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A NOTE ON IMPRECISE GROUP AND ITS PROPERTIES†

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ABSTRACT. In this paper, using the notion of the imprecise set, the idea of an imprecise group is introduced including some examples. The two key rules of classical set theory are obeyed by this extended version of fuzzy sets, which the existing complement definition of a fuzzy set failed to do. With the support from general group theory, the paper also provides some fundamental properties of an imprecise group here. Additionally, it includes a few characteristics of imprecise subgroups, and abelian imprecise group.

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Key words : Imprecise set, imprecise group, imprecise subgroup, Abelian imprecise group.

List of Abbreviations

FS	Fuzzy Set
MF	Membership Function
RF	Reference Function
MV	Membership Value
FG	Fuzzy Group
FS_G	Fuzzy Subgroup
IS	Imprecise Set
IG	Imprecise Group
IS_G	Imprecise Subgroup
FCs	Fuzzy Cosets
FNS_G	Fuzzy Normal Subgroup

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FOs	Fuzzy Orders
FCS_G	Fuzzy Complex Subgroup
KS_G	Kernel Subgroup
AFG	Anti Fuzzy Group
$IVFM$	Interval Valued Fuzzy Matrix

1. Introduction

The theory of FSs is a generalisation of the theory of classical sets. This theory was introduced by Zadeh [24] in the year 1965. It has been merged with various uncertainty techniques and is extended to a wide range in mathematics by many authors. One of the remarkable application of FS theory is Rosenfeld's [3] fuzzy group theory. In 1971, Rosenfeld [3] used the concept of a fuzzy subset of a set to introduce the notion of a FS_G. Rosenfeld's [3] work motivated the development of fuzzy abstract algebra. This study has been carried out further by Mukherjee and Bhattacharya [31], Bhattacharya [32, 33] and Bhattacharya and Mukherjee [34]. In 2013, Li et al. [51] did a detailed investigation on (λ, μ) FS_Gs, specially (λ, μ) FCs and (λ, μ)-FN_Gs with some basic properties. In 2016, Jun et al. [50] introduced a notion of $(\in, \wedge q)$-FS_Gs which is a generalization of Rosenfeld's FS_G [3]. In 2019, Hussain and Palaniyandi [38] implemented fuzzy set theory and fuzzy group theory in Q-fuzzy groups. In 2019, Ardanza-Trevijano et al. [41] implemented the idea of different type of annihilator on FS_Gs which is essential in classical duality theory and extended widely to apply the concept of orthogonal complement in Euclidean spaces. They discovered that in natural duality of a group, a fuzzy subgroup can be recovered after taking the inverse annihilator of it. In 2021, Bejines et al. [8] proved that using an aggregation function on two FS_Gs is always a FS_G if the cardinality of group is of prime power. In 1994, Kim [16, 17] introduced the idea of fuzzy orders of the element of a group. In 2021, Prasanna et al. [2] studied about the new concept of K-Q-FOs of a group. In 2022, Masmali et al. [15] characterized the notion of μ-FS_Gs and proved many fundamental algebraic properties. One of the important expansion of FS theory is intuitionistic FS theory introduced by Atanassov [23] in 1986. This theory has a wide range of application specially in medical, neural networks and history of time travel. In 1996, Biswas [37] extended the concept of intuitionistic FS to intuitionistic FS_G. In 2019, Alolaiyan et al. [10] defined t-intuitionistic FO and investigated different algebraic properties of it. Further, he extended this work to establish t-intuitionistic fuzzification of Lagrange's theorem. In 2020, Alghazzawi et al. [9] introduced the notion of ρ anti-intuitionistic FSs, ρ anti-intuitionistic FC, ρ anti-intuitionistic FNS_G, quotient group of a group induced by ρ anti-intuitionistic FNS_G and established a group isomorphism between these newly defined quotient group of a group G relative to its particular normal subgroup. In 2020, Gulzar et al. [29] studied about normalizer, centralizer, abelian and cyclic subgroups of t-intuitionistic FS_G and investigated its properties. It is shown that under group homomorphism the image and pre image of t-intuitionistic FS_G of Abelian (cyclic) subgroups are t-intuitionistic fuzzy Abelian(cyclic) subgroups. In 2020, Gulzar et al. [30] initiated a new concept of complex intuitionistic FS_G and studied its various characteristics. In 2021, Bhunia et al. [42] introduced the idea of Pythagorean FS_G and studied many properties. In 2022, Bal et al. [25] defined KS_G of an intuitionistic FG and proved that this is again a subgroup having same properties of general group. In the same year, Ahmad et al. [20] defined KS_G of FG, AFG and studied some of its properties. In 2023, Rasuli [39, 40] studied Intuitionistic FS_G using norms over intuitionistic FCS_G and Q-intuitionistic FS_G along with their properties respectively.

However, Zadeh's [24] formulation of fuzzy set complement did not obey the notion of the two universal law of the classical set: non contradiction and excluded middle, which contradicts the statement that FS theory is a generalization of classical set theory. In this regard, Baruah [11] concluded that this drawback in fuzzy complement definition is due to the fact that the existing definition of FS has defined for only MF. And, this led to the conclusion that Zadeh's fuzzy complement set definition do not follow the two important laws of classical set theory: law of exclusion and law of conclusion. Then Baruah [11, 12, 13] and [14] forwarded a new definition of FSs in terms of MF and RF, which enabled us to get a new definition of fuzzy complement of a FS. And, this extended definition of FS can overcome the drawback of Zadeh's [24] FS and can give us union and intersection of a FS and its complement as universal set and null set respectively. The IS is the term used to describe this extended definition of a FS with a new complement form. This set satisfies many properties of the classical set theory and is discussed by many authors. The theory is later employed in a variety of extension studies of fuzzy numbers. For instance, in 2011 Neog et al. [46] generalized the concept of complement of a FS by taking non-zero fuzzy RF with some examples and showed that this generalization of fuzzy complement satisfies all those properties of union and intersection of classical set. In 2012, Dhar [28] highlighted the shortcomings of Zadeh's [24] FS definition and proved by geometrical representation of FSs that Baruah's new FS definition is more acceptable to answer various questions that would arise in FS theory. In 2013, Dhar [26] studied determinant of fuzzy matrices with respect to MF and RF, and investigated some properties that are analogous to the properties of determinant of classical matrix. In the same year, Dhar [27] also proposed a new definition for the cardinality of FSs with respect to MF and RF to give a proper cardinality of FS while dealing with the complement of a FS. Further, some results are proven with this new definition and found that results are analogous to that of the existing definition of FS. In 2015, Borgoyary [43] applied Baruah's [11] extended definition of FSs in usual matrices and named it as imprecise matrices with new notations. Using min and max operators, some new definitions of matrices are also obtained. It is found that the properties that hold good in classical matrices also hold good in these new matrices which is called imprecise matrices. In the same year, Borgoyary

[44] studied 2 and 3 dimensional fuzzy number in terms of MF and RF. It is seen that most of the properties from classical set theory that hold good in this study. In 2015 and 2016, Basumatary [4, 7] redefined fuzzy closure on the basis of extended definition of Baruah [11] and fuzzy closure with reference to fuzzy boundary respectively. In this study the author discussed some properties of fuzzy closure using this extended definition with some supported numerical examples. In 2016, Borgoyary [45] has talked about how the MF and RF represent a special imprecise number. Therefore, every imprecise number is also an imprecise set, though the converse may not be true. Again in 2017, Borgoyary [21] studied about normal imprecise functions with the help of sine and cosine functions. In 2023, Pushpalatha and Chandra [48] introduced a new concept of $IVFM$ matrices on the basis of RF and studied some properties related to arithmetic, geometric and harmonic mean of the matrices. The main aim of this study was to convert the uncontrollable function to controllable function and undesigned function to designed function using sine and cosine functions. In 2017, Basumatary et al. [6] redefined fuzzy boundary definition using Baruah's [11] FS definition and fuzzy complement definition with respect to fuzzy MF and fuzzy RF. Here the authors observed that there are some boundary properties of classical set that are not satisfied in fuzzy definition. But in this article, it is shown that those properties can hold good in the proposed definition of fuzzy boundary with respect to RF. In 2018, Basumatary and Mwchahary [5] applied the extended definition of Baruah's [11] fuzzy set in intuitionistic FS and studied the characteristic of this new concept.

