, then the attractor is totally disconnected. The attractor is arc wise connected, if for every $x, y \in K$ there exist a continuous function $\phi : [0, 1] \to K$ such that $\phi(0) = x$ and $\phi(1) = y$. The attractor is locally connected at a point $x \in K$, if for every neighborhood N of x, there exists a connected neighborhood M of x such that $M \subset N$. A is said to be locally connected if it is locally connected at all of its points.

Fractal interpolation generates complex structures using a set of given data. A set of data is a set of points of the form $\{(x_i, F_i) \in \mathbb{R}^2 : i = 0, 1, 2, \ldots, N\}$, where $x_0 < x_1 < x_2 < x_3 < \ldots < x_N$. An interpolation function corresponding to this set of data is a continuous function $f : [x_0, x_N] \to \mathbb{R}$ such that $f(x_i) = F_i \quad$ for $i = 1, 2, \ldots, N$. The points $(x_i, F_i) \in \mathbb{R}^2$ are called the interpolation points. We say that the function f interpolates the data and that f passes through the interpolation points. The process of fractal interpolation involves repeatedly using transformations that copy and build upon themselves. Let $\{\mathbb{R}^2; w_n, n = 1, 2, \ldots, N\}$ be the IFS associated with the data set $\{(x_n, F_n) : n = 1, 2, \ldots, N\}$. Let the vertical scaling factor d_n satisfy $0 \leq d_n < 1$ for $n = 1, 2, \ldots, N$. Then there is a metric d equivalent to the Euclidean metric on $\mathbb{R}^2$, such that the IFS is hyperbolic with respect to d. Consequently, there is a unique nonempty compact set $G \subset \mathbb{R}^2$ such that $G = \cup_{n=1}^{N} w_n(G)$.

The product of two IFSs is defined as an IFS on the product space with contractions as the product of corresponding contraction mappings. Also, the attractor of a product IFS coincides with the product of corresponding attractors of IFSs in the product space. A particular type of product IFS namely the Type 1 IFS is studied in the following section and its separation and topological properties are discussed in the following sections.

3. Type 1 hyperbolic IFS and its attractors

Attractors in product spaces and their topological and separation properties are discussed in the literature[27, 2, 9, 4, 5]. There are hyperbolic IFS in product spaces that are not product IFS. Those attractors in product spaces may not result from a product IFS. A specific collection of such IFSs is studied in this paper. We start with a study on a collection of a special type of IFS in product space called Type 1 IFSs constructed using two distinct complete metric spaces with a finite number of contraction maps. The collection of all Type 1 IFSs contains the set of all product IFSs. i.e, every product IFS is a Type 1 IFS. Since every IFS in higher dimensional space is not a product IFSs, the study on these generalized IFSs is relevant in higher dimensional spaces. Type 1 IFS helps to construct the projections of attractors in their corresponding spaces using the projection maps. The existence of a non empty compact subset on product space is established, and general results for Type 1 IFS are proved.

Definition 3.1. Let X and Y be complete metric spaces. Let S_{i1} and $S_{i2}, i = \{1, 2, \cdots, m\}$ be contraction mappings on X and Y respectively with contractivity factors $s_{ij}, j = \{1, 2\}$. The IFS on the product space $X \times Y$ with contraction mappings of the form $S_i = (S_{i1}, S_{i2})$ is called a Type 1 IFS and its contractivity factor is given by $r = \max\limits_{1 \leq i \leq m} \max\limits_{j=1,2} s_{ij}$.

Consider $H(X)$ and $H(Y)$, the space of all non empty compact subsets of X and Y with Hausdorff metric h_1 and h_2 respectively. The attractor of a Type 1 IFS lies in the product space, $\tilde{H} = H(X) \times H(Y)$.

Definition 3.2. Let $\{X \times Y; S_{ij}, i = \{1, 2, \cdots, m\}, j = \{1, 2\}\}$ be a Type 1 IFS with contractivity factor r. Let $\tilde{h}$ be the metric on $\tilde{H} = H(X) \times H(Y)$ defined as $\tilde{h}((A_1, A_2), (B_1, B_2)) = \max\{h_1(A_1, B_1), h_2(A_2, B_2)\}$ for all $(A_1, A_2), (B_1, B_2) \in \tilde{H}$. Define the map $\tilde{W}$ by $\tilde{W}(A_1, A_2) = (\mathop{\cup}\limits_{i=1}^{m} S_{i1}(A_1), \mathop{\cup}\limits_{i=1}^{m} S_{i2}(A_2))$ for all $(A_1, A_2) \in \tilde{H}$.

The metric $\tilde{h}$ is the Hausdorff metric on $\tilde{H}$ and the above defined $\tilde{W}$ is a contraction mapping on the complete metric space $\tilde{H}$, which is proved in the following lemma.

Lemma 3.3. *Let X and Y be complete metric spaces. Let S_{i1} and $S_{i2}, i = \{1, 2, \cdots, m\}$ be contraction mappings on X and Y respectively with contractivity factors $r_{ij}, j = \{1, 2\}$. Let $\tilde{W}(A_1, A_2) = (\mathop{\cup}\limits_{i=1}^{m} S_{i1}(A_1), \mathop{\cup}\limits_{i=1}^{m} S_{i2}(A_2))$ for all $(A_1, A_2) \in \tilde{H}$. Then $\tilde{W}$ is a contraction mapping on $\tilde{H}$ with contractivity factor r, where $r = \max\limits_{1 \leq i \leq m} \max\limits_{j=1,2}\{r_{ij}\}$. In other words, $\tilde{W} : \tilde{H} \to \tilde{H}$ satisfies, $\tilde{h}(\tilde{W}(A), \tilde{W}(B)) \leq r.\tilde{h}(A, B)$.*

Proof. Let $A = (A_1, A_2)$ and $B = (B_1, B_2)$ belongs to $\tilde{H}$.

$$\begin{aligned}\tilde{h}(\tilde{W}(A), (\tilde{W}(B)) &= \tilde{h}((\mathop{\cup}_{i=1}^{m} S_{i1}(A_1), \mathop{\cup}_{i=1}^{m} S_{i2}(A_2)), (\mathop{\cup}_{i=1}^{m} S_{i1}(B_1), \mathop{\cup}_{i=1}^{m} S_{i2}(B_2))) \\ &= \max\{h_1(\mathop{\cup}_{i=1}^{m} S_{i1}(A_1), \mathop{\cup}_{i=1}^{m} S_{i2}(A_2)), h_2(\mathop{\cup}_{i=1}^{m} S_{i1}(B_1), \mathop{\cup}_{i=1}^{m} S_{i2}(B_2))\} \\ &\leq \max_i \max_j\{h_j(S_{ij}(A_j), S_{ij}(B_j))\}.\end{aligned}$$

Since each S_{ij} is a contraction with contractivity factor r_{ij}, $\tilde{h}(\tilde{W}(A), (\tilde{W}(B)) \leq \max\limits_i \max\limits_j\{r_{ij}.h_j(A_j, B_j)\} \leq r. \max\limits_j\{h_j(A_j, B_j)\} = r.\tilde{h}(A, B)$. □

This contraction map leads to the existence of the unique fixed point called the attractor of the Type 1 IFS on $\tilde{H}$.

