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VARIOUS DENSES INDUCED BY BI-PARTIALLY ORDERED SETS†

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Abstract. We introduce the concepts of right join-dense, right meet-dense, left join-dense and left meet-dense induced by bi-partially ordered sets on complete generalized residuated lattices. We investigate properties of these concepts and give an example related to them.

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1. Introduction

In this paper, we propose the concepts that characterize bi-partially ordered sets on complete generalized residuated lattices, namely right join-dense, right meet-dense, left join-dense, and left meet-dense. We conduct a comprehensive investigation into the properties of these concepts, and provide a relevant example related to them. The purpose of the paper is to contribute to the understanding of the structural and behavioral characteristics of bi-partially ordered sets on complete generalized residuated lattices through the exploration of these concepts.

2. Preliminaries

In this section, we present some preliminary concepts and properties.

Definition 2.1. [2, 6, 7, 8, 9] A structure $(L, \vee, \wedge, \odot, \rightarrow, \Rightarrow, \perp, \top)$ is called a *generalized residuated lattice* if it satisfies the following three conditions:
(GR1) $(L, \vee, \wedge, \top, \perp)$ is bounded where $\top$ is the upper bound and $\perp$ is the universal lower bound,

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(GR2) $(L, \odot, \top)$ is a monoid where $\top$ is the identity,
(GR3) it satisfies a residuation ; i.e., $a \odot b \leq c$ if and only if $a \leq b \to c$ if and only if $b \leq a \Rightarrow c$.

In this paper, we always assume that $(L, \wedge, \vee, \odot, \to, \Rightarrow, \top, \bot)$ is a complete generalized residuated lattice.

Lemma 2.2. [1, 4, 5, 6, 8] *Let $x, y, z \in L$. Let $\{x_i\}_{i\in\Gamma}, \{y_i\}_{i\in\Gamma} \subseteq L$. Then the following hold.*
(1) *If $y \leq z$, then $x \odot y \leq x \odot z$, $x \to y \leq x \to z$, $z \to x \leq y \to x$, $x \Rightarrow y \leq x \Rightarrow z$ and $z \Rightarrow x \leq y \Rightarrow x$.*
(2)

$$\begin{array}{ll} x \to \left(\bigwedge_{i\in\Gamma} y_i\right) & = \bigwedge_{i\in\Gamma} (x \to y_i), \left(\bigvee_{i\in\Gamma} x_i\right) \to y = \bigwedge_{i\in\Gamma}(x_i \to y), \\ \left(\bigvee_{i\in\Gamma} x_i\right) \to \left(\bigvee_{i\in\Gamma} y_i\right) & \geq \bigwedge_{i\in\Gamma} (x_i \to y_i), \left(\bigwedge_{i\in\Gamma} x_i\right) \to \left(\bigwedge_{i\in\Gamma} y_i\right) \geq \bigwedge_{i\in\Gamma} (x_i \to y_i), \\ x \Rightarrow \left(\bigwedge_{i\in\Gamma} y_i\right) & = \bigwedge_{i\in\Gamma} (x \Rightarrow y_i), \left(\bigvee_{i\in\Gamma} x_i\right) \Rightarrow y = \bigwedge_{i\in\Gamma}(x_i \Rightarrow y), \\ \left(\bigvee_{i\in\Gamma} x_i\right) \Rightarrow \left(\bigvee_{i\in\Gamma} y_i\right) & \geq \bigwedge_{i\in\Gamma} (x_i \Rightarrow y_i), \left(\bigwedge_{i\in\Gamma} x_i\right) \Rightarrow \left(\bigwedge_{i\in\Gamma} y_i\right) \geq \bigwedge_{i\in\Gamma} (x_i \Rightarrow y_i). \end{array}$$

(3) $(x \odot y) \to z = x \to (y \to z)$ *and* $(x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$.