In this article, our interest is to study the formation of an IG under multiplicative operation. Here, we attempted to use our group definition to explain the fundamental properties, theorems, and examples. In general, the IG is an extended concept to study Rosenfeld's [3] fuzzy subgroup theory. When the FS definition is imprecise, the elements are defined in terms of two MF and RF functions. In this case, the RF is assumed to be zero everywhere and the MF is taken throughout the unit interval [0,1]. In our study, the Rosenfeld's [3] work is used to design an IG using the definition of the FS_G.

2. Motivation

In past years, the application of FS theory has generated some debate among the researchers, as it was observed by some authors that FS theory cannot deal with certain uncertainty boundary problems of real world. Among them, Piegat [1] mentioned about the shortcomings in Zadeh's [24] definition of fuzzy arithmetic for solving some practical problems. To eliminate such shortcomings, many researchers induced new formulation for fuzzy arithmetic operations; for example Kosinski et al. [49]. Shi gao et al. [36] found some other drawbacks in Zadeh's [24] fuzzy complement definition for fuzzy number. This is why, the authors proposed an extended definition called C-FS theory which is free from Zadeh's [24] FS's shortcomings. They point out that if the complement of a FS

is defined as $1 - \mathfrak{u}_{\mathfrak{f}}(x)$ where $\mathfrak{u}_{\mathfrak{f}}$ is MF then the complement of a set may not exist in Zadeh's [24] FS theory.

For example: According to the existing definition of FS if $A_f = \{x, \mathfrak{u}_{\mathfrak{f}}(x)\} = \{x, 0.5\}$ is a FS and its complement is $A_f^c = \{x, 1 - \mathfrak{u}_{\mathfrak{f}}(x)\} = \{x, 1 - 0.5\} = \{x, 0.5\}$
Then $A_f \cup A_f^c = \{x, 0.5\} \cup \{x, 0.5\} = \{x, 0.5\} \neq \{x, 1\}$ (universal set)
And, $A_f \cap A_f^c = \{x, 0.5\} \cap \{x, 0.5\} = \{x, 0.5\} \neq \{x, 0\}$ (null set)

3. Novelty

Baruah [11, 12, 13] and [14] also pointed out some other drawbacks of the theory of FS. He noted that the complement definition of FS and *Probability-Possibility Consistency Principle* are not defined well. He defined the fuzzy set definition in a new way which is in terms of the two functions namely fuzzy MF and fuzzy RF instead of a single MF. This extended definition is termed as IS and is defined in such a way that if $\mathfrak{u}_{\mathfrak{m}}^1(\mathfrak{t}_1)$ is a fuzzy MF and $\mathfrak{u}_{\mathfrak{r}}^2(\mathfrak{t}_1)$ is a fuzzy RF such that $0 \leqslant \mathfrak{u}_{\mathfrak{r}}^2(\mathfrak{t}_1) \leqslant \mathfrak{u}_{\mathfrak{m}}^1(\mathfrak{t}_1) \leqslant 1$, then $\mathfrak{u}_A^i = \{\mathfrak{t}_1, \mathfrak{u}_{\mathfrak{m}}^1(\mathfrak{t}_1), \mathfrak{u}_{\mathfrak{r}}^2(\mathfrak{t}_1); \mathfrak{t}_1 \in X\}$ where X is the universal set and $\mathfrak{u}_v^i(\mathfrak{t}_1) = \mathfrak{u}_{\mathfrak{m}}^1(\mathfrak{t}_1) - \mathfrak{u}_{\mathfrak{r}}^2(\mathfrak{t}_1)$ is the actual MV for all $\mathfrak{t}_1 \in X$.

Now, if $A_f = \{\mathfrak{t}_1, \mathfrak{u}_{\mathfrak{f}}(\mathfrak{t}_1)\}$ is the existing FS and $A_f^c = \{\mathfrak{t}_1, 1 - \mathfrak{u}_{\mathfrak{f}}(\mathfrak{t}_1)\}$ is its fuzzy complement, then according to this extended definition, it would be $\mathfrak{u}_A^i = \{\mathfrak{t}_1, \mathfrak{u}_{\mathfrak{m}}^1(\mathfrak{t}_1), 0\}$ and the complement of $\mathfrak{u}_A^i$ would be $\mathfrak{u}_A^{i^c} = \{\mathfrak{t}_1, 1, \mathfrak{u}_{\mathfrak{m}}^1(\mathfrak{t}_1)\}$.

$$\begin{aligned}\text{Then } \mathfrak{u}_A^i \cup \mathfrak{u}_A^{i^c} &= \{\mathfrak{t}_1, \mathfrak{u}_{\mathfrak{m}}^1(\mathfrak{t}_1), 0\} \cup \{\mathfrak{t}_1, 1, \mathfrak{u}_{\mathfrak{m}}^1(\mathfrak{t}_1)\} \\ &= \{\mathfrak{t}_1, 1, 0\} \\ &= X(\text{universal set})\end{aligned}$$

$$\begin{aligned}\text{And, } \mathfrak{u}_A^i \cap \mathfrak{u}_A^{i^c} &= \{\mathfrak{t}_1, \mathfrak{u}_{\mathfrak{m}}^1(\mathfrak{t}_1), 0\} \cap \{\mathfrak{t}_1, 1, \mathfrak{u}_{\mathfrak{m}}^1(\mathfrak{t}_1)\} \\ &= \{\mathfrak{t}_1, \mathfrak{u}_{\mathfrak{m}}^1(\mathfrak{t}_1), \mathfrak{u}_{\mathfrak{m}}^1(\mathfrak{t}_1)\} \\ &= \phi(\text{null set})\end{aligned}$$

This is why the above definition of Baruah [11] is more acceptable and logical than Zadeh's FS theory.

The main focus of this article is to adopt the extended definition of fuzzy set in order to develop a new methodology to discuss the fuzzy group more appropriately so that the result can be applied in different areas.

Narzary *et al.* [19] currently used Baruah's [11] and Rosenfeld's [3] definition on normal fuzzy subgroup and this work is presently accepted for publication.

In this article the proposed imprecise group definition is obtained within a suitable mathematical framework and it is defined in accordance with Baruah's [11] extended FS definition.