Theorem 3.4. *Let $\{X \times Y; S_{ij}, i = \{1, 2, \cdots, m\}, j = \{1, 2\}\}$ be a Type 1 IFS having contractivity factor r. Then there exists a unique non empty compact fixed point for the Type 1 IFS.*

Proof. The completeness of the parent spaces ensures the completeness of the spaces $(H(X), h_1)$ and $(H(Y), h_2)$. Each component space of $H(X) \times H(Y)$ is complete, giving the completeness of this product space. The map on $\tilde{H}$ is defined by $\tilde{W}(A_1, A_2) = (\overset{m}{\underset{i=1}{\cup}} S_{i1}(A_1), \overset{m}{\underset{i=1}{\cup}} S_{i2}(A_2))$ for all $(A_1, A_2) \in \tilde{H}$. The continuous mappings S_{ij} maps the compact sets (A_j) to the compact set $S_{ij}(A_j), j = \{1, 2\}$. The finite union of these sets gives the compact set $\overset{m}{\underset{i=1}{\cup}} S_{ij}(A_j), j = \{1, 2\}$. The Tychonoff theorem ensures the compactness of $\tilde{W}(A_1, A_2)$. Thus the map $\tilde{W}$ is a well defined contraction mapping on $\tilde{H}$. Finally, the contraction mapping theorem guarantees the existence of the unique fixed point for the map $\tilde{W}$ on $\tilde{H}$. □

Next example illustrates a Type 1 IFS in $\mathbb{R}^2$ and its corresponding two projection IFSs in both the coordinate axis.

Example 3.5. Consider the Type 1 IFS $\Phi = \{\mathbb{R}^2; S_1 = (\frac{x}{3}, \frac{y}{3}), S_2 = (\frac{2x+1}{4}, \frac{y+2}{3}), S_3 = (\frac{x+2}{3}, \frac{y}{3})\}$ with attractor A as shown in Figure 1. The corresponding projection IFSs are $\Pi_1(\Phi) = \{\frac{x}{3}, \frac{2x+1}{4}, \frac{x+2}{3}\}$ and $\Pi_2(\Phi) = \{\frac{y}{3}, \frac{y+2}{3}\}$. The attractor of the projection IFS on X axis is an overlapping set with two copies of the attractor and the attractor of the projection IFS on Y axis is a totally disconnected cantor set with two copies of the attractor.

3.1. General results on Type 1 hyperbolic IFS.

In higher dimensional spaces, especially in product spaces, to visualize and to study the attractors is challenging. Thus studies of attractors on more elevated dimensional spaces have more relevance. Constructing an IFS that results in an attractor in product space has much more importance. In this section, we validate the Collage theorem for Type 1 IFS, which constructs a Type 1 IFS whose attractor is the given fractal. Before that, we show that the mapping of a Type 1 IFS to its attractor is continuous.

Theorem 3.6. ***(Continuity theorem for Type 1 IFS.)*** *Let $(X \times Y, S_{ij}, i = \{1, 2, \cdots, m\}, j = \{1, 2\})$ and $(X \times Y, S'_{ij}, i = \{1, 2, \cdots, m\}, j = \{1, 2\})$ be two Type 1 IFSs with attractors A and A' and contractivity factors r and r' respectively. Then, $\tilde{h}(A, A') \leq \frac{\max_k(\max_i(h_k(S_{ij}(A), S'_{ij}(A'))))}{1-\min(r_1, r_2)}$, where $\tilde{h}$ is the Hausdorff metric on $\tilde{H}$.*

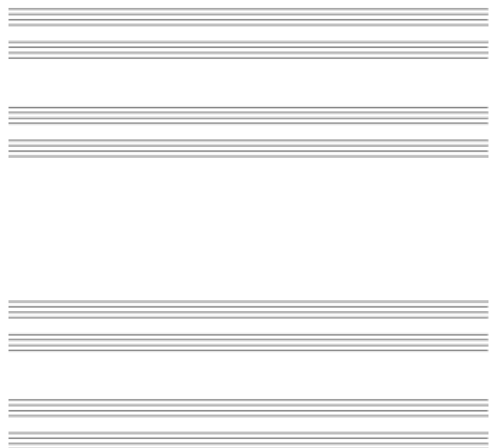

FIGURE 1. Attractor of Type 1 IFS Φ in Example 3.5

Proof. Let $A = (A_1, A_2)$ and $A' = (A'_1, A'_2)$ belong to H'. Then

$$\begin{aligned}\tilde{h}(A, A') &= \max\{h_1(A_1, A'_1), h_2(A_2, A'_2)\}\\ &\leq \frac{\max_k\{\max_i\{h_1(S_{i1}(A), S'_{i1}(A')), h_2(S_{i2}(A), S'_{i2}(A'))\}\}}{1-\min\{r_k, r'_k\}}\\ &= \frac{\max_{k=1,2}\{\max_{1\leq i\leq m}\{h_k(S_{i1}(A), S'_{i1}(A'))\}\}}{1-\{\max_{k=1,2}\min\{r_k, r'_k\}\}}.\end{aligned}$$

Let $r_1 = \max(r_{ij})$ and $r_2 = \max(r'_{ij})$, where r_{ij} and r'_{ij} are contractivity factors of S_{ij} and S'_{ij} respectively. Let $r = \max\{r_1, r_2\}$. So, $1 - \max_{k=1,2}\{\min\{r_k, r'_k\}\} \geq 1 - \min\{r_1, r_2\}$. Thus, $\tilde{h}(A_1, A_2) \leq \frac{\max(\max(h(S_{ij}(A), S'_{ij}(A'))))}{1-\min(r_1, r_2)}$. □

Next, we prove the Collage theorem for Type 1 IFS which constructs a Type 1 IFS whose attractor is very close to any arbitrarily chosen set with respect to the Hausdorff metric.

Theorem 3.7. ***(Collage theorem for Type 1 IFS.)*** *$L \in \tilde{H}$, $\epsilon > 0$ be given. A Type 1 IFS, $\Phi = (X \times Y; S_{ij}, i = \{1, 2, \cdots, m\}, j = \{1, 2\})$ with contractivity factor t is chosen such that $\tilde{h}(L, \tilde{W}(L)) \leq \epsilon$. Then, $\tilde{h}(L, K) \leq \frac{\epsilon}{1-t}$, $K = (K_1, K_2)$ is the attractor of the Type 1 IFS, where $\tilde{h}$ is the Hausdorff metric on $\tilde{H}$.*

Proof. Let $K = (K_1, K_2)$ and $L = (L_1, L_2)$. Then $\tilde{h}(K, L) = \max_{i=1,2}(h_i(K_i, L_i))$. Applying the Collage theorem for the projection IFSs Π_{Φ_1} and Π_{Φ_2} of the Type 1 IFS, we get, $h(L, K) \leq \frac{h\left(L, \overset{n}{\underset{i=1}{\cup}} S_i(L)\right)}{1-\max r_i}$. Then $h_k(K_k, L_k) \leq \frac{h_k(K_k, \cup S_{i1}(L_k))}{1-\max r_k}$, where r_{ij} is the contractivity factors of S_{ij}. Thus, $\tilde{h}(K, L) \leq \max_{i=1,2} \frac{h_i(L_i, \overset{n}{\underset{i=1}{\cup}} S_{i1})}{1 - \max_{1 \leq j \leq m} r_{ij}} \leq$

$$\frac{\max\{h_i(L_i, \overset{n}{\underset{i=1}{\cup}} S_{i1})\}}{1 - \{\max_{i=1,2}\{\max_{1 \leq j \leq m} r_{ij}\}\}} \leq \frac{\tilde{h}(L, \tilde{W}(L))}{(1-r)} \leq \frac{\epsilon}{1-t}.$$ □

The general results like continuity theorem and Collage theorem are valid for Type 1 IFS. In the next section we discuss the separation and topological properties of Type 1 IFS with the help of projection IFSs. These properties reveals how the attractor is occupied in the space. We relates the separation and topological properties in the coming section.

4. Separation Properties of Type 1 IFS

In this section, we discuss the separation properties of Type 1 IFS using the projection IFSs. Totally disconnectedness just touchingness and overlapping conditions of Type 1 IFSs are studied using the separation properties of projection IFSs. Strong separation condition (SSC) and open set conditions for Type 1 IFS are reviewed, and results concerning the conditions of corresponding projection IFSs are obtained. A Type 1 IFS is said to satisfy SSC if the distinct copies of attractors by the contraction maps are disjoint. It is shown that the Type 1 IFS meets SSC if and only if any of the corresponding projection IFSs satisfy SSC.

