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POSNER'S THEOREM FOR GENERALIZED DERIVATIONS ASSOCIATED WITH A MULTIPLICATIVE DERIVATION

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Abstract. Let R be a ring and P be a prime ideal of R. A mapping $d : R \to R$ is called a multiplicative derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In this paper, our main motive is to obtain the well-known theorem due to Posner in the ring R/P for generalized derivations associated with a multiplicative derivation defined by an additive mapping $F : R \to R$ such that $F(xy) = F(x)y + xd(y)$, where $d : R \to R$ is a multiplicative derivation not necessarily additive. This article discusses the use of generalized derivations associated with a multiplicative derivation to investigate the commutativity of the quotient ring R/P.

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1. Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$. The ring R is called a prime ring if $a, b \in R, aRb = (0)$ implies $a = 0$ or $b = 0$. An ideal P of R is said to be prime if for $a, b \in R$, $aRb \subseteq P$ implies that $a \in P$ or $b \in P$. The commutator of two elements x and y of R is defined as $[x, y] = xy - yx$, while the symbol $x \circ y$ denotes the anti commutator of two elements x and y of R, which is defined as $xy + yx$. An additive mapping $d : R \to R$ is a derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive map $F : R \to R$ is said to be a generalized derivation associated with a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. The study of generalized derivation in prime rings was initiated by Hvala in [1], where he introduced the concept of generalized derivations. Obviously, every derivation is a generalized derivation, but the converse is not true in general. An additive map $f : R \to R$ is said to be left multiplier $f(xy) = f(x)y$ for all $x, y \in R$.

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Hence, generalized derivation covers both the concepts of derivations and left multipliers.

The concept of multiplicative derivation was first introduced by Daif [3]. The motivation behind the concept was the work of Martindale [4]. According to Daif, the mapping $D : R \rightarrow R$ is said to be a multiplicative derivation if $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. Particularly, in the case of multiplicative derivations, the mappings are not assumed to be additive. Later, the complete description of these mappings was given by Goldmann and Semrl [5]. The concept of multiplicative derivations was further extended to multiplicative generalized derivations by Daif and Tammam El-Saiyad [6]. Further, a more general definition of multiplicative (generalized)-derivation was given by Dhara and Ali [9] as follows: A mapping $F : R \rightarrow R$ is said to be a multiplicative (generalized)-derivation if there exists a map g on R such that $F(xy) = F(x)y + xg(y)$ for all $x, y \in R$, where g is any mapping on R not necessarily additive.

A mapping $f : R \rightarrow R$ is called centralizing on a non empty set H if $[f(x), x] \in Z(R)$ for all $x \in H$ and is called commuting if $[f(x), x] = 0$ for all $x \in H$. The first investigation in this direction was made by Posner [7]. Regarding commutativity in prime ring, Posner [7] studied as follows: A prime ring R admitting a nonzero centralizing derivation is commutative.

The relation between the existence of a derivation of a ring R and its commutativity has been a topic of continuous research in the last several years (see [8], [10], [11], [12]). Many authors have obtained the commutativity of prime and semi prime rings with generalized derivations satisfying certain differential constraints. In this paper, we are interested in the study of rings given as a quotient R/P, where R is an arbitrary ring and P is a prime ideal of R. The symbol $\overline{x}$ denotes the element $x + P$ in R/P. We are using a generalized derivation associated with a multiplicative derivation on R. More precisely, we obtain Posner's first and second theorems. Using a generalized derivation associated with a multiplicative derivation, in this paper, we investigate the commutativity of the factor ring R/P.