Definition 2.3. [5, 6] Let X be a set. A map $e^r_X : X \times X \to L$ is called an *r-partial order* (or *right-partial order*) if it satisfies the following three conditions :
(O1) $e^r_X(x, x) = \top$ for all $x \in X$,
(O2) If $e^r_X(x, y) = e^r_X(y, x) = \top$ where $x, y \in X$, then $x = y$,
(R) $e^r_X(x, y) \odot e^r_X(y, z) \leq e^r_X(x, z)$ for all $x, y, z \in X$.

A map $e^l_X : X \times X \to L$ is called an *l-partial order* (or *left partial order*) if it satisfies the following three conditions :
(O1) $e^l_X(x, x) = \top$ for all $x \in X$,
(O2) If $e^l_X(x, y) = e^l_X(y, x) = \top$ where $x, y \in X$, then $x = y$,
(L) $e^l_X(y, z) \odot e^l_X(x, y) \leq e^l_X(x, z)$ for all $x, y, z \in X$.

The triple $\left(X, e^r_X, e^l_X\right)$ is called a *bi-partially ordered set.*

Definition 2.4. [6] Let $\left(X, e^r_X, e^l_X\right)$ be a bi-partially ordered set. Let $A \in L^X$.
(1) A point x_0 is called an *r-join* (or *right-join*) of A, denoted by $x_0 = \sqcup_r A$, if it satisfies
(RJ1) $A(x) \leq e^r_X(x, x_0)$ for all $x \in X$,
(RJ2) $\bigwedge_{x\in X} [A(x) \Rightarrow e^r_X(x, y)] \leq e^r_X(x_0, y)$ for all $y \in X$.
(2) A point x_1 is called an *r-meet* (or *right-meet*) of A, denoted by $x_1 = \sqcap_r A$, if it satisfies
(RM1) $A(x) \leq e^r_X(x_1, x)$ for all $x \in X$,
(RM2) $\bigwedge_{x\in X} [A(x) \to e^r_X(y, x)] \leq e^r_X(y, x_1)$ for all $y \in X$.
(3) A point x_0 is called an *l-join* (or *left-join*) of A, denoted by $x_0 = \sqcup_l A$, if it satisfies
(LJ1) $A(x) \leq e^l_X(x, x_0)$ for all $x \in X$,
(LJ2) $\bigwedge_{x\in X} \left[A(x) \to e^l_X(x, y)\right] \leq e^l_X(x_0, y)$ for all $y \in X$.

(4) A point x_1 is called an *l-meet* (or *left-meet*) of A, denoted by $x_1 = \sqcap_l A$, if it satisfies
(LM1) $A(x) \leq e^l_X(x_1, x)$ for all $x \in X$,
(LM2) $\bigwedge_{x\in X}\left[A(x) \Rightarrow e^l_X(y,x)\right] \leq e^l_X(y, x_1)$ for all $y \in X$.
(5) X is *r-join complete* (resp. *r-meet complete*) if there exists $\sqcup_r A$ (resp. $\sqcap_r A$) for all $A \in L^X$.
(6) X is *l-join complete* (resp. *l-meet complete*) if there exists $\sqcup_l A$ (resp. $\sqcap_l A$) for all $A \in L^X$.

Lemma 2.5. [6] *Let $\left(X, e^r_X, e^l_X\right)$ be a bi-partially ordered set. Let $x_0, x_1 \in X$. Let $A \in L^X$. Then the following hold.*
(1) $x_0 = \sqcup_r A$ if and only if $\bigwedge_{x\in X}\left[A(x) \Rightarrow e^r_X(x,y)\right] = e^r_X(x_0, y)$ for all $y \in X$.
(2) $x_1 = \sqcap_r A$ if and only if $\bigwedge_{x\in X}\left[A(x) \rightarrow e^r_X(y,x)\right] = e^r_X(y, x_1)$ for all $y \in X$.
(3) $x_0 = \sqcup_l A$ if and only if $\bigwedge_{x\in X}\left[A(x) \rightarrow e^l_X(x,y)\right] = e^l_X(x_0, y)$ for all $y \in X$.
(4) $x_1 = \sqcap_l A$ if and only if $\bigwedge_{x\in X}\left[A(x) \Rightarrow e^l_X(y,x)\right] = e^l_X(y, x_1)$ for all $y \in X$.
(5) $\sqcup_r A$, $\sqcap_r A$, $\sqcup_l A$ and $\sqcap_l A$ are unique if each exists.