Definition 4.1. A Type 1 IFS $\Phi = \{X \times Y; S_i = (S_{i1}, S_{i2}), i = 1, 2, \cdots, m\}$ is said to satisfy SSC if $S_i \cap S_j = \phi, i \neq j \in \{i = 1, 2, \cdots, m\}$.

Theorem 4.2. *A Type 1 IFS* $\Phi = \{X \times Y; S_i = (S_{i1}, S_{i2}), i = 1, 2, \cdots, m\}$ *satisfies SSC if and only if any of the projection IFSs* $\Pi_{\Phi_1} = \{X; S_{i1}\}$ *or* $\Pi_{\Phi_2} = \{Y; S_{i2}\}$ *satisfies SSC.*

Proof. Let $K = (K_1, K_2)$, K_{Φ_1} and K_{Φ_2} be the attractors of the Type 1 IFS and projection IFSs respectively. $S_i(K) \cap S_j(K) = (S_{i1}, S_{i2})(K_1) \cap (S_{j1}, S_{j2})(K_2) = (S_{i1}(K_1) \cap S_{j1}(K_1), S_{i2}(K_2) \cap S_{j2}(K_2))$. Since $K_1 \subseteq K_{\Phi_1}$ and $K_2 \subseteq K_{\Phi_2}$, if any of the projection IFSs say Π_{Φ_1} satisfies SSC then, $S_{i1}(K_{\Phi_1}) \cap S_{j1}(K_{\Phi_1}) = \phi, i \neq j \in \{1, 2, \cdots, m\}$ implies $S_{i1}(K_1) \cap S_{j1}(K_1) = \phi, i \neq j \in \{1, 2, \cdots, m\}$. Then $S_i(K) \cap S_j(K) = (S_{i1}, S_{i2})(K_1) \cap (S_{j1}, S_{j2})(K_2) = \phi$. Thus the Type 1 IFS satisfies SSC. Conversely, the Type 1 IFS satisfies the SSC then $S_i(K) \cap S_j(K) = (S_{i1}, S_{i2})(K_1) \cap (S_{j1}, S_{j2})(K_2) = (S_{i1}(K_1) \cap S_{j1}(K_1), S_{i2}(K_2) \cap S_{j2}(K_2)) = \phi$. Thus any of the projection IFSs satisfy SSC. □

4.1. Open set condition for Type 1 IFS. Attractors constructed using iterated function systems show totally disconnectedness, just touchingness and overlapping separation conditions. These separation properties are described according to the existence of a non empty open set satisfying the open set condition(OSC). OSC is an accepted criterion controlling the overlapping of fractal structures. In this subsection, we define the OSC for Type 1 IFS and study an algebraic equivalence for this condition using neighbor maps for Type 1 IFS. The conditions of the Type 1 IFS such that the corresponding projection IFSs satisfy the OSC is also studied.

Definition 4.3. A Type 1 IFS satisfies OSC if and only if there exists open set $V = (V_1, V_2)$ in $X \times Y$ such that

(1) $\overset{m}{\underset{i=1}{\cup}} S_{il}(V_l) \subset V_l, l = \{1, 2\}$.

(2) $S_{il}(V_l) \cap S_{jl}(V_l) = \phi$ for all $i \neq j \in \{1, 2, \cdots, m\}, l = \{1, 2\}$.

The following result shows a relation between the OSC of the Type 1 IFS with the OSC of the corresponding projection IFS. It is shown that either of the projection IFS satisfies the OSC when the Type 1 IFS meets OSC and the Type 1 IFS satisfies OSC if both the projection IFS satisfies OSC.

Theorem 4.4. *If Type 1 IFS* $\Phi = \{X \times Y; S_i = (S_{i1}, S_{i2}), i = 1, 2, \cdots m\}$ *satisfies OSC then at least one of the projection IFS satisfies OSC.*

Proof. The first condition of the OSC of the Type 1 IFS Φ implies that there exist an open set $V = (V_1, V_2) \subset (X_1, X_2)$ such that $\overset{m}{\underset{i=1}{\cup}} (S_{i1}, S_{i2})(V_1, V_2) \subset (V_1, V_2)$. Then $\overset{m}{\underset{i=1}{\cup}} S_{il}(V_l) \subset V_l, l = \{1, 2\}$. Thus $\overset{m}{\underset{i=1}{\cup}} S_{i1}(V_1) \subset V_1$ and $\overset{m}{\underset{i=1}{\cup}} S_{i2}(V_2) \subset V_2$. This establishes the first condition of OSC for the projection IFSs Π_{Φ_1} and Π_{Φ_2} of the Type 1 IFS Φ. The second condition of the OSC for Type 1 IFS gives, $S_{il}(V_l) \cap S_{jl}(V_l) = \phi$ for all $i \neq j \in \{1, 2, \cdots, m\}, l = \{1, 2\}$. i.e., $(S_{i1}(V_1) \cap S_{j1}(V_1), S_{i2}(V_2) \cap S_{j2}(V_2)) = \phi$ for all $i \neq j \in \{1, 2, \cdots, m\}$. Then either $S_{i1}(V_1) \cap S_{j1}(V_1) = \phi$ or $S_{i2}(V_2) \cap S_{j2}(V_2)) = \phi$ for all $i \neq j \in \{1, 2, \cdots, m\}$ implies either Π_{Φ_1} or Π_{Φ_2} satisfies the second condition for the OSC. So, either of the projection IFSs satisfies the OSC. □

The converse of the above result is not valid. Any projection IFS that satisfies OSC need not imply the OSC of the Type 1 IFS. It is enough that any of the projection IFS satisfies the OSC for the Type 1 IFS to satisfy the second condition of OSC. But the first condition for OSC is met only if both the projection IFS satisfy the OSC. Thus the result is true if both of the projection IFSs satisfy the OSC. This is proved in the following theorem.

Theorem 4.5. *Let* $\Phi = \{X \times Y; S_i = (S_{i1}, S_{i2}), i = 1, 2, \cdots, m\}$ *be a Type 1 IFS. If the corresponding IFSs* Π_{Φ_1} *and* Π_{Φ_2} *satisfy the OSC, then so does the Type 1 IFS,* Φ.

Proof. OSC of the projection IFSs implies there exist $V_{\Pi_{\Phi_1}} \subseteq X_1$ and $V_{\Pi_{\Phi_2}} \subseteq X_2$ such that $\overset{m}{\underset{i=1}{\cup}} S_i(V_{\Pi_{\Phi_1}}) \subset V_{\Pi_{\Phi_1}}$ and $\overset{m}{\underset{i=1}{\cup}} S_i(V_{\Pi_{\Phi_2}}) \subset V_{\Pi_{\Phi_2}}$. Then $(\overset{m}{\underset{i=1}{\cup}} S_i(V_{\Pi_{\Phi_1}}), \overset{m}{\underset{i=1}{\cup}} S_i(V_{\Pi_{\Phi_2}})) \subset (V_{\Pi_{\Phi_1}}, V_{\Pi_{\Phi_2}})$. Thus the Type 1 IFS satisfies the first condition for the OSC with the open set $V = (V_{\Pi_{\Phi_1}}, V_{\Pi_{\Phi_2}})$. Now, since $S_i(V_{\Pi_{\Phi_l}}) \cap S_{jl}(V_{\Pi_{\Phi_l}}) = \phi$ for all $i \neq j \in \{1, 2, \cdots, m\}, l = \{1, 2\}$ implies $(S_{i1}(V_{\Pi_{\Phi_1}}) \cap S_{j1}(V_{\Pi_{\Phi_1}}), S_{i2}(V_{\Pi_{\Phi_2}}) \cap S_{j2}(V_{\Pi_{\Phi_2}})) = \phi$. Thus, the Type 1 IFS satisfies the second condition of the OSC with the open set $V = (V_{\Pi_{\Phi_1}}, V_{\Pi_{\Phi_2}})$. This completes the proof. □