2. Preliminary Results

In order to study our main theorem, first we will establish some results:

Lemma 2.1. *Let R be a ring and P be a prime ideal of R. Let F be a generalized derivation associated with a multiplicative derivation d of R. If $a \in R$ such that $aF(x) \in P$, then $a \in P$ or $d(R) \subseteq P$.*

Proof. For any a in R, we have

$$aF(x) \in P, \quad \text{for all } x \in R. \tag{1}$$

Replacing x by xy, we get

$$aF(x)y + axd(y) \in P, \quad \text{for all } x, y \in R. \tag{2}$$

Using (1), we get

$$axd(y) \in P, \quad \text{for all } x, y \in R. \tag{3}$$

Which is

$$aRd(y) \in P, \quad \text{for all } y \in R. \tag{4}$$

Now, using the fact that P is prime, we have

$$a \in p \quad \text{or} \quad d(R) \subseteq P.$$

□

Lemma 2.2. *Let R be a ring and P be a prime ideal of R. Let F be a generalized derivation associated with a multiplicative (not additive) derivation d and $G = \{x \in R | d(x) \in P\}$ be a subset of R, then G is an additive subgroup of R.*

Proof. Consider

$$G = \{x \in R : d(x) \in P\}.$$

Let $x \in R$ and $y, z \in G$, which implies $d(y) \in P$ and $d(z) \in P$. Now,

$$F(x(y-z)) = F(x)(y-z) + xd(y-z).$$

Which gives

$$F(xy - xz) = F(x)(y-z) + xd(y-z), \quad \text{for all } \; x, y, z \in R.$$

That is

$$F(x)y + xd(y) - F(x)z - xd(z) = F(x)(y-z) + xd(y-z), \quad \text{for all } \; x, y, z \in R.$$

Which results in

$$F(x)(y-z) + xd(y) - xd(z) = F(x)(y-z) + xd(y-z), \quad \text{for all } x, y, z \in R.$$

That is

$$xd(y) - xd(z) = xd(y-z), \quad \text{for all } x, y, z \in R.$$

Since, $d(y) \in P$ and $d(z) \in P$, therefore, $xd(y-z) \in P$. Which implies

$$d(y-z)xd(y-z) \in P, \quad \text{for all } \; x, y, z \in R.$$

Therefore,

$$d(y-z) \in P, \quad \text{for all } \; y, z \in R.$$

Hence,

$$y - z \in G, \quad \text{for all } \; y, z \in R.$$

Therefore, G is an additive subgroup of R. □

Lemma 2.3. *Let R be a ring and P be a prime ideal of R. Let F be a generalized derivation associated with a multiplicative derivation d of R. If $[[x, F(x)], y] \in P$ for all $y \in R$, then we have $\overline{x[x, d(x)]x} = \overline{x^2[x, d(x)]}$.*

Proof. For any $x, y \in R$, we have
$$[[x, F(x)], y] \in P. \tag{5}$$
Therefore,
$$\overline{[x, F(x)]} \in Z(R/P), \quad \text{for all } x \in R. \tag{6}$$
On linearizing
$$\overline{[x, F(y)]} + \overline{[y, F(x)]} \in Z(R/P), \quad \text{for all } x, y \in R. \tag{7}$$
Replacing y by x^2 in (7), we get
$$\overline{[x, F(x)]x} + \overline{x[x, d(x)]} + \overline{x[x, F(x)]} + \overline{[x, F(x)]x} \in Z(R/P), \quad \text{for all } x \in R. \tag{8}$$
Using (6), we get
$$\overline{3x[x, F(x)]} + \overline{x[x, d(x)]} \in Z(R/P), \quad \text{for all } x \in R. \tag{9}$$
Which results in
$$\overline{[3x[x, F(x)] + x[x, d(x)], x]} = \overline{0}, \quad \text{for all } x \in R. \tag{10}$$
This gives
$$\overline{x[x, d(x)]x} - \overline{x^2[x, d(x)]} = \overline{0}, \quad \text{for all } x \in R.$$
Which implies
$$\overline{x[x, d(x)]x} = \overline{x^2[x, d(x)]}, \quad \text{for all } x \in R.$$
□