3. Various denses on generalized residuated lattices

Definition 3.1. [6] Let $\left(X, e^r_X, e^l_X\right)$ and $\left(Y, e^r_Y, e^l_Y\right)$ be bi-partially ordered sets. Let $f : X \to Y$ be a map. Define four maps $f^{\rightarrow}_{r*}$, $f^{*\rightarrow}_{r}$, $f^{\rightarrow}_{l*}$, $f^{*\rightarrow}_{l} : L^X \to L^Y$ by

$$\begin{aligned}
f^{\rightarrow}_{r*}(A)(y) &= \bigvee_{x\in X}\left[e^r_Y(y, f(x)) \odot A(x)\right],\\
f^{*\rightarrow}_{r}(A)(y) &= \bigvee_{x\in X}\left[A(x) \odot e^r_Y(f(x), y)\right],\\
f^{\rightarrow}_{l*}(A)(y) &= \bigvee_{x\in X}\left[A(x) \odot e^l_Y(y, f(x))\right],\\
f^{*\rightarrow}_{l}(A)(y) &= \bigvee_{x\in X}\left[e^l_Y(f(x), y) \odot A(x)\right]
\end{aligned}$$

where $A \in L^X$.

Definition 3.2. Let $\left(X, e^r_X, e^l_X\right)$ be a bi-partially ordered set. Let $Y \subseteq X$. Let $i : Y \to X$ be the inclusion map.
(1) Y is called an *r-join-dense* (or *right join-dense*) in X if for all $x \in X$, there exists $A \in L^Y$ such that $x = \bigsqcup_r i^{\rightarrow}_{r*}(A)$.
(2) Y is called an *r-meet-dense* (or *right meet-dense*) in X if for all $x \in X$, there exists $A \in L^Y$ such that $x = \sqcap_r i^{*\rightarrow}_{r}(A)$.
(3) Y is called an *l-join-dense* (or *left join-dense*) in X if for all $x \in X$, there exists $A \in L^Y$ such that $x = \bigsqcup_l i^{\rightarrow}_{l*}(A)$.
(4) Y is called an *l-meet-dense* (or *left meet-dense*) in X if for all $x \in X$, there exists $A \in L^Y$ such that $x = \sqcap_l i^{*\rightarrow}_{l}(A)$.

Lemma 3.3. *Let $\left(X, e^r_X, e^l_X\right)$ be a bi-partially ordered set. Then the following hold.*
(1) $\bigwedge_{x\in X}\left[e^r_X(x,y) \Rightarrow e^r_X(x,z)\right] = e^r_X(y,z)$ for all $y, z \in X$.
(2) $\bigwedge_{x\in X}\left[e^r_X(y,x) \rightarrow e^r_X(z,x)\right] = e^r_X(z,y)$ for all $y, z \in X$.
(3) $\bigwedge_{x\in X}\left[e^l_X(x,y) \rightarrow e^l_X(x,z)\right] = e^l_X(y,z)$ for all $y, z \in X$.
(4) $\bigwedge_{x\in X}\left[e^l_X(y,x) \Rightarrow e^l_X(z,x)\right] = e^l_X(z,y)$ for all $y, z \in X$.