4. Preliminaries

Definition 4.1 (Rosenfeld [3])**.** Rosenfeld [3] defined FS_G of a group using the notion of Zadeh's [24] fuzzy subset of a set in the year 1971. Rosenfeld [3] defined a fuzzy subset $\mathfrak{u}_{\mathfrak{f}}$ of a group G to be a FS_G of G if

(i) $\mathfrak{u}_{\mathfrak{f}}(\mathfrak{t}_1\mathfrak{t}_2) \geq \mathfrak{u}_{\mathfrak{f}}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{f}}(\mathfrak{t}_2); \forall \mathfrak{t}_1, \mathfrak{t}_2 \in G$

(ii) $\mathfrak{u}_{\mathfrak{f}}({\mathfrak{t}_1}^{-1}) \geq \mathfrak{u}_{\mathfrak{f}}(\mathfrak{t}_1); \forall \mathfrak{t}_1 \in G$
(iii) $\mathfrak{u}_{\mathfrak{f}}(e) \geq \mathfrak{u}_{\mathfrak{f}}(\mathfrak{t}_1); \forall \mathfrak{t}_1 \in G$.

Definition 4.2 ([3]). If $\mathfrak{u}_{\mathfrak{f}_1}$ and $\mathfrak{u}_{\mathfrak{f}_2}$ are two FS_G then their product is defined as $\mathfrak{u}_{\mathfrak{f}_1} \circ \mathfrak{u}_{\mathfrak{f}_2}(\mathfrak{t}_3) = \vee\{\mathfrak{u}_{\mathfrak{f}_1}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{f}_2}(\mathfrak{t}_2) | \mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3 \in G, \mathfrak{t}_1\mathfrak{t}_2 = \mathfrak{t}_3\}$.

Definition 4.3 ([14]). If $\mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1)$ is a fuzzy MF and $\mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1)$ is a fuzzy RF such that $0 \leqslant \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1) \leqslant \mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1) \leqslant 1$, then the IS is defined as $\mathfrak{u}^i_A = \{\mathfrak{t}_1, \mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1), \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1); \mathfrak{t}_1 \in X\}$ where X is the universal set and $\mathfrak{u}^i_v(\mathfrak{t}_1) = \mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1) - \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1)$ gives the actual MV for all $\mathfrak{t}_1 \in X$.
Here, $\mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1)$ gives the greatest possible grade of membership of $\mathfrak{t}_1$ and $\mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1)$ gives the least possible grade of reference of $\mathfrak{t}_1$ derived from the 'presence of $\mathfrak{t}_1$ in the set'. Thus the grade of actual presence of $\mathfrak{t}_1$ in the set represents a subregion in the unit interval enclosed within a single brackets $(\mathfrak{u}^1_{\mathfrak{m}}, \mathfrak{u}^2_{\mathfrak{r}})(\mathfrak{t}_1)$ where $\mathfrak{u}^1_{\mathfrak{m}} - \mathfrak{u}^2_{\mathfrak{r}}$ gives the actual membership value of $\mathfrak{t}_1$.
The imprecise set is written as $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1) = \{\mathfrak{t}_1, \mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1), \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1) : \mathfrak{t}_1 \in X\}$.
For convenient of writing above IS is denoted by $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1); \forall \mathfrak{t}_1 \in X$.

Definition 4.4 ([14]). If $A = \{\mathfrak{t}_1, \mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1), \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1); \mathfrak{t}_1 \in X\}$ and $B = \{\mathfrak{t}_1, \mathfrak{u}^3_{\mathfrak{m}}(\mathfrak{t}_1), \mathfrak{u}^4_{\mathfrak{r}}(\mathfrak{t}_1); \mathfrak{t}_1 \in X\}$ are two ISs then

$A \overset{i}{\cup} B = \{\mathfrak{t}_1, max(\mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1), \mathfrak{u}^3_{\mathfrak{m}}(\mathfrak{t}_1)), min(\mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1), \mathfrak{u}^4_{\mathfrak{r}}(\mathfrak{t}_1)); \mathfrak{t}_1 \in X\}$

$A \overset{i}{\cap} B = \{\mathfrak{t}_1, min(\mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1), \mathfrak{u}^3_{\mathfrak{m}}(\mathfrak{t}_1)), max(\mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1), \mathfrak{u}^4_{\mathfrak{r}}(\mathfrak{t}_1)); \mathfrak{t}_1 \in X\}$

Definition 4.5 ([26]). If $A = \{\mathfrak{t}_1, \mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1), \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1); \mathfrak{t}_1 \in X\}$ and $B = \{\mathfrak{t}_1, \mathfrak{u}^3_{\mathfrak{m}}(\mathfrak{t}_1), \mathfrak{u}^4_{\mathfrak{r}}(\mathfrak{t}_1); \mathfrak{t}_1 \in X\}$ are two ISs then $A \overset{i}{\times} B = \{\mathfrak{t}_1, \mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1) \times \mathfrak{u}^3_{\mathfrak{m}}(\mathfrak{t}_1), \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1) \times \mathfrak{u}^4_{\mathfrak{r}}(\mathfrak{t}_1); \mathfrak{t}_1 \in X\}$. It can also be presented as $A \overset{i}{\times} B$

The symbol of MF and the RF are denoted by $\mathfrak{u}^1_{\mathfrak{m}}$ and $\mathfrak{u}^2_{\mathfrak{r}}$ respectively in our study. And, the variables are denoted by $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3$, etc.

5. Imprecise Group (IG)

Definition 5.1. Let an imprecise subset $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}$ of a group $[G, *]$ together with a binary composition '*' be called an IG $[\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}, *]$, if $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2) = \{\mathfrak{t}_1\mathfrak{t}_2, \mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2), \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2); \forall \mathfrak{t}_1, \mathfrak{t}_2 \in G\}$, $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1) = \{\mathfrak{t}_1, \mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1), \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1)\}, \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2) = \{\mathfrak{t}_2, \mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_2), \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_2)\}$ and $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}({\mathfrak{t}_1}^{-1}) = \{{\mathfrak{t}_1}^{-1}, \mathfrak{u}^1_{\mathfrak{m}}({\mathfrak{t}_1}^{-1}), \mathfrak{u}^2_{\mathfrak{r}}({\mathfrak{t}_1}^{-1})\}$ satisfies the following conditions:

(i) $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2) \geq \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1) \wedge \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2); \forall \mathfrak{t}_1, \mathfrak{t}_2 \in G$
(ii) $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}({\mathfrak{t}_1}^{-1}) \geq \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1); \forall \mathfrak{t}_1 \in G$
(iii) $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(e) \geq \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1); \forall \mathfrak{t}_1 \in G$

Where $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1) \wedge \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2) = (\mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1) \wedge \mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_2), \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1) \vee \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_2))$
$= (min(\mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1), \mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_2)), max(\mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1), \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_2))); \forall \mathfrak{t}_1, \mathfrak{t}_2 \in [G, *]$, ${\mathfrak{t}_1}^{-1}$ is an inverse of $\mathfrak{t}_1$, $\mathfrak{u}^1_{\mathfrak{m}}$ is the MF, $\mathfrak{u}^2_{\mathfrak{r}}$ is the RF which is considered to be 0 in our study i.e., $0 \leq \mathfrak{u}^1_{\mathfrak{m}} \leq 1$ and $\mathfrak{u}^2_{\mathfrak{r}} = 0$ and $\mathfrak{u}^i_v = \mathfrak{u}^1_{\mathfrak{m}} - \mathfrak{u}^2_{\mathfrak{r}}$ gives the MV for all $\mathfrak{t}_1 \in G$.
We denote the ordinary group by $[G, *]$ and $e \in [G, *]$ as an identity element

throughout the discussion.
In our study, instead of above symbol for an IG we use to denote it by $[\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}, [G,*]]$ where the actual value of $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ is given by $\mathfrak{u}_{\mathfrak{m}}^{1} - \mathfrak{u}_{\mathfrak{r}}^{2}$.