Lemma 2.4. *Let R be a ring and P be a prime ideal of R. Let F be a generalized derivation of R, associated with a multiplicative derivation d. If $[x, F(x)] \in P$ for all $x \in R$, then $d(R) \subseteq P$ or R/P is commutative.*

Proof. For all $x \in R$, we have
$$[x, F(x)] \in P. \tag{11}$$
On linearizing (11), we get
$$[x, F(y)] + [y, F(x)] \in P, \quad \text{for all } x, y \in R. \tag{12}$$
Now, replacing y by yx in (12), we get
$$([x, F(y)] + [y, F(x)])x + y[x, d(x)] + [x, y]d(x) \in P, \quad \text{for all } x, y \in R. \tag{13}$$
Using (12), we get
$$y[x, d(x)] + [x, y]d(x) \in P, \quad \text{for all } x, y \in R. \tag{14}$$
Replacing y by zy in (14), we get
$$zy[x, d(x)] + z[x, y]d(x) + [x, z]yd(x) \in P, \quad \text{for all } x, y, z \in R.$$
Using (14), we get
$$[x, y]yd(x) \in P, \quad \text{for all } x, y \in R. \tag{15}$$
Which is
$$[x, y]Rd(x) \in P, \quad \text{for all } x, y \in R.$$

Using the fact that P is a prime ideal of R, we get

$$d(x) \in P \quad \text{or} \quad [x, y] \in P, \quad \text{for all } x, y \in R.$$

Consequently, R is a union of two subgroups, G_1 and G_2, where

$$G_1 = \{x \in R | d(x) \in P\} \text{ and } G_2 = \{x \in R | [R, y] \subset P\}.$$

Now, G_1 is clearly an additive subgroup of R and using Lemma 2.2, G_2 is also an additive subgroup of R. Since a group cannot be a union of two of its proper subgroups, we conclude that either $R = G_1$ or $R = G_2$. That is, $d(R) \subseteq P$ or R/P is commutative. □

3. Main Results

In this work, we explore Posner's theorem on the ring R/P in the context of generalized derivation associated with multiplicative derivation. The obtained results are stated below:

Theorem 3.1. *Let R be a ring and P be a prime ideal of R. Let F_1 and F_2 be two generalized derivations of R associated with multiplicative derivations d_1 and d_2, respectively. If $F_1(x)d_2(y) + F_2(x)d_1(y) \in P$, then we have one of the following assertions:*

(i) *char $R/P = 2$,*
(ii) *$F_1(R) \subseteq P$,*
(iii) *$d_1(R) \subseteq P$,*
(iv) *$d_2(R) \subseteq P$.*

Proof. Assume that char$R/P \neq 2$, we are given that

$$F_1(x)d_2(y) + F_2(x)d_1(y) \in P, \quad \text{for all } x, y \in R. \tag{16}$$

Replacing y by ry in (16), we get

$$F_1(x)d_2(r)y + F_2(x)d_1(r)y + F_1(x)rd_2(y) + F_2(x)rd_1(y) \in P, \quad \text{for all } x, y, r \in R. \tag{17}$$

Using (16), we have

$$F_1(x)rd_2(y) + F_2(x)rd_1(y) \in P, \quad \text{for all } x, y, r \in R. \tag{18}$$

Now replace x by rx in (16)

$$F_1(r)xd_2(y) + rd_1(x)d_2(y) + F_2(r)xd_1(y) + rd_2(x)d_1(y) \in P, \quad \text{for all } x, y, r \in R. \tag{19}$$

Using (18), we get

$$r(d_1(x)d_2(y) + d_2(x)d_1(y)) \in P, \quad \text{for all } x, y, r \in R. \tag{20}$$

Since P is prime, we have

$$d_1(x)d_2(y) + d_2(x)d_1(y) \in P, \quad \text{for all } x, y \in R. \tag{21}$$