The set $V = (V_1, V_2)$ satisfying the OSC is called a feasible open set for the Type 1 IFS or for the attractor. Band and Graf give an algebraically equivalent condition for OSC [7]. The map S_i^{-1} transforms the pieces $S_i(A)$ and $S_j(A)$ to attractor and $h(A) = S_i^{-1}S_j(A)$ respectively. The other condition for the OSC can be treated as $S_{il}(V_l) \cap S_{jl}(V_l) = \phi$ for all $i = j$ and is equivalent to $V \cap S_i^{-1}S_j(V) = \phi$. Thus if the Type 1 IFS satisfies the open set condition, then the map $h = S_i^{-1}S_j$ cannot be near the identity map id. Let $S^* = \underset{n \geq 1}{\cup} S^n$. The maps in the set $N = \{h = S_i^{-1}S_j; i, j \in S^*, i_1 \neq j_1\}$ $= \{(S_{i1}(x), S_{i1}(y))^{(-1)}(S_{j1}(x), S_{j1}(y)); i, j \in \{1, 2, \cdots, m\}, i \neq j\}$ are called neighbor maps. If a point $(x, y) \in X \times Y$ is contained in a feasible open set $V = (V_1, V_2)$ then (x, y) is said to be forbidden point for $A = (A_1, A_2)$. The points of the neighbor map set, $H = \{h(A); h \in N\}$, are not contained in any of the feasible sets. Next, we discuss the points on the closure of the collection of fixed points of neighbor maps. Let J denotes the collection of fixed points of neighbor maps. i.e, $J = \{(x, y); (S_{i1}^{-1}(S_{j1}), S_{i2}^{-1}(S_{j2}))(x, y) = (x, y)\}$.

Theorem 4.6. *Every limit of fixed points of neighbour maps of a Type 1 IFS are forbidden points of its attractor, $K = (K_1, K_2)$.*

Proof. If $(x, y) \in \overline{J}$ is contained in an open set $V = (V_1, V_2)$, then by definition of J, for a neighbor map say, $(S_{i1}^{-1}(S_{j1}), S_{i2}^{-1}(S_{j2}))$ its fixed point is contained in this open set. V contains the fixed point (y_1, y_2) of a neighbor map, say, $(S_{i1}^{-1}(S_{j1}), S_{i2}^{-1}(S_{j2}))$. ie., $(S_{i1}^{-1}(S_{j1}), S_{i2}^{-1}(S_{j2}))(y_1, y_2) = (y_1, y_2)$. Thus $(S_{j1}, S_{j2})(y_1, y_2) = (S_{i1}(y_1), S_{i2}(y_2))$. So, $S_{i1}(V_1) \cap S_{j1}(V_1) = \phi$ and $S_{i2}(V_2) \cap S_{j2}(V_2) = \phi$, implies $S_{j1}(y_1) = S_{i1}(y_1)$ and $S_{j2}(y_2) = S_{i2}(y_2)$. Thus V is not feasible, and hence (x, y) is a forbidden point of the attractor. □

Theorem 4.7. *The neighbor map set of a Type 1 IFS is a subset of the closure of the fixed points of its neighbor maps. i.e, $H \subset \overline{J}$.*

Proof. Let $b \in H$, say $(b_1, b_2) \in S_{i1}^{-1}(S_{j1}), S_{i2}^{-1}(S_{j2})(K_1, K_2)$. Choose contractive map h such that $S_{i1}(b_1) \in S_{j1}(K_1)$ and $S_{i2}(b_1) \in S_{j2}(K_2)$. Consider the fixed point (c_1, c_2) for the contractive mapping $h = S_i^{-1}S_j$. Let $a = h^{-1}(b)$. Since $|c - b| = r_h|c - a|$. The sequence h_n such that $b \in h_n(A)$ shows that r_h limits to 0 and a is finite since it is in K. So the fixed points of h_n limit to b. □

Now we will discuss the points other than the closure of fixed points of neighbour maps which are forbidden points. This result shows that there are no such points.

Theorem 4.8. *For an $\epsilon > 0$, there exists a neighbor map, h for any given forbidden point $x = (x_1, x_2)$ of the attractor, $K = (K_1, K_2)$ of a Type 1 IFS such that $|h(x) - x| < \epsilon$.*

Proof. Consider the open ball B having center (x_1, x_2) with radius $\frac{\epsilon}{2}$. Since the open set $V = \underset{\mathbf{i}\in S^*}{\cup} S_{\mathbf{i}}(B)$ is not a feasible open set, $S_i(V) \cap S_j(V) \neq \phi$ for some $i \neq j$. Then there exist $\mathbf{i},\mathbf{j} \in S^*$ satisfying $S_i S_{\mathbf{i}}(B) \cap S_j S_{\mathbf{j}}(B) \neq \phi$. So the contractive mapping $h = S_{i\mathbf{i}}^{-1}$ or its inverse satisfies the desired condition. □

Next example illustrates a Type 1 IFS in $\mathbb{R}^2$ and the separation properties of its corresponding two projection IFS attractors in both the coordinate axis.

Example 4.9. Consider the Type 1 IFS $\{\mathbb{R}^2; S_1 = (\frac{x}{3}, \frac{y}{3}), S_2 = (\frac{2x+1}{4}, \frac{y+2}{3}), S_3 = (\frac{x+2}{3}, \frac{y}{3})\}$. The corresponding projection IFSs $\Pi_1(\Phi) = \{\frac{x}{3}, \frac{2x+1}{4}, \frac{x+2}{3}\}$ has an overlapping attractor and $\Pi_2(\Phi) = \{\frac{y}{3}, \frac{y+2}{3}\}$ has a totally disconnected attractor.

Next, we propose constructing a feasible open set satisfying the open set condition.

Theorem 4.10. *Let $\{X \times Y; S_{ij}, i = \{1, 2, \cdots, m\}, j = \{1, 2\}\}$ be a Type 1 IFS. If $V = (V_1, V_2) \subset V_C$ be non empty open set then $U = \bigcup_l S_l((V_1, V_2))$ where union is taken over all words l satisfies the OSC.*

Proof. By definition of the proposed open set, $U = (U_1, U_2)$, $S_1(U_1, U_2) \subset (U_1, U_2)$. Since points of $S_i^{-1}S_j(V_C)$ are near to $S_i^{-1}S_j(A)$ than to K and $V \subset V_C$ implies $V \cap S_i^{-1}S_j(V) = \phi$ for every neighbor maps $h = S_i^{-1}S_j$. Now if suppose $x = (x_1, x_2) \in S_i(U) \cap S_j(U)$, $x \in (S_j S_l(V) \cap S_j S_l'(V))$ for some words l, l' which is not possible since $V \cap S_j^{-1} S_j V = \phi$ for every neighbor maps $h = S_i^{-1}S_j$. Thus the proposed open set U satisfies the OSC. □

Separation properties of Type 1 IFS is studied using the corresponding projection IFSs and an equivalent condition for the separation property OSC is studied in Type 1 IFS. The connectedness property of attractor of Type 1 IFS will be discussed in the next session.

5. Connectedness of attractor of Type 1 hyperbolic IFS

Topology is an efficient tool to study how a structure behaves spatially. This section equips with the topological aspects of sets defined by Type 1 IFS. Type 1 IFS is studied as a sub-IFS of the corresponding product IFS of the projection maps[2]. Here we deal with the connectedness properties of Type 1 IFS attractors.