Example 5.2.

Let $G = \{1, -1, i, -i\}$ be the multiplicative group then we define a mapping $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} : G \to [0,1]$ by

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \begin{cases} (1,0) \;\; for \;\; \mathfrak{t}_1 = 1, -1 \\ (0.92, 0) \;\; for \;\; \mathfrak{t}_1 = i, -i \end{cases} \tag{5.1}$$

Where $\mathfrak{u}_v^i(1) = 1, \mathfrak{u}_v^i(-1) = 1, \mathfrak{u}_v^i(i) = 0.92, \mathfrak{u}_v^i(-i) = 0.92$
Then

(i) For $\mathfrak{t}_1 = 1, \mathfrak{t}_2 = -1$

$$\begin{aligned}\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(1.-1) &= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(-1) \\ &= (1,0) \\ &= (min(\mathfrak{u}_{\mathfrak{m}}^{1}(1), \mathfrak{u}_{\mathfrak{m}}^{1}(-1)), max(\mathfrak{u}_{\mathfrak{r}}^{2}(1), \mathfrak{u}_{\mathfrak{r}}^{2}(-1))) \\ &= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(1) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(-1)\end{aligned}$$

Similarly, $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2) \geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2); \forall \;\; \mathfrak{t}_1, \mathfrak{t}_2 \in G$

(ii) For $\mathfrak{t}_1 = i$

$$\begin{aligned}\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(i^{-1}) &= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(-i) \\ &= (0.92, 0) \\ &= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(i)\end{aligned}$$

Similarly, it follows for $\mathfrak{t}_1 = -i, 1, -1$.
Therefore $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1{}^{-1}) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1); \forall \mathfrak{t}_1 \in G$

(iii) $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(e = 1) \geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1); \forall \mathfrak{t}_1 \in G$
Therefore $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ is an IG of $[G,*]$.

Definition 5.3. If $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ is an IG of a group $[G,*]$ then the inverse of IG $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ under multiplication operator is defined by ${\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}}^{-1}(\mathfrak{t}_1) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1{}^{-1})$. Where ${\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}}^{-1}$ is the inverse of $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ and $\mathfrak{t}_1{}^{-1}$ is the inverse of $\mathfrak{t}_1$.

Example 5.4.

Consider the IG of Example 5.2
Here we have,

$${\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}}^{-1}(\mathfrak{t}_1) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1{}^{-1}) = \begin{cases} (1,0) \;\; for \;\; \mathfrak{t}_1 = 1, -1 \\ (0.92, 0) \;\; for \;\; \mathfrak{t}_1 = i, -i \end{cases} \tag{5.2}$$

Therefore, in this particular example we get the same IG after taking the inverse of the IG considered in Example 5.2.

Definition 5.5. If $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ and $\mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}$ are IGs of a group $[G,*]$ where $\mathfrak{u}_{\mathfrak{m}}^{1}$ and $\mathfrak{u}_{\mathfrak{r}}^{2}$ are MF and RF of the IG $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ respectively; $\mathfrak{u}_{\mathfrak{m}}^{3}$ and $\mathfrak{u}_{\mathfrak{r}}^{4}$ are MF and RF of the IG $\mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}$ respectively; $\mathfrak{u}_v^i = \mathfrak{u}_{\mathfrak{m}}^{1} - \mathfrak{u}_{\mathfrak{r}}^{2}$ is the MV of $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ and $\mathfrak{u}_v^i = \mathfrak{u}_{\mathfrak{m}}^{3} - \mathfrak{u}_{\mathfrak{r}}^{4}$ is the MV of $\mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}$ then their product is defined as

$(\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}})(\mathfrak{t}) = \{(\vee(\mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{m}}^{3}(\mathfrak{t}_2)), \vee(\mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_1) \vee \mathfrak{u}_{\mathfrak{r}}^{4}(\mathfrak{t}_2)))|\mathfrak{t}_1, \mathfrak{t}_2 \in G, \mathfrak{t}_1\mathfrak{t}_2 = \mathfrak{t} \in G\}$.
Where $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}$ is the product of two IG $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ and $\mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}$.
And, $(\vee(\mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{m}}^{3}(\mathfrak{t}_2)), \vee(\mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_1) \vee \mathfrak{u}_{\mathfrak{r}}^{4}(\mathfrak{t}_2))) = (max(min(\mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_1), \mathfrak{u}_{\mathfrak{m}}^{3}(\mathfrak{t}_2))),$
$max(max(\mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_1), \mathfrak{u}_{\mathfrak{r}}^{4}(\mathfrak{t}_2))))$.

Example 5.6.

Let us define two IG $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ and $\mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}$ over a multiplicative group $G = \{1, \omega, \omega^2\}$ by:

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \begin{cases} (0.91, 0) & for \ \ \mathfrak{t}_1 = 1 \\ (0.81, 0) & for \ \ \mathfrak{t}_1 = \omega, \omega^2 \end{cases} \tag{5.3}$$

$$\mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2) = \begin{cases} (0.63, 0) & for \ \ \mathfrak{t}_2 = 1, \omega, \omega^2 \end{cases} \tag{5.4}$$

Then their product is $(\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}})(\mathfrak{t}) = \{(\vee(\mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{m}}^{3}(\mathfrak{t}_2)), \vee(\mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_1) \vee \mathfrak{u}_{\mathfrak{r}}^{4}(\mathfrak{t}_2)))|\mathfrak{t}_1, \mathfrak{t}_2 \in G, \mathfrak{t}_1\mathfrak{t}_2 = \mathfrak{t} \in G\}$
Therefore, we get

$$(\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}})(\mathfrak{t}) = \begin{cases} (0.63, 0) & for \ \ \mathfrak{t}_1 = 1 \\ (0.63, 0) & for \ \ \mathfrak{t}_1 = \omega \\ (0.63, 0) & for \ \ \mathfrak{t}_1 = \omega^2 \end{cases} \tag{5.5}$$

which is clearly again an IG over $[G, *]$.

Lemma 5.7. *Let* $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ *be an IG of* $[G, *]$. *Then* $\forall \mathfrak{t}_1 \in G$
(i) $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(e) \geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1)$
(ii) $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}({\mathfrak{t}_1}^{-1})$; ${\mathfrak{t}_1}^{-1}$ *is the inverse of* $\mathfrak{t}_1$.

6. Some Basic Properties of IG

Property 6.1. If $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ is an imprecise subgroup of a group $[G, *]$, then
$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_3)$
$\Rightarrow \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_3); \forall \mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3 \in G$
and $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2\mathfrak{t}_1) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_3\mathfrak{t}_1)$
$\Rightarrow \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_3); \forall \mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3 \in G$ for identity $e \in G$ and all other $\mathfrak{t}_1 \in G$.