Proof. (1) Note that $\bigwedge_{x\in X}[e^r_X(x,y)\Rightarrow e^r_X(x,z)]\le e^r_X(y,y)\Rightarrow e^r_X(y,z)=e^r_X(y,z)$. On the other hand, since $e^r_X(x,y)\odot e^r_X(y,z)\le e^r_X(x,z)$ for all $x\in X$, we have by residuation that $e^r_X(y,z)\le e^r_X(x,y)\Rightarrow e^r_X(x,z)$ for all $x\in X$, which implies that

$$e^r_X(y,z)\le\bigwedge_{x\in X}[e^r_X(x,y)\Rightarrow e^r_X(x,z)].$$

(2) Note that $\bigwedge_{x\in X}[e^r_X(y,x)\to e^r_X(z,x)]\le e^r_X(y,y)\to e^r_X(z,y)=e^r_X(z,y)$. On the other hand, since $e^r_X(z,y)\odot e^r_X(y,x)\le e^r_X(z,x)$ for all $x\in X$, we have by residuation that $e^r_X(z,y)\le e^r_X(y,x)\to e^r_X(z,x)$ for all $x\in X$, which implies that

$$e^r_X(z,y)\le\bigwedge_{x\in X}[e^r_X(y,x)\to e^r_X(z,x)].$$

(3) Note that $\bigwedge_{x\in X}\left[e^l_X(x,y)\to e^l_X(x,z)\right]\le e^l_X(y,y)\to e^l_X(y,z)=e^l_X(y,z)$. On the other hand, since $e^l_X(y,z)\odot e^l_X(x,y)\le e^l_X(x,z)$ for all $x\in X$, we have by residuation that $e^l_X(y,z)\le e^l_X(x,y)\to e^l_X(x,z)$ for all $x\in X$, which implies that

$$e^l_X(y,z)\le\bigwedge_{x\in X}\left[e^l_X(x,y)\to e^l_X(x,z)\right].$$

(4) Note that $\bigwedge_{x\in X}\left[e^l_X(y,x)\Rightarrow e^l_X(z,x)\right]\le e^l_X(y,y)\Rightarrow e^l_X(z,y)=e^l_X(z,y)$. On the other hand, since $e^l_X(y,x)\odot e^l_X(z,y)\le e^l_X(z,x)$ for all $x\in X$, we have by residuation that $e^l_X(z,y)\le e^l_X(y,x)\Rightarrow e^l_X(z,x)$ for all $x\in X$, which implies that

$$e^l_X(z,y)\le\bigwedge_{x\in X}\left[e^l_X(y,x)\Rightarrow e^l_X(z,x)\right].$$

□

Theorem 3.4. *Let $\left(X,e^r_X,e^l_X\right)$ be a bi-partially ordered set. Let $Y\subseteq X$. Let $A\in L^Y$. Then the following hold.*
(1) $\bigwedge_{x\in X}[i^{\rightarrow}_{r*}(A)(x)\Rightarrow e^r_X(x,z)]=\bigwedge_{y\in Y}[A(y)\Rightarrow e^r_X(y,z)]$ *for all* $z\in X$.
(2) $\bigwedge_{x\in X}[i^{*\rightarrow}_{r}(A)(x)\to e^r_X(z,x)]=\bigwedge_{y\in Y}[A(y)\to e^r_X(z,y)]$ *for all* $z\in X$.
(3) $\bigwedge_{x\in X}\left[i^{\rightarrow}_{l*}(A)(x)\to e^l_X(x,z)\right]=\bigwedge_{y\in Y}\left[A(y)\to e^l_X(y,z)\right]$ *for all* $z\in X$.
(4) $\bigwedge_{x\in X}\left[i^{*\rightarrow}_{l}(A)(x)\Rightarrow e^l_X(z,x)\right]=\bigwedge_{y\in Y}\left[A(y)\Rightarrow e^l_X(z,y)\right]$ *for all* $z\in X$.