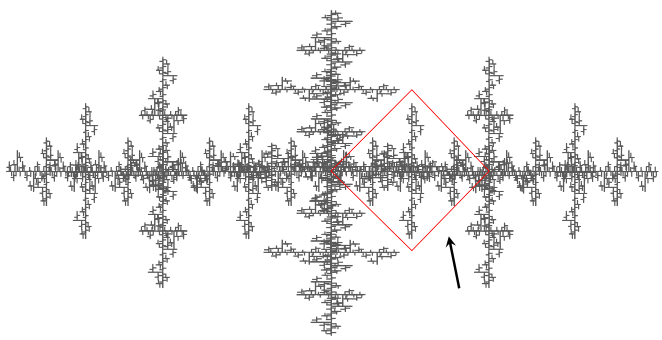

FIGURE 2. A line fractal with a polygonal set for the construction of feasible open set in Theorem 4.10

Theorem 5.1. *A Type 1 IFS attractor, K_Φ is a subset of the attractor of product IFS of the projection IFSs $\Pi_1(\Phi)$ and $\Pi_2(\Phi)$.*

A subspace of a totally disconnected space is also totally disconnected, giving the following analogy.

Corollary 5.2. *Let Φ be a Type 1 IFS. If the attractor of product IFS of the projection IFSs $\Pi_1(\Phi)$ and $\Pi_2(\Phi)$ is totally disconnected, then the attractor of Φ is totally disconnected.*

5.1. Connected attractors of Type 1 IFS. In general, topological spaces arc wise connected imply connectedness. But the converse is not valid. Here we show that connectedness and arc wise connectedness is equivalent to a condition.

Theorem 5.3. *If the attractor $K = (K_1, K_2)$ of a Type 1 IFS, $\Phi = \{X \times Y; S_1, S_2, \cdots, S_m\}$ where each $S_i(x, y) = (S_{i1}(x), S_{i2}(y))$ is connected then*

$$S_{il}(K_l) \cap S_{jl}(K_l) \neq \phi,\ i \neq j \in \{1, 2, \cdots, m\},\ l = \{1, 2\}.$$

Proof. Suppose that K is connected. Then both K_1 and K_2 are connected. Then $K_l = U \cup V$, where U and V are some closed subsets of $K_l, l = \{1, 2\}$ such that $U \cap V \neq \phi$. Also $K_i = \overset{m}{\underset{j=1}{\cup}} S_{ij}(K_i)$. Since continuous image of closed set is closed and finite union of closed sets is closed, we can let $U = S_{il}(K_l)$ and $V = \overset{m}{\underset{j=1, j\neq i}{\cup}} S_{jl}(K_l)$. Also $S_{il}(K_l) \cap \overset{m}{\underset{j=1, j\neq i}{\cup}} S_{jl}(K_l) \neq \phi$. Since $U \cap V \neq \phi$, we get $S_{il}(K_l) \cap S_{jl}(K_l) \neq \phi$ for some $i \neq j \in \{1, 2, \cdots, m\}$. The result is true for $l = \{1, 2\}$. So $S_{il}(K_l) \cap S_{jl}(K_l) \neq \phi,\ i \neq j\ \in \{1, 2, \cdots, m\},\ l = \{1, 2\}$. □

5.2. Arc wise connected attractors of Type 1 IFS. Another related condition is arc wise connectedness. Connectedness neither implies nor implied by arc wise connectedness in general. This section deals with the arc wise connectedness property of Type 1 IFS attractors.

Theorem 5.4. *If* $S_{il}(K_l) \cap S_{jl}(K_l) \neq \phi,\ i \neq j\ \in \{1, 2, \cdots, m\},\ l = \{1, 2\}$, *then the attractor* $K = (K_1, K_2)$ *of the Type 1 IFS is arc wise connected.*

Proof. To prove that the attractor $K = (K_1, K_2)$ is arc wise connected, it is enough to prove that each of its components is arc wise connected. For this, we prove that each component K_l is arc wise connected. Define, $A = \{f_l : K_l^2 \times [0, 1] \to K_l; f_l(p_l, q_l, 0) = p_l \ and \ f_l(p_l, q_l, 1) = q_l,\ p_l, q_l \in K_l\}$. For the metric ρ on the collection of all continuous functions on X by, $\rho(f, g) = \sup_{x\in X}\{d(f(x), g(x))\}$, the metric space (A, ρ) is a complete metric space. Consider a sequence $\{x_{l0}, x_{l1}, \cdots, x_{lk}\} \in K_l$ such that $x_{l0} = p_l$ and $x_{lk} = q_l$. Suppose that $x_{l_i} \in S_{il}(K_l)$ and $x_{l_j} \in S_{jl}(K_l)$. For $f_l \in A$, define $S_l f_l \in A$ as, $S_l f_l(p_l, q_l, t) = S_{lj}(f_l(S_{lj}^{-1}(x_{lj}), S_{lj}^{-1}(x_{lj}), jt - k))$, for $\frac{k}{j} \leq t \leq \frac{k+1}{j},\ 0 \leq j \leq k$. Then $\{S_l^m f_l\}_{m=1}^{\infty}$ is a Cauchy sequence. Therefore there exist an $f_l^* \in A$ such that $S_l^m f_l$ converges to f_l^* where $f^* : K_l^2 \times [0, 1] \to K_l$. Consider the metric, $D(S_l^m f_l, u) = \sup\{\lim_{n\to\infty} d(S_l^m f_l(r_n), S_l^m f_l(s_n)), \lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = u\}$ for $r_n, s_n \in K_l^2 \times [0, 1]$. Then $D(S_l^m f_l, u) = 0$. Therefore $f_l^* \in A$ is continuous at u. That is, f_l^* is a continuous arc between p_l and q_l. So K_l is arc wise connected. Hence K is arc wise connected. □

5.3. Locally connected attractors of Type 1 IFS. This section deals with the locally connectedness property of Type 1 IFS attractors. The finite connectedness property is studied to show the equivalence of connectedness and locally connectedness of Type 1 IFS attractors.

Definition 5.5. (Finite connectedness property.) A product space C satisfies finite connected property if for any $\epsilon > 0$, each factor of C is a finite union of connected sets, each of whose diameter is less than ϵ.

A necessary condition of a Type 1 IFS to satisfy finite connectedness property is obtained in the next theorem.

Theorem 5.6. *If* $S_{il}(K_l) \cap S_{jl}(K_l) \neq \phi,\ i \neq j\ \in \{1, 2, \cdots, m\},\ l = \{1, 2\}$, *then the attractor* $K = (K_1, K_2)$, *of the Type 1 IFS satisfies the finite connectedness property.*

Proof. K is connected, since $S_{il}(K_l) \cap S_{jl}(K_l) \neq \phi,\ i \neq j\ \in \{1, 2, \cdots, m\},\ l = \{1, 2\}$. So each of the factors K_l is connected. $K = (\underset{q\in J}{\cup} S_{1(e)}(K_1), \underset{q\in J}{\cup} S_{2(e)}(K_2))$ where $S_{l(e)} = S_{j_1 l} \circ S_{j_2 l} \circ \cdots \circ S_{j_q l},\ j_k \in \{1, 2, \cdots, m\}$. Since connectedness is preserved under continuous functions, $S_{l(e)}(K_l)$ is connected. So $\underset{q\in J}{\cup} S_{1(e)}(K_1)$ is a factor of K, is a finite union of connected sets. $diam(S_{l(e)}(K_l)) = diam(S_{j_1 l} \circ$

$S_{j_2 l} \circ \cdots \circ S_{j_q l}(K_l)) \leq t_{lj_1} * t_{lj_2} \cdots * t_{lj_q} * diam(K_l) \leq t_l * t_l \cdots * t_l * diam(K_l) = t_l^q * diam(K_l)$.
where $*$ denotes the usual multiplication. Since $0 \leq t_l = \max\ t_{lj} < 1$, without loss of generality we choose an $\epsilon > 0$ for a large q such that $diam(S_{l(e)}(K_l)) < \epsilon$. Thus, each factor of K is a finite union of connected sets, each of whose diameter is less than ϵ. So K satisfies the finite connectedness property. □

A direct consequence result for a necessary condition for finite connectedness of a Type 1 IFS using connectedness property follows from the above result.