Property 6.2. If $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ is an imprecise subgroup of a group $[G, *]$, and $({\mathfrak{t}_1}^{-1})^{-1} = \mathfrak{t}_1; \forall \mathfrak{t}_1 \in G$ then $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(({\mathfrak{t}_1}^{-1})^{-1}) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1)$; for all $\mathfrak{t}_1 \in G$ where ${\mathfrak{t}_1}^{-1}$ is the inverse of $\mathfrak{t}_1$.

Property 6.3. If $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ is an imprecise subgroup of a group $[G, *]$, and $(\mathfrak{t}_1\mathfrak{t}_2)^{-1} = {\mathfrak{t}_2}^{-1}{\mathfrak{t}_1}^{-1}; \forall \mathfrak{t}_1, \mathfrak{t}_2 \in G$ then $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}((\mathfrak{t}_1\mathfrak{t}_2)^{-1}) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}({\mathfrak{t}_2}^{-1}{\mathfrak{t}_1}^{-1})$; for all $\mathfrak{t}_1, \mathfrak{t}_2 \in G$ where ${\mathfrak{t}_1}^{-1}$ and ${\mathfrak{t}_2}^{-1}$ are the inverses of $\mathfrak{t}_1$ and $\mathfrak{t}_2$ respectively.
The minimum operator and maximum operator which we are using in the following proof are already explained in Definition 5.1.

Proposition 6.4. *Necessary and sufficient condition for an IG of a group* $[G, *]$ *to be an* IS_G *is that* $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1{\mathfrak{t}_2}^{-1}) \geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2); \forall \mathfrak{t}_1, \mathfrak{t}_2 \in G$.

Proof. Let $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ be an IG of $[G, *]$.
Then,

$$\begin{aligned} \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1}) &\geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2{}^{-1}) \\ &\geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2); \forall \mathfrak{t}_1, \mathfrak{t}_2 \in G \end{aligned}$$

Conversely,let

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1}) \geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2)$$

Then

$$\begin{aligned} &\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2\mathfrak{t}_2{}^{-1}) \geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2{}^{-1}); \\ \Rightarrow &\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(e) \geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2) \\ \Rightarrow &\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(e) \geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2); \forall \mathfrak{t}_2 \in G \qquad \text{(i)} \end{aligned}$$

Now,

$$\begin{aligned} \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(e\mathfrak{t}_2{}^{-1}) &\geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(e) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2{}^{-1}) \\ &\geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(e) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2) \\ &= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2) \\ \Rightarrow \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2{}^{-1}) &\geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2); \forall \mathfrak{t}_2 \in G \qquad \text{(ii)} \end{aligned}$$

And, $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2) \geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2); \mathfrak{t}_1, \mathfrak{t}_2 \in G$ (iii)
From (i), (ii) and (iii), $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ is an imprecise subgroup.
This is an extended definition of subgroup in general group theory □

Theorem 6.5. *Cancellation laws may not hold in an IS_G.*

Proof. Let us consider an imprecise subset M of all 2×2 imprecise matrices over integers under matrix multiplication, which forms an IS_G.
Let

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} = \begin{bmatrix} (0.21, 0) & (0, 0) \\ (0, 0) & (0, 0) \end{bmatrix}, \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}} = \begin{bmatrix} (0, 0) & (0, 0) \\ (0, 0) & (0.31, 0) \end{bmatrix} \text{ and } \mathfrak{w}_{\mathfrak{m}}^{\mathfrak{r}} = \begin{bmatrix} (0, 0) & (0, 0) \\ (0.41, 0) & (0, 0) \end{bmatrix}$$

Then,

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}} = \begin{bmatrix} (\vee\{\wedge(0.21, 0), \wedge(0, 0)\}, & \\ \quad \wedge\{\vee(0, 0), \vee(0, 0)\}) & (\vee\{\wedge(0.21, 0), \wedge(0, 0)\}, \wedge\{\vee(0, 0), \vee(0, 0)\}) \\ (\vee\{\wedge(0, 0), \wedge(0, 0)\}, & \\ \quad \wedge\{\vee(0, 0), \vee(0, 0)\}) & (\vee\{\wedge(0, 0), \wedge(0, 0.31)\}, \wedge\{\vee(0, 0), \vee(0, 0)\}) \end{bmatrix}$$

(The maximum $'\vee'$ and minimum $'\wedge'$ operators are already explained in Definition 5.5 with Example 5.6).

$$\begin{aligned} &= \begin{bmatrix} (0, 0) & (0, 0) \\ (0, 0) & (0, 0) \end{bmatrix} \\ &= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{w}_{\mathfrak{m}}^{\mathfrak{r}} \end{aligned}$$

But $\mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}} \neq \mathfrak{w}_{\mathfrak{m}}^{\mathfrak{r}}$ □

Theorem 6.6. *Let $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ be an IS_G of a group $[G, *]$. Then*

$$\begin{aligned} &\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1}) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(e) \\ \Rightarrow &\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2) \textit{ for any } \mathfrak{t}_1, \mathfrak{t}_2 \in G \textit{ and } \mathfrak{t}_2{}^{-1} \textit{ is the inverse of } \mathfrak{t}_2. \end{aligned}$$

Proof. Let us consider $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1}) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(e)$ (*)
Then,

$$\begin{aligned} \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) &= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1 e) \\ &= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1}\mathfrak{t}_2) \end{aligned}$$

$$\geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1}) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2);\ \text{[Definition 5.1]}$$
$$= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(e) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2)$$
$$= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2)$$

Therefore $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) \geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2)$ (i)

Now, interchanging $\mathfrak{t}_1$ and $\mathfrak{t}_2$ in (*) we have

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2\mathfrak{t}_1{}^{-1}) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(e)$$

And,

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2) \geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) \quad \text{(ii)}$$

From (i) and (ii) $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2)$ □

Remark 6.7. But the converse of the above result is not true which is shown by Example 6.8

Example 6.8.

Let us consider a multiplicative group $G = \{1, \omega, \omega^2\}$ where ω is the cube root of unity.

Then the mapping $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} : G \longrightarrow [0, 1]$ defined by

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \begin{cases} (0.96, 0) & for \ \ \mathfrak{t}_1 = 1 \\ (0.66, 0) & for \ \ \mathfrak{t}_1 = \omega, \omega^2 \end{cases} \tag{6.1}$$

is an IS_G over G under multiplication using Proposition 6.4.
where $\mathfrak{u}_v^i(1) = 0.96, \mathfrak{u}_v^i(\omega) = 0.66, \mathfrak{u}_v^i(\omega^2) = 0.66$ are the MVs of the respective elements.

Now,

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\omega) = (0.66, 0)$$
$$= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\omega^2)$$

But,

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\omega^2(\omega)^{-1}) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\omega^2\omega^{-1})$$
$$= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\omega)$$
$$= (0.66, 0)$$
$$\neq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(1).$$

Theorem 6.9. *Let* $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ *be an* IS_G *of a group* $[G, *]$ *and let* $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) \leq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2); \forall \mathfrak{t}_1 \in G$ *and fixed* $\mathfrak{t}_2 \in G$ *then* $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2\mathfrak{t}_1)$.