Now replace x by $rd_1(z)$ in (16)

$$(F_1(r)d_1(z) + rd_1(d_1(z)))d_2(y) + (F_2(r)d_1(z) + rd_2(d_1(z)))d_1(y) \in P, \tag{22}$$

for all $r, y, z \in R$.
Using (21) with $x = d_1(z)$, we get

$$F_1(r)d_1(z)d_2(y) + F_2(r)d_1(z)d_1(y) \in P, \quad \text{for all } z, y, r \in R. \tag{23}$$

Using (16), we can write

$$F_1(r)d_2(z)d_1(y) + F_2(r)d_1(z)d_1(y) \in P, \quad \text{for all } z, y, r \in R. \tag{24}$$

Using (21) and (23), we get

$$F_1(r)(d_2(z)d_1(y) - d_1(z)d_2(y)) \in P, \quad \text{for all } z, y, r \in R. \tag{25}$$

Using Lemma 2.1, we get

$$F_1(r) \in P \text{ or } d_2(z)d_1(y) - d_1(z)d_2(y) \in P, \quad \text{for all } y, z \in R. \tag{26}$$

Using (21) and (26), we get

$$2d_2(z)d_1(y) \in P, \quad \text{for all } y, z \in R.$$

Since R/P is not of char 2, for all $y, z \in P$ we have

$$\overline{2d_2(z)d_1(y)} = \overline{0}.$$

Which gives

$$\overline{d_2(z)d_1(y)} = \overline{0}.$$

That is

$$d_2(z)d_1(y) \in P.$$

On replacing $d_2(z) = a$, we have

$$ad_1(y) \in P, \quad \text{for all } a, y \in R.$$

Using Lemma 2.1, we get

$$d_1(y) \subseteq P \quad \text{or} \quad a \in P, \quad \text{for all } a, y \in R.$$

Which results in

$$d_1(y) \subseteq P \quad \text{or} \quad d_2(z) \subseteq P, \quad \text{for all } y, z \in R.$$

□

Using similar arguments and in equation (16) replacing x by $rd_2(z)$ instead of $rd_1(z)$ and in equation (21) replacing x by $d_2(z)$ instead of $d_1(z)$, we get

Theorem 3.2. *Let R be a ring and P be a prime ideal of R. Let F_1 and F_2 be two generalized derivations of R associated with multiplicative derivations d_1 and d_2, respectively. If $F_1(x)d_2(y) + F_2(x)d_1(y) \in P$, then we have one of the following assertions:*

(i) *char $R/P = 2$,*
(ii) *$F_2(R) \subseteq P$,*
(iii) *$d_1(R) \subseteq P$,*
(iv) *$d_2(R) \subseteq P$.*

If we take R to be a prime ring and $P = (0)$, then we can obtain Posner's first theorem:

Corollary 3.3 (7, Theorem 1)**.** *Let R be a 2 torsion free prime ring. If d_1 and d_2 are derivations on R such that the iterate of d_1 and d_2 is also a derivation of R, then $d_1 = 0$ or $d_2 = 0$.*

Proof. Assume that the iterate of two derivations d_1 and d_2 is also a derivation, then

$$d_1d_2(xy) = d_1d_2(x)y + xd_1d_2(y), \quad \text{for all } x, y \in R. \tag{27}$$

As d_2 is a derivation, hence

$$d_1d_2(xy) = d_1(d_2(x)y + xd_2(y)), \quad \text{for all } x, y \in R.$$

As d_1 is a derivation, which gives

$$d_1d_2(xy) = d_1d_2(x)y + d_2(x)d_1(y) + d_1(x)d_2(y) + xd_1d_2(y), \quad \text{for all } x, y \in R. \tag{28}$$

Using (27) and (28), we get

$$d_1(x)d_2(y) + d_2(x)d_1(y) = 0, \quad \text{for all } x, y \in R.$$

Hence, the above theorem follows.