Proof. (1) Note that

$\bigwedge_{x\in X}[i^{\rightarrow}_{r*}(A)(x)\Rightarrow e^r_X(x,z)]=\bigwedge_{x\in X}\left[\bigvee_{y\in Y}[e^r_X(x,y)\odot A(y)]\Rightarrow e^r_X(x,z)\right]$
$=\bigwedge_{x\in X}\bigwedge_{y\in Y}[[e^r_X(x,y)\odot A(y)]\to e^r_X(x,z)]$ (by Lemma 2.2(2))
$=\bigwedge_{x\in X}\bigwedge_{y\in Y}[A(y)\Rightarrow[e^r_X(x,y)\Rightarrow e^r_X(x,z)]]$ (by Lemma 2.2(3))
$=\bigwedge_{y\in Y}\left[A(y)\Rightarrow\bigwedge_{x\in X}[e^r_X(x,y)\Rightarrow e^r_X(x,z)]\right]$ (by Lemma 2.2(2))
$=\bigwedge_{y\in Y}[A(y)\Rightarrow e^r_X(y,z)]$ (by Lemma 3.3(1)).

(2) Note that

$$\begin{aligned}
&\bigwedge_{x\in X} [i_r^{*\to}(A)(x) \to e_X^r(z,x)] = \bigwedge_{x\in X} \left[\bigvee_{y\in Y} [A(y)\odot e_X^r(y,x)] \to e_X^r(z,x)\right]\\
&= \bigwedge_{x\in X}\bigwedge_{y\in Y} [[A(y)\odot e_X^r(y,x)] \to e_X^r(z,x)] \text{ (by Lemma 2.2(2))}\\
&= \bigwedge_{x\in X}\bigwedge_{y\in Y} [A(y) \to [e_X^r(y,x) \to e_X^r(z,x)]] \text{ (by Lemma 2.2(3))}\\
&= \bigwedge_{y\in Y} \left[A(y) \to \bigwedge_{x\in X} [e_X^r(y,x) \to e_X^r(z,x)]\right] \text{ (by Lemma 2.2(2))}\\
&= \bigwedge_{y\in Y} [A(y) \to e_X^r(z,y)] \text{ (by Lemma 3.3(2))}.
\end{aligned}$$

(3) Note that

$$\begin{aligned}
&\bigwedge_{x\in X} \left[i_{l*}^{\to}(A)(x) \to e_X^l(x,z)\right] = \bigwedge_{x\in X} \left[\bigvee_{y\in Y} \left[A(y)\odot e_X^l(x,y)\right] \to e_X^l(x,z)\right]\\
&= \bigwedge_{x\in X}\bigwedge_{y\in Y} \left[\left[A(y)\odot e_X^l(x,y)\right] \to e_X^l(x,z)\right] \text{ (by Lemma 2.2(2))}\\
&= \bigwedge_{x\in X}\bigwedge_{y\in Y} \left[A(y) \to \left[e_X^l(x,y) \to e_X^l(x,z)\right]\right] \text{ (by Lemma 2.2(3))}\\
&= \bigwedge_{y\in Y} \left[A(y) \to \bigwedge_{x\in X} \left[e_X^l(x,y) \to e_X^l(x,z)\right]\right] \text{ (by Lemma 2.2(2))}\\
&= \bigwedge_{y\in Y} \left[A(y) \to e_X^l(y,z)\right] \text{ (by Lemma 3.3(3))}.
\end{aligned}$$

(4) Note that

$$\begin{aligned}
&\bigwedge_{x\in X} \left[i_{l*}^{\to}(A)(x) \Rightarrow e_X^l(z,x)\right] = \bigwedge_{x\in X} \left[\bigvee_{y\in Y} \left[e_X^l(y,x)\odot A(y)\right] \Rightarrow e_X^l(z,x)\right]\\
&= \bigwedge_{x\in X}\bigwedge_{y\in Y} \left[\left[e_X^l(y,x)\odot A(y)\right] \Rightarrow e_X^l(z,x)\right] \text{ (by Lemma 2.2(2))}\\
&= \bigwedge_{x\in X}\bigwedge_{y\in Y} \left[A(y) \Rightarrow \left[e_X^l(y,x) \Rightarrow e_X^l(z,x)\right]\right] \text{ (by Lemma 2.2(3))}\\
&= \bigwedge_{y\in Y} \left[A(y) \Rightarrow \bigwedge_{x\in X} \left[e_X^l(y,x) \Rightarrow e_X^l(z,x)\right]\right] \text{ (by Lemma 2.2(2))}\\
&= \bigwedge_{y\in Y} \left[A(y) \Rightarrow e_X^l(z,y)\right] \text{ (by Lemma 3.3(4))}.
\end{aligned}$$