Proof. Suppose $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) \leq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2)$ (*)

Then,

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2) \geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2)$$
$$\geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1);\ \text{[using *]}$$
$$= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1)$$

Therefore $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2) \geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1)$

Again, replacing, $\mathfrak{t}_1$ by $\mathfrak{t}_1\mathfrak{t}_2$ we have,

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2) \geq \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2) \quad (**)$$

Now,

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_2{}^{-1})$$

$$\geq \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2) \wedge \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}({\mathfrak{t}_2}^{-1});\ [\text{ Definition 5.1 }]$$
$$\geq \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2) \wedge \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2)$$
$$\geq \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2) \wedge \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2);\ [\text{using } **]$$
$$\geq \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2)$$

Therefore $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1) \geq \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2)$
Thus, $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1) = \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2)$
Similarly, it can be prove that $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2\mathfrak{t}_1) = \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1)$ □

Theorem 6.10. *Let $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}$ be an IS_G of a group $[G,*]$ and let $\mathfrak{t}_1 \in G$. Then*

$$\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2) = \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2); \forall \mathfrak{t}_2 \in G$$
$$\Leftrightarrow \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1) = \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(e).$$

Proof. Let $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2) = \{\mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2), \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2)\} = \{\mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_2), \mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_2)\} = \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2)$
Let,

$$\mathfrak{t}_2 = e$$

Then,

$$\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1.e) = \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(e)$$
$$\Rightarrow \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1) = \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(e)$$

Therefore $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1) = \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(e)$ (*)
Conversely, let $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1) = \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(e)$ for any $\mathfrak{t}_1 \in G$
Then

$$\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2) \geq \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1) \wedge \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2);\ [\text{ Definition 5.1 }]$$
$$= \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(e) \wedge \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2);\ [\text{using } *]$$
$$= \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2); \forall \mathfrak{t}_2 \in G$$

Therefore $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2) \geq \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2); \forall \mathfrak{t}_2 \in G$
And,

$$\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2) = \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2 e)$$
$$\geq \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2) \wedge \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(e)$$
$$= \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2) \wedge \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1)$$
$$= \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2)$$

Therefore $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2) \geq \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2)$
Thus $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2) = \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}(\mathfrak{t}_2)$ □

Theorem 6.11. *Product of two IS_G is again an imprecise subgroup.*

Proof. Let $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}}$ and $\mathfrak{v}^{\mathfrak{r}}_{\mathfrak{m}}$ be two IS_G of $[G,*]$
Let $[\mathfrak{t}_1 = \mathfrak{t}_{1_1}\mathfrak{t}_{2_1}, \mathfrak{t}_2 = \mathfrak{t}_{1_2}\mathfrak{t}_{2_2}]$, then

$$[\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}} \circ \mathfrak{v}^{\mathfrak{r}}_{\mathfrak{m}}](\mathfrak{t}_1) = \{\vee(\mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_{1_1}) \wedge \mathfrak{u}^3_{\mathfrak{m}}(\mathfrak{t}_{2_1})), \vee(\mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_{1_1}) \vee \mathfrak{u}^4_{\mathfrak{r}}(\mathfrak{t}_{2_1}))\}$$
$$[\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}} \circ \mathfrak{v}^{\mathfrak{r}}_{\mathfrak{m}}](\mathfrak{t}_2) = \{\vee(\mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_{1_2}) \wedge \mathfrak{u}^3_{\mathfrak{m}}(\mathfrak{t}_{2_2})), \vee(\mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_{1_2}) \vee \mathfrak{u}^4_{\mathfrak{r}}(\mathfrak{t}_{2_2}))\}$$

(For the product of two IGs are already discussed in Definition 5.5 with Example 5.6)
Now by Proposition 6.4,

$$[\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}} \circ \mathfrak{v}^{\mathfrak{r}}_{\mathfrak{m}}](\mathfrak{t}_1{\mathfrak{t}_2}^{-1}) \geq [\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}} \circ \mathfrak{v}^{\mathfrak{r}}_{\mathfrak{m}}](\mathfrak{t}_1\mathfrak{t}_2)$$
$$= \{\vee(\mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_1) \wedge \mathfrak{u}^3_{\mathfrak{m}}(\mathfrak{t}_2)), \vee(\mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_1) \vee \mathfrak{u}^4_{\mathfrak{r}}(\mathfrak{t}_2))\}$$
$$= \{\vee(\mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_{1_1}\mathfrak{t}_{2_1}) \wedge \mathfrak{u}^3_{\mathfrak{m}}(\mathfrak{t}_{1_2}t_{2_2})), \vee(\mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_{1_1}\mathfrak{t}_{2_1}) \vee \mathfrak{u}^4_{\mathfrak{r}}(\mathfrak{t}_{1_2}\mathfrak{t}_{2_2}))\}$$
$$\geq \{\vee(\mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_{1_1}) \wedge \mathfrak{u}^3_{\mathfrak{m}}(\mathfrak{t}_{2_1})) \wedge (\vee(\mathfrak{u}^1_{\mathfrak{m}}(\mathfrak{t}_{1_2}) \wedge \mathfrak{u}^3_{\mathfrak{m}}(\mathfrak{t}_{2_2}))), \vee(\mathfrak{u}^2_{\mathfrak{r}}(\mathfrak{t}_{1_1}) \vee$$

$$\begin{aligned}
&\mathfrak{u}_{\mathfrak{r}}^{4}(\mathfrak{t}_{2_1})) \vee (\vee(\mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_{1_2}) \vee \mathfrak{u}_{\mathfrak{r}}^{4}(\mathfrak{t}_{2_2}))))\} \\
&\geq [\vee\{(\mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_{1_1}) \wedge \mathfrak{u}_{\mathfrak{m}}^{3}(\mathfrak{t}_{2_1})), \vee(\mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_{1_1}) \vee \mathfrak{u}_{\mathfrak{r}}^{4}(\mathfrak{t}_{2_1}))\} \wedge \{\vee(\mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_{1_2}) \wedge \\
&\quad \mathfrak{u}_{\mathfrak{m}}^{3}(\mathfrak{t}_{2_2})), \vee(\mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_{1_2}) \vee \mathfrak{u}_{\mathfrak{r}}^{4}(\mathfrak{t}_{2_2}))\}] \\
&= [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_{1_1}\mathfrak{t}_{2_1}) \wedge [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_{1_2}\mathfrak{t}_{2_2}) \\
&= [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_1) \wedge [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_2)
\end{aligned}$$

Therefore $[\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_1\mathfrak{t}_2^{-1}) \geq [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_1) \wedge [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_2)$

Thus the product of two IS_G is again an IS_G □

Proposition 6.12. *Let $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ be an IS_G of $[G, *]$. Then $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ is an IS_G of $[G, *]$ iff $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$*
i.e., $(\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}})^2 = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ and $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1^{-1}); \forall \mathfrak{t}_1 \in G$ where $\mathfrak{t}_1^{-1}$ is the inverse of $\mathfrak{t}_1$.