Corollary 5.7. *If the attractor K of Type 1 IFS is connected, then K satisfies the finite connected property.*

Another consequence result for a necessary condition for finite connectedness of a Type 1 IFS using arc wise connectedness property follows from the above result.

Corollary 5.8. *If the attractor K of Type 1 IFS is arc wise connected, then K satisfies the finite connected property.*

Theorem 5.9. *If the attractor $K = (K_1, K_2)$ of a Type 1 IFS satisfies the finite connected property, then both K_1 and K_2 are locally connected.*

Proof. If the attractor $K = (K_1, K_2)$ satisfies the finite connected property, then for every $\epsilon > 0, K_i = \bigcup_{k=1}^{p} M_{i_k}$, where M_{i_k} is a connected set with diameter less than ϵ. Consider $x_i \in K_i$. Then x_i belongs to some of the $M'_{ik}s$ for $k = \{1, 2, , \cdots, p\}$. Let C_i be the union of $M'_{ik}s$ that contains x_i. Since a finite union of connected sets with a common point is connected, C_i is connected. Clearly x_i does not belongs to $K_i \setminus C_i$ and $diam(C_i) < \epsilon$. Then $diam(x_i, K_i \setminus C_i) > 0$. Therefore, for each $\epsilon > 0$, there exist $\delta < \epsilon$ such that $B_i(x_i, \delta) \cap (K_i \setminus C_i) = \phi$, where $B_i(x_i, \delta)$ is the $\delta-$neighbourhood of x_i. Then, $B_i(x_i, \delta) \cap K_i = B_i(x_i, \delta) \cap C_i$. Since $x_i \in C_i$ and C_i is connected, $B_i(x_i, \delta) \cap C_i$ is connected. Therefore, $B_i(x_i, \delta) \cap K_i$ is connected. Thus, for every $\epsilon-$neighbourhood of x_i, there exists a connected $\delta-$neighbourhood of x_i such that $B_i(x_i, \delta) \cap C_i \subset B_i(x_i, \delta)$. Therefore, K_i is locally connected at x_i. Since x_i is arbitrary, the factor K_i is locally connected. The result holds for all $i \in \{1, 2\}$. □

A characterisation result of locally connectedness of a product space using locally connectedness of each coordinate space is given by K. D. Joshi [18]

Theorem 5.10. *A product space is locally connected if and only if each coordinate space is locally connected and all except finitely many of them are connected.*

Theorem 5.11. *If the attractor $K = (K_1, K_2)$ of a Type 1 IFS is connected, then K is locally connected.*

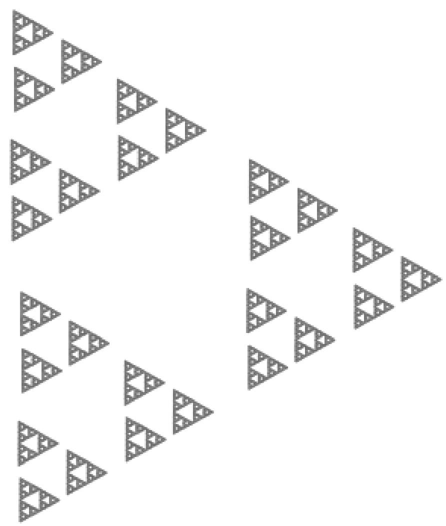

FIGURE 3. Attractor of Type 1 IFS Ψ in Example 5.13

Proof. Suppose that $K = (K_1, K_2)$ is connected. Then both K_1 and K_2 are connected. Also, K satisfies finite product property. Therefore, each factor K_i is locally connected. So K is locally connected. □

The proved results and the result that a compact locally connected continuum is arc wise connected gives the following theorem.

Theorem 5.12. *Let K be an attractor of a Type 1 IFS, $\{X \times Y; S_{ij}, i = \{1, 2, \cdots, m\}, j = \{1, 2\}\}$. Then the following are equivalent:*

(1) $S_{il}(K_l) \cap S_{jl}(K_l) \neq \phi,\ i \neq j\ \in \{1, 2, \cdots, m\},\ l = \{1, 2\}$.
(2) *K is connected.*
(3) *K is arc wise connected.*
(4) *K is locally connected.*

Example 5.13. Consider the Type 1 IFS $\Psi = \{\mathbb{R}^2; S_1, S_2\}$ where $S_1 = (\frac{x}{2}, \frac{y}{3})$ and $S_2 = (\frac{x}{2} + \frac{1}{2}, \frac{y}{3} + \frac{2}{3})$. The projection IFSs are $\Pi_1 = \{\mathbb{R}; \frac{x}{2}, \frac{x}{2} + \frac{1}{2}\}$ and $\Pi_2 = \{\mathbb{R}; \frac{y}{3}, \frac{y}{3} + \frac{2}{3}\}$. The attractor of the Type 1 IFS is $A = (A_1, A_2)$ where $A_1 = [0, 1]$ and A_2 is the classical cantor set. $S_{11}(A_1) \cap S_{21}(A_1) \neq \phi$ but $S_{12}(A_2) \cap S_{22}(A_2) = \phi$. So the attractor of this Type 1 IFS Ψ is not connected. The attractor of this Type 1 IFS Ψ is shown in Figure 3.

6. Relating separation properties and topological properties

It is commonly accepted that the overlapping of self similar fractals is controlled by the separation properties of the corresponding iterated function system. The separation properties of Type 1 IFS are studied using the separation properties of the corresponding projection IFSs. A Type 1 IFS is totally disconnected if both the projection IFSs are totally disconnected. A Type 1 IFS is just touching if any of the projection IFS is just touching and the other is totally disconnected. A Type 1 IFS is overlapping if any corresponding projection IFS is overlapping. Now we study how the OSC affects the connectedness of an attractor of Type 1 IFS. The notion of addressing points on the attractor is carried over to the attractors of product spaces to study the particular case of Type 1 IFS. The relation between the separation property, totally disconnectedness of an IFS, and the topological property, totally disconnectedness of an IFS attractor, is also studied.

Let $\{X \times Y; S_1, S_2, \cdots, S_N\}$, $S_i = (S_{i1}(x), S_{i2}(y))$ be a Type 1 IFS on a complete metric space $(X \times Y, d)$. Let $(\Sigma \times \Sigma, d_c)$ be the associated code space of the Type 1 IFS. For $(\sigma, \omega) \in \Sigma \times \Sigma$, $n \in N$, and $(x, y) \in X \times Y$. Define $\phi : \Sigma \times \Sigma \to X \times Y$ by $\phi((\sigma, \omega), n, (x, y)) = (S_{\sigma_1} S_{\sigma_2} S_{\sigma_3} \cdots (x), S_{\omega_1} S_{\omega_2} S_{\omega_3} \cdots (y))$. Then there is a real constant D such that $d(\phi((\sigma, \omega), m, (x_1, y_1)), \phi((\sigma, \omega), n, (x_2, y_2))) \leq D * S^{m \wedge n}$. Then, $\phi(\sigma, \omega) = \lim_{n \to \infty} \phi((\sigma, \omega), n, (x, y))$ exists and belongs to $X \times Y$ and is independent of (x, y). An address of a point $(a_1, a_2) \in X \times Y$ is an element of the collection $\phi^{-1}(x, y) = \{(\sigma, \omega) \in \Sigma \times \Sigma; \phi(\sigma, \omega) = (x, y)\}$.

Theorem 6.1. *An attractor is topologically totally disconnected if the corresponding IFS satisfies the totally disconnectedness separation property.*

Proof. An IFS with attractor K is totally disconnected implies $\omega_i(K) \cap \omega_j(K) = \phi$ for every $i, j \in \{1, 2, \cdots, n\}$. Then, $\omega_i(K) \cap \omega_j(K) = \phi$ for every $i, j \in \{1, 2, \cdots, n\}$ implies K is topologically totally disconnected. □

The converse of the theorem is not valid. An attractor on a product IFS which is topologically totally disconnected but does not satisfy totally disconnectedness separation property is illustrated in the next example.