□

Theorem 3.4. *Let R be a ring and $F : R \to R$ be a generalized derivation associated with a multiplicative derivation d. If $[[x, F(x)], y] \in P$ for all $y \in R$, then we have one of the following:*

(i) *char $R/P = 2$,*
(ii) *R/P is commutative.*

Proof. Assume that char $R/P \neq 2$, for all $x, y \in R$ we have

$$[[x, F(x)], y] \in P. \tag{29}$$

Which gives

$$\overline{[x, F(x)]} \in Z(R/P). \tag{30}$$

Linearizing (30), we get

$$\overline{[x, F(y)] + [y, F(x)]} \in Z(R/P). \tag{31}$$

Replacing y by xy in (31), we get

$$\overline{F(x)[x, y] + [x, F(x)]y + x[x, d(y)] + x[y, F(x)] + [x, F(x)]y} \in Z(R/P).$$

Which gives

$$\overline{2[x, F(x)]y + F(x)[x, y] + x[x, d(y)] + x[y, F(x)]} \in Z(R/P).$$

Substituting x^2 in place of y gives

$$\overline{4x^2[x, F(x)] + x[x, d(x)]x + x^2[x, d(x)]} \in Z(R/P). \tag{32}$$

Using Lemma (2.3), we get

$$\overline{4x^2[x,F(x)]+2x^2[x,d(x)]} \in Z(R/P). \tag{33}$$

By hypothesis, we have

$$\overline{[x^2,F(x^2)]} \in Z(R/P).$$

Which gives

$$\overline{x[x,F(x)]x+x^2[x,d(x)]+[x,F(x)]x^2+x[x,d(x)]x} \in Z(R/P).$$

Using (30) and Lemma (2.3), we get

$$\overline{2x^2[x,F(x)]+2x^2[x,d(x)]} \in Z(R/P). \tag{34}$$

Using (33) and (34), we get

$$\overline{2x^2[x,F(x)]} \in Z(R/P).$$

Particularly,

$$\overline{[2x^2[x,F(x)],F(x)]} = \overline{0}.$$

Since R/P is 2 torsion free

$$\overline{x[x,F(x)]^2} = \overline{0}. \tag{35}$$

Hence, by (35) and $\overline{[x,F(x)]} \in Z(R/P)$, we get

$$\overline{x} = \overline{0} \quad \text{or} \quad \overline{[x,F(x)]} = \overline{0}.$$

As R/P is prime, Lemma 2.4 finishes the proof. □

We can obtain Posner's second theorem for $P = (0)$ as:

Corollary 3.5 (7, Theorem 2)**.** *Let R be a 2 torsion free prime ring. Let d be a derivation on R such that $[d(x),x] \in Z(R)$ for all $x,y \in R$, then R is commutative.*

The following examples demonstrate that prime ideal is a necessary condition in the hypotheses of both theorems.

Example 3.6. Let $R = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : a,b,c \in Z \right\}$ be the ring of 3×3 matrices, define $F,d : R \longrightarrow R$ as follows

$$d\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } F\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, F is a generalized derivation associated with the multiplicative derivation d. Let $P = \left\{ \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : a \in Z \right\}$, which is not a prime ideal of R. Now, it can be verified that the condition $[[x,F(x)],y] \in P$ in Theorem 3.4 holds true, but neither char $R/P = 2$ nor R/P is commutative.

Example 3.7. Let $R = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in \Re \right\}$, be the ring of 3×3 matrices where $\Re$ is a non-commutative ring, define $F, d : R \longrightarrow R$ as follows

$$d \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b^2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } F \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -b \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, F is a generalized derivation associated with the multiplicative derivation d. Let $P = \left\{ \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : a \in \Re \right\}$, clearly, P is not a prime ideal of R. We can see that the condition $F_1(x)d_2(y) + F_2(x)d_1(y) \in P$ in Theorem 3.1 holds for $F_1 = F_2 = F$ and $d_1 = d_2 = d$. Now, we can easily verify that any assertion of Theorem 3.1 does not hold true.

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