Proof. If $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ is an IS_G of $[G, *]$ then clearly $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$i.e., $(\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}})^2 = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ and $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1^{-1}); \forall \mathfrak{t}_1 \in G$

Conversely, let $(\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}})^2 = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ and $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1^{-1}); \forall \mathfrak{t}_1 \in G$

Now, let $\mathfrak{t}_1, \mathfrak{t}_2 \in G$ then $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2) = (\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}})^2(\mathfrak{t}_1\mathfrak{t}_2)$

Therefore
$$\begin{aligned}
\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2^{-1}) &= (\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}})^2(\mathfrak{t}_1\mathfrak{t}_2^{-1}) \\
&= [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_1\mathfrak{t}_2^{-1}) \\
&\geq [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_1\mathfrak{t}_2) \\
&= \{\vee(\mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_2)), \vee(\mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_1) \vee \mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_2))\}; \text{[Definition 5.5]} \\
&= \{\vee(\mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_{1_1}\mathfrak{t}_{2_1}) \wedge \mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_{1_2}\mathfrak{t}_{2_2})), \vee(\mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_{1_1}t_{2_1}) \vee \mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_{1_2}\mathfrak{t}_{2_2}))\} \\
&\geq \{\vee(\mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_{1_1}) \wedge \mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_{2_1})) \wedge (\vee(\mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_{1_2}) \wedge \mathfrak{u}_{\mathfrak{m}}^{3}(\mathfrak{t}_{2_2}))), \vee(\mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_{1_1}) \vee \\
&\quad \mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_{2_1})) \vee (\vee(\mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_{1_2}) \vee \mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_{2_2}))))\} \\
&\geq [\vee\{(\mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_{1_1}) \wedge \mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_{2_2})), \vee(\mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_{1_1}) \vee \mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_{2_1}))\} \wedge \{\vee(\mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_{1_2}) \wedge \\
&\quad \mathfrak{u}_{\mathfrak{m}}^{1}(\mathfrak{t}_{2_2})), \vee(\mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_{1_2}) \vee \mathfrak{u}_{\mathfrak{r}}^{2}(\mathfrak{t}_{2_2}))\}] \\
&= [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_{1_1}\mathfrak{t}_{2_1}) \wedge [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_{1_2}\mathfrak{t}_{2_2}) \\
&= [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_1) \wedge [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_2) \\
&= {\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}}^2(\mathfrak{t}_1) \wedge {\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}}^2(\mathfrak{t}_2) \\
&= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) \wedge \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2)
\end{aligned}$$

Therefore by Proposition 6.4, $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ is an IS_G □

Theorem 6.13. *Let $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ and $\mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}$ be two IS_Gs such that $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}} = \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$. Then $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}$ is an IS_G of $[G, *]$.*

Proof.
$$\begin{aligned}
\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}} &= {\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}}^2 \circ {\mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}}^2 \text{ [Proposition 6.12]} \\
&= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}] \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}} \\
&= \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ [\mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}] \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}} \\
&= [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}] \circ [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}] \\
&= [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}]^2
\end{aligned}$$

Therefore by Proposition 6.12, $[\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \circ \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}]$ is an IS_G of $[G, *]$ □

Theorem 6.14. *If $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ and $\mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}$ be two IS_Gs of a group $[G, *]$. Then thier intersection $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \cap \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}$ is also an IS_G of $[G, *]$.*

Proof. To show:
$[\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \cap \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_1\mathfrak{t}_2^{-1}) \geq [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \cap \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_1) \wedge [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \cap \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_2^{-1})$

$$\geq [\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}} \cap \mathfrak{v}^{\mathfrak{r}}_{\mathfrak{m}}](\mathfrak{t}_1) \wedge [\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}} \cap \mathfrak{v}^{\mathfrak{r}}_{\mathfrak{m}}](\mathfrak{t}_2)$$

$$[\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}} \cap \mathfrak{v}^{\mathfrak{r}}_{\mathfrak{m}}](\mathfrak{t}_1\mathfrak{t}_2{}^{-1}) = (\mathfrak{u}^{1}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1}) \wedge \mathfrak{u}^{3}_{\mathfrak{m}}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1}), \mathfrak{u}^{2}_{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1}) \vee \mathfrak{u}^{4}_{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1})) \text{ [Definition 4.4]}$$

$$\geq ((\mathfrak{u}^{1}_{\mathfrak{m}}(\mathfrak{t}_1) \wedge \mathfrak{u}^{1}_{\mathfrak{m}}(\mathfrak{t}_2)) \wedge (\mathfrak{u}^{3}_{\mathfrak{m}}(\mathfrak{t}_1) \wedge \mathfrak{u}^{3}_{\mathfrak{m}}(\mathfrak{t}_2)), (\mathfrak{u}^{2}_{\mathfrak{r}}(\mathfrak{t}_1) \vee \mathfrak{u}^{2}_{\mathfrak{r}}(\mathfrak{t}_2)) \vee (\mathfrak{u}^{4}_{\mathfrak{r}}(\mathfrak{t}_1) \vee \mathfrak{u}^{4}_{\mathfrak{r}}(\mathfrak{t}_2)))$$

$$\geq ((\mathfrak{u}^{1}_{\mathfrak{m}}(\mathfrak{t}_1) \wedge \mathfrak{u}^{3}_{\mathfrak{m}}(\mathfrak{t}_1)) \wedge (\mathfrak{u}^{1}_{\mathfrak{m}}(\mathfrak{t}_2) \wedge \mathfrak{u}^{3}_{\mathfrak{m}}(\mathfrak{t}_2)), (\mathfrak{u}^{2}_{\mathfrak{r}}(\mathfrak{t}_1) \vee \mathfrak{u}^{4}_{\mathfrak{r}}(\mathfrak{t}_1)) \vee (\mathfrak{u}^{2}_{\mathfrak{r}}(\mathfrak{t}_2) \vee \mathfrak{u}^{4}_{\mathfrak{r}}(\mathfrak{t}_2)))$$

$$= ((\mathfrak{u}^{1}_{\mathfrak{m}}(\mathfrak{t}_1) \wedge \mathfrak{u}^{3}_{\mathfrak{m}}(\mathfrak{t}_1)), (\mathfrak{u}^{2}_{\mathfrak{r}}(\mathfrak{t}_1) \vee \mathfrak{u}^{4}_{\mathfrak{r}}(\mathfrak{t}_1))) \wedge ((\mathfrak{u}^{1}_{\mathfrak{m}}(\mathfrak{t}_2) \wedge \mathfrak{u}^{3}_{\mathfrak{m}}(\mathfrak{t}_2)), (\mathfrak{u}^{2}_{\mathfrak{r}}(\mathfrak{t}_2) \vee \mathfrak{u}^{4}_{\mathfrak{r}}(\mathfrak{t}_2)))$$

$$= [\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}} \cap \mathfrak{v}^{\mathfrak{r}}_{\mathfrak{m}}](\mathfrak{t}_1) \wedge [\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}} \cap \mathfrak{v}^{\mathfrak{r}}_{\mathfrak{m}}](\mathfrak{t}_2)$$

Therefore by Proposition 6.4, $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}} \cap \mathfrak{v}^{\mathfrak{r}}_{\mathfrak{m}}$ is an IS_G □

Theorem 6.15. *The intersection of any collection of imprecise subgroups is itself an imprecise subgroup.*

Proof. Let G be a group and let $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}1}, \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}2}, \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}3}, \cdots$ be any collection of normal imprecise subgroup of G

Let $\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}1} \cap \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}2} \cap \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}3} \cap \cdots \cap \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}i} = \bigcap_{i=n, n\in\mathbb{N}} \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}i}$

To show, $\bigcap_{i=n, n\in\mathbb{N}} \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}i}$ is an imprecise subgroup of G.