Example 6.2. Consider the product of IFSs $\psi = \{\mathbb{R}^2; s_1 = (\frac{x}{3}, \frac{y}{3})$, $s_2 = (\frac{x+1}{3}, \frac{y+2}{3}), s_3 = (\frac{x+2}{3}, \frac{y}{3})\}$ and $\phi = \{\mathbb{R}^2; s'_1 = (\frac{x}{3}, \frac{y}{3}), s'_2 = (\frac{2x+1}{4}, \frac{y+2}{3}), s'_3 = (\frac{x+2}{3}, \frac{y}{3})\}$. The Type 1 IFS ψ constructs an attractor in $\mathbb{R}^2$ with the totally disconnected classical Cantor set in the first coordinate axis and another totally disconnected attractor of two copies in the second coordinate axis. Thus the Type 1 IFS ψ is totally disconnected. Thus the product IFS of the totally disconnected IFS ψ and the overlapping Type 1 IFS ϕ generates an overlapping IFS with attractor having a point with two different addresses. But since one of the attractor is a totally disconnected space, the product IFS $\psi \times \phi$ has a totally disconnected attractor.

It is clear that an IFS satisfying strong OSC constructs totally disconnected attractors. Also, the overlapping IFSs do not assure OSC. Next, we study the totally disconnected and just touching IFSs having obstacle OSC. We define doubly recurrent addresses in the product spaces to examine such cases.

Definition 6.3. An address in product space $(s_{11}, s_{12}, s_{21}, s_{22}, s_{13}, s_{23}, \cdots)$ is said to be doubly recurrent if for $K_1, K_2 \geq 1$ there are $n_1, n_2 \geq 1$ with $s_{1i}, s_{2i}, \cdots, s_{K_j i} = s_{(n+1)i}, s_{(n+2)i} \cdots, s_{(n+K_j)i}$.

Now we show an obstacle condition for the OSC in product spaces. Identifying a doubly recurrent address is an obstacle to the open set condition.

Theorem 6.4. *Let $\{X \times Y; S_{ij}, i = \{1, 2, \cdots, m\}, j = \{1, 2\}\}$ be a Type 1 IFS. If a point on the attractor, $(a_i, a_j) \in K_{S_i} \cap K_{S_j}$ has a doubly recurrent address, then the Type 1 IFS cannot hold the OSC.*

Proof. It is enough to show that the attractor satisfies the finite clustering property. By finite clustering property, an attractor K cannot satisfy OSC if for any $n \in N$ there is a copy K_i intersecting with at least n other non comparable copies K_j with $diam(K_j) \geq diam(K_j)$. For a given n, let a point say, $(a_1, a_2) \in K_{S_1} \cap K_{S_2}$, $S_1 \neq S_2$ have a doubly recurrent address $s = s_{11}, s_{12}, s_{21}, s_{22}, \cdots$ and another address t. Fix the initial word $i = i_{11}, i_{12}, i_{21}, i_{22}, \cdots, i_{n1}, i_{n2}$ of the sequence s such that there are $i_{l_i k_j}, i_{l_{i+1} k_{j+1}}, \cdots, i_{n1}, i_{n2} = s_{11}, s_{12}, \cdots, s_{l_i(n-k_j)}, s_{l_{i+1}(n-k_{j+1})}$ for $l_i, i = 1, 2$. Define $j_k = i_{11}, i_{12}, \cdots, i_{l_i k_j}, i_{l_{i+1} k_{j+1}}, t$. Then, $S_{i_{11} i_{12} \cdots i_{l_i k_j} i_{l_{i+1} k_{j+1}}}(a_1, a_2) \in K_i \cap K_{j_k}$. Thus K_i intersects with n non-comparable copies of K_{j_k}. So the Type 1 IFS does not satisfy OSC. □

A totally disconnected attractor need not necessarily result from a totally disconnected IFS. In product spaces, separation properties of product spaces were explained in terms of their corresponding coordinate IFSs in [2]. It is shown that a Type 1 IFS is totally disconnected if both the projection IFS are totally disconnected. An example of a Type 1 IFS which is neither connected nor totally disconnected is provided.

Example 6.5. Consider the IFS $\chi = \{X \times Y; S_1, S_2, S_3\} = \phi$ such that $S_1(x, y) = (\frac{x}{2} + \frac{1}{2}, 0), S_2(x, y) = (\frac{x}{2} - \frac{1}{2}, 0), S_3(x, y) = (\frac{x}{2}, \frac{y}{2} + \frac{1}{2})$. Let K be the square $[-1, 1] \times [-1, 1]$ then, $S_1(K) = [0, 1] \times \{0\}, S_2(K) = [-1, 0] \times \{0\}, S_3(K) = [\frac{-1}{2}, \frac{1}{2}] \times [\frac{3}{4}, \frac{5}{4}]$. The projection IFSs $\Pi_{\chi_1} = \{\frac{x}{2} + \frac{1}{2}, \frac{x}{2} - \frac{1}{2}, \frac{x}{2}\}$ and $\Pi_{\chi_2} = \{\frac{y}{2} + \frac{1}{2}\}$. The attractors of the projection IFSs Φ_{χ_1} and Φ_{χ_2} are the connected subsets $[-1, 1] \times \{0\}$ and $\{0\} \times [\frac{3}{4}, \frac{5}{4}]$ respectively. The Type 1 IFS attractor is neither connected nor totally disconnected. The first iterate to the attractor of the Type 1 IFS to determine the connectedness of the attractor is shown in Figure 4.

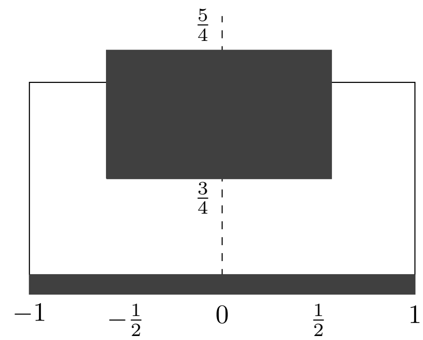

FIGURE 4. First iterate to the attractor of the Type 1 IFS χ to determine the connectedness of the attractor in Example 6.5

7. Application

Fractal geometry has considerable applications in various fields, especially in modeling irregular structures in higher dimensional spaces and thereby to study these complex structures. Major applications of fractal geometry include extracting data from images and analyzing to get conclusions. Nature uses fractal modeling for its construction. Self similar structures in nature can be approximated using fractal sets. In finance, modeling the fractal structure of financial assets predicts prices and crises in the market. Soft computing uses fractal parameters to explain transmission dynamics of various diseases[28]. Fractional properties like space fillingness, fractional dimensions, symmetricity, complexity, etc., are utilized for these analyses. Intense research is carried out in Fractal radios and Fractal Antennas[14]. Fractal image compression gives the most acceptable compression ratios compared to the standard image compression techniques.

In 2010, Bulusu Rama and Jibitesh Mishra introduced a construction of the 3D form of the popular fractals, the Mandelbrot Set and Julia Set which resembles the real world structures[26]. In 2014, Ankit Garg et al. designed an algorith to create 3D models using a basic model which is a cube or a sphere. A reccursive is iterated a specific number of times to create a 3D fractal model which is illustrated in Figure 5[12].