□

Let (X, e_X^r, e_X^l) be a bi-partially ordered set. Let $x \in X$. Define four maps $(e_X^r)^x$, $(e_X^l)_x$, $(e_X^l)^x$, $(e_X^r)_x : X \to L$ by

$$(e_X^r)^x(y) = e_X^r(y,x), \qquad (e_X^l)_x(y) = e_X^l(x,y),$$
$$(e_X^l)^x(y) = e_X^l(y,x), \qquad (e_X^r)_x(y) = e_X^r(x,y)$$

where $y \in X$.

Let $(e_X^r)^x|_Y$, $(e_X^l)_x|_Y$, $(e_X^l)^x|_Y$ and $(e_X^r)_x|_Y$ be the restrictions to Y of $(e_X^r)^x$, $(e_X^l)_x$, $(e_X^l)^x$ and $(e_X^r)_x$, respectively.

Theorem 3.5. *Let (X, e_X^r, e_X^l) be a bi-partially ordered set. Let $Y \subseteq X$. Then the following hold.*
(1) Y is an r-join-dense in X if and only if $x = \bigsqcup_r i_{r*}^{\to}((e_X^r)^x|_Y)$ for all $x \in X$.
(2) Y is an r-meet-dense in X if and only if $x = \sqcap_r i_r^{*\to}\left((e_X^r)_x|_Y\right)$ for all $x \in X$.
(3) Y is an l-join-dense in X if and only if $x = \bigsqcup_l i_{l*}^{\to}\left((e_X^l)^x|_Y\right)$ for all $x \in X$.
(4) Y is an l-meet-dense in X if and only if $x = \sqcap_l i_l^{*\to}\left((e_X^l)_x|_Y\right)$ for all $x \in X$.

Proof. (1) Assume that Y is an r-join-dense in X. Let $x \in X$. Then there exists $A \in L^Y$ such that $x = \bigsqcup_r i_{r^*}^{\to}(A)$. By (RJ1), we have

$$i_{r^*}^{\to}(A)(t) \le e_X^r(t,x) \quad \text{for all } t \in X. \tag{1}$$

By (RJ2), $\bigwedge_{w\in X}[i^{\rightarrow}_{r^*}(A)(w) \Rightarrow e^r_X(w,t)] \leq e^r_X(x,t)$ for all $t \in X$. Since

$$\bigwedge_{w\in X}[i^{\rightarrow}_{r^*}(A)(w) \Rightarrow e^r_X(w,t)] = \bigwedge_{y\in Y}[A(y) \Rightarrow e^r_X(y,t)] \text{ by Theorem 3.4(1),}$$

we have

$$\bigwedge_{y\in Y}[A(y) \Rightarrow e^r_X(y,t)] \leq e^r_X(x,t) \quad \text{for all } t \in X. \tag{2}$$

Note that

$$\begin{aligned}
&\bigwedge_{w\in X}[i^{\rightarrow}_{r*}((e^r_X)^x|_Y)(w) \Rightarrow e^r_X(w,z)] \\
&= \bigwedge_{y\in Y}[((e^r_X)^x|_Y)(y) \Rightarrow e^r_X(y,z)] \text{ (by Theorem 3.4(1))} \\
&= \bigwedge_{y\in Y}[e^r_X(y,x) \Rightarrow e^r_X(y,z)] \\
&\leq \bigwedge_{y\in Y}[i^{\rightarrow}_{r^*}(A)(y) \Rightarrow e^r_X(y,z)] \text{ (by Eq. (1) and Lemma