$$\bigcap_{i=n, n\in\mathbb{N}} \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}i}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1}) = [\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}1} \cap \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}2} \cap \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}3} \cap \cdots \cap \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}i}](\mathfrak{t}_1\mathfrak{t}_2{}^{-1})$$

$$= \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}1}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1}) \cap \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}2}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1}) \cap \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}3}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1}) \cap \cdots \cap \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}i}(\mathfrak{t}_1\mathfrak{t}_2{}^{-1})$$

$$\geq \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}1}(\mathfrak{t}_1\mathfrak{t}_2) \cap \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}2}(\mathfrak{t}_1\mathfrak{t}_2) \cap \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}3}(\mathfrak{t}_1\mathfrak{t}_2) \cap \cdots \cap \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}i}(\mathfrak{t}_1\mathfrak{t}_2)$$

$$\geq (\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}1}(\mathfrak{t}_1) \wedge \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}1}(\mathfrak{t}_2)) \cap (\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}2}(\mathfrak{t}_1) \wedge \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}2}(\mathfrak{t}_2)) \cap (\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}3}(\mathfrak{t}_1) \wedge \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}3}(\mathfrak{t}_2)) \cap \cdots \cap (\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}i}(\mathfrak{t}_1) \wedge \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}i}(\mathfrak{t}_2))$$

$$= \bigcap_{i=n, n\in\mathbb{N}} (\mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}i}(\mathfrak{t}_1) \wedge \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}i}(\mathfrak{t}_2))$$

Hence $\bigcap_{i=n, n\in\mathbb{N}} \mathfrak{u}^{\mathfrak{r}}_{\mathfrak{m}i}$ is an imprecise subgroup of G □

Theorem 6.16. *Union of two IG is not necessarily an IG.*

Example 6.17.

Let us consider a Klein 4-group G

TABLE 1

*	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

Let us define imprecise subgroups on G by

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \begin{cases} (0.9, 0) & for \;\; \mathfrak{t}_1 = e, ab \\ (0.6, 0) & for \;\; \mathfrak{t}_1 = a, b \end{cases} \tag{6.2}$$

where $\mathfrak{u}_v^i(e) = 0.9, \mathfrak{u}_v^i(ab) = 0.9, \mathfrak{u}_v^i(a) = 0.6, \mathfrak{u}_v^i(b) = 0.6$ are the MVs of the respective elements.

$$\mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \begin{cases} (1, 0) & for \;\; \mathfrak{t}_1 = e, ab \\ (0.71, 0) & for \;\; \mathfrak{t}_1 = a \\ (0.6, 0) & for \;\; \mathfrak{t}_1 = b \end{cases} \tag{6.3}$$

where $\mathfrak{u}_v^i(e) = 1, \mathfrak{u}_v^i(ab) = 1, \mathfrak{u}_v^i(a) = 0.71, \mathfrak{u}_v^i(b) = 0.6$ are the MVs of the respective elements.
Clearly $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ and $\mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}$ are the IS_Gs.
Then by using the Definition 4.4, the union of $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ and $\mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}$ is

$$[\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \cup \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](\mathfrak{t}_1) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} = \begin{cases} (1, 0) & for \;\; \mathfrak{t}_1 = e, ab \\ (0.71, 0) & for \;\; \mathfrak{t}_1 = a \\ (0.6, 0) & for \;\; \mathfrak{t}_1 = b \end{cases} \tag{6.4}$$

where $\mathfrak{u}_v^i(e) = 1, \mathfrak{u}_v^i(ab) = 1, \mathfrak{u}_v^i(a) = 0.71, \mathfrak{u}_v^i(b) = 0.6$ are the MVs of the respective elements.
But,

$$\begin{aligned} [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \cup \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](a.ab) &= [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \cup \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](a^2b) \\ &= [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \cup \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](b) \\ &= (0.6, 0) \end{aligned}$$

And,

$$\begin{aligned} [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \cup \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](a) \wedge [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \cup \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](ab) &= (0.71, 0) \wedge (1, 0) \\ &= (0.71 \wedge 1, 0 \vee 0) \\ &= (0.71, 0); \text{ } (min \text{ operator } '\wedge' \text{ is} \\ &\text{already discussed in Definition 5.1)} \end{aligned}$$

Therefore $[\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \cup \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](a.ab) \ngeq [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \cup \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](a) \wedge [\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}} \cup \mathfrak{v}_{\mathfrak{m}}^{\mathfrak{r}}](ab)$.

7. Abelian IG

Definition 7.1. An IG $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ of a group $[G, *]$ is said to be an abelian imprecise group if $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2\mathfrak{t}_1); \forall \mathfrak{t}_1, \mathfrak{t}_2 \in G$.

Remark 7.2. If G is an abelian group, then every imprecise subgroup $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ of G is an imprecise abelian subgroup of G, but the converse may not be true.

Example 7.3.

Let $G = \{e, a, b, c\}$ be an abelian group under multiplication.

TABLE 2

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Then we define a mapping on G by

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \begin{cases} (0.9, 0) & for \quad \mathfrak{t}_1 = e, a \\ (0.8, 0) & for \quad \mathfrak{t}_1 = b, c \end{cases} \tag{7.1}$$

such that $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ is an imprecise subgroup by Proposition 6.4.
Now,

$$\begin{aligned}
\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(ea) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(a) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(ae)\\
\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(eb) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(b) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(be)\\
\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(ec) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(c) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(ce)\\
\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(ab) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(c) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(ba)\\
\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(ac) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(b) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(ca)\\
\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(bc) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(a) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(cb)
\end{aligned}$$

Clearly, $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ is an abelian imprecise subgroup of G.
For the converse part, let us consider the Example 7.4:

Example 7.4.

We know $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with respect to multiplication is a group where $i \cdot j = k, j \cdot k = i, k \cdot i = j, i^2 = j^2 = k^2 = -1$.
Let $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}$ be an IG of $[G, *]$ defined by

$$\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1) = \begin{cases} (1, 0) & for \quad \mathfrak{t}_1 = 1 \\ (0.71, 0) & for \quad \mathfrak{t}_1 = -1 \\ (0.6, 0) & for \quad \mathfrak{t}_1 = \pm i, \pm j, \pm k \end{cases} \tag{7.2}$$

where $\mathfrak{u}_v^i(1) = 1, \mathfrak{u}_v^i(-1) = 0.71, \mathfrak{u}_v^i(\pm i) = 0.6, \mathfrak{u}_v^i(\pm j) = 0.6, \mathfrak{u}_v^i(\pm k) = 0.6$
Then clearly $[\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}, Q_8]$ is an IG if we proceed as in Example 5.2
Also, $\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_1\mathfrak{t}_2) = \mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}(\mathfrak{t}_2\mathfrak{t}_1); \forall \mathfrak{t}_1, \mathfrak{t}_2 \in Q_8$
Therefore $[\mathfrak{u}_{\mathfrak{m}}^{\mathfrak{r}}, Q_8]$ is an abelian IG but Q_8 is itself not an abelian group.

8. Conclusion

We presented the IS_G in our study based on the IS criteria. It is discovered that many of the IS_G properties, which are analogous of the ordinary group, can be discussed in the present group. These properties are supported by specific examples. There are, however, a great deal of theories and properties that need to be looked into for the study of IS_G. Further, we will apply this new concept in defining normal imprecise cosets, cyclic imprecise subgroup and study their behavior in imprecise form. Also, we shall include the study of anti IS_G and their properties in our future work.

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