In this section, an application of simulation of a fractal structure in $\mathbb{R}^3$ is proposed using the theory of Type 1 IFS and fractal interpolation function. Simulation of a fractal structure using Iterated function system commences with an initial image and thereupon a series of contractive mappings are applied systematically. Thus obtained attractor of the IFS represents a complex image. We use Barnsley's interpolation method to find a best fit affine transformation for a given set of $N+1$ data points say, $(x_0, y_0), (x_1, y_1), \cdots, (x_N, y_N)$. An IFS

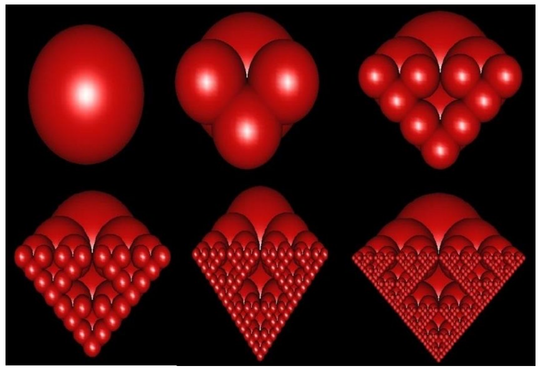

Figure 5. Construction of 3D models using a basic model which is a sphere

$\{\mathbb{R}^2; S_1, S_2, \cdots, S_N\}$ of affine transformations is obtained. The affine transformations,

$$S_i\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a_i & 0\\c_i & d_i\end{pmatrix} + \begin{pmatrix}e_i\\f_i\end{pmatrix}, \text{ for } i = \{1, 2, \cdots, N\}.$$

and the conditions

$$S_i\begin{pmatrix}x_0\\y_0\end{pmatrix} = \begin{pmatrix}x_{i-1}\\y_{i-1}\end{pmatrix} \text{ and } S_i\begin{pmatrix}x_N\\y_N\end{pmatrix} = \begin{pmatrix}x_i\\y_i\end{pmatrix}$$

form four linear equations whose solutions give,

$$a_i = \tfrac{x_i - x_{i-1}}{x_N - x_0}, \qquad c_i = \tfrac{y_i - y_{i-1}}{x_N - x_0} - d_i(\tfrac{y_N - y_0}{x_N - x_0}),$$

$$e_i = \tfrac{x_N x_{i-1} - x_0 x_i}{x_N - x_0}, \qquad f_i = \tfrac{x_N y_{i-1} - x_0 y_i}{x_N - x_0} - d_i(\tfrac{x_N y_0 - x_0 y_N}{x_N - x_0})$$

for $i = 1, 2, \cdots, N$. The mappings $S_i s$ are contractive since the free parameter d_i satisfies $0 \leq d_i < 1$. The other real parameters a_i, c_i, e_i and f_i are calculated using the interpolation data.

Now, there exist a unique non empty compact subset $K \subset \mathbb{R}^2$ corresponding to this hyperbolic IFS such that $K = \bigcup_{i=1}^{n} S_i(K)$ and the curve K of the continuous function $f : [x_0, x_N] \to \mathbb{R}$, interpolates the given data $\{(x_0, y_0), (x_1, y_1), \cdots, (x_N, y_N)\}$. This continuous map $f(x)$ is called the fractal interpolation function (FIF).

We propose constructing 3D fractal structures using projection IFSs of a Type 1 IFS. Consider two different 2D images of a 3D structure from different positions. The outline of the 2D images are analysed using the CAD program to obtain a data sets, $\{(x_{1_0}, y_{1_0}), (x_{1_1}, y_{1_1}), \cdots (x_{1_{n1}}, y_{1_{n1}})\}$ and $\{(x_{2_0}, y_{2_0}), (x_{2_1}, y_{2_1}), \cdots, (x_{2_{n_2}}, y_{2_{n2}})\}$. The method of fractal interpolation function gives IFSs, IFS_1 and IFS_2 corresponding to the obtained data sets. Let $IFS_1 = \{\mathbb{R}^2; S_{11}, S_{12}, \cdots, S_{1n_1}\}$ with contractivity factor $s_1 = \max_{1 \leq i \leq n_1} s_{1i}$ where s_{1i} is the contractivity factor of S_{1i} and $IFS_2 = \{\mathbb{R}^2; S_{21}, S_{22}, \cdots, S_{2n_2}\}$ with contractivity factor

$s_2 = \max_{1 \le i \le n_2} s_{2i}$ where s_{2i} is the contractivity factor of S_{2i}. The affine transformations

$$S_{ij}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{ij} & 0 \\ c_{ij} & d_{ij} \end{pmatrix} + \begin{pmatrix} e_{ij} \\ f_{ij} \end{pmatrix}, \text{ for, } i = \{1, 2\}, j = \{1, 2, \cdots, n_i\}$$

Then, the unique compact sets $K_1 = \bigcup_{i=1}^{n_1} S_{1i}(K_1)$ and $K_2 = \bigcup_{i=1}^{n_2} S_{2i}(K_2)$ are approximations of given 2D images.

Now, two projection IFSs of the Type 1 IFS which construct the 3D fractal image are constructed. The first projection IFS on $\mathbb{R}$ is constructed as follows: Using the parameters of FIF, choose the parameters $r_1 = \frac{s_{11}+s_{12}+\cdots+s_{1n_1}}{n_1}$, $r_2 = \frac{s_{21}+s_{22}+\cdots+s_{2n_2}}{n_2}$, $x_1 = e_{11} + e_{12} + \cdots + e_{1n_1}, y_1 = f_{11} + f_{12} + \cdots + f_{1n_1}, x_2 = e_{21}+e_{22}+\cdots+e_{2n_2}, y_2 = f_{21}+f_{22}+\cdots+f_{2n_2}$. Let the IFS takes the form $\{\mathbb{R}; S_1\}$. The contraction map $S : \mathbb{R} \to \mathbb{R}$ is defined as $S(x) = r_{\max}x + R_{\max}$, where $r_{\max} = \max\{r_1, r_2\}$ and $R_{\max} = \max\{x_1, y_1, x_2, y_2\}$. The second projection IFS is taken from one of the two IFSs, IFS_1 and IFS_2, considering the radius of the bounding circle of the attractor, contractivity factor of IFS, and the fractal dimension of the attractor. Let the chosen IFS be $\{\mathbb{R}^2; S_{21}, S_{22}, \cdots, S_{2N}\}$. Thus, a Type 1 IFS attractor is constructed in $\mathbb{R}^3$ corresponding to the given 2D images whose projection IFSs are obtained above. For the Type 1 IFS $\{\mathbb{R}^3; S_i\}$, where $S_i = (S_1, S_{2i})$, there exist a unique non empty compact attractor K in $\mathbb{R}^3$ such that $K = (S_1(K), \bigcup_{j=1}^{n} S_{2j}(K))$. A compact subset in $\mathbb{R}^3, G = B \times [a, b]$ where B is a box in $\mathbb{R}^2$ containing the 2D image results to the 3D fractal image, K under Type 1 IFS iterations. Thus this Type 1 IFS attractor approximates the 3D fractal image of the two given 2D images.

8. Conclusion

Self similar sets and their applications can be seen in natural science, medical science, and all areas where irregularity is to be modeled. The study of the separation and topological properties of these types of structures in product spaces has much more importance. In this paper, a special type of attractor in product space, named Type 1 IFS is studied. The existence of the unique fixed point called the attractor of the Type 1 IFS is proved. The general results, Collage theorem and continuity theorem is verified for Type 1 IFS. Separation property, OSC, and its topological property, connectedness are discussed. It is shown that the Type 1 IFS meets SSC if and only if any of the corresponding projection IFSs satisfy SSC. The conditions of the Type 1 IFS such that the corresponding projection IFSs satisfy the OSC is also proved. An algebraic equivalence for OSC is studied to characterize the feasible open set satisfying OSC. Connectedness properties of attractors of Type 1 IFS are established, and an equivalence condition for these properties is obtained. These properties are related by a generalized address system in product spaces. A 3D fractal image is constructed as an attractor of Type 1 IFS using two 2D images from different positions. However generating fractal interpolants with good precision require computational resources. As a future work, studies on the constructed 3D fractal image can also be conducted to analyze many fractal properties. More applications in the field of digital topology, ocean sciences, natural science and medical science are possible. The wide range of applications of fractal theory can leads to the development in many areas of science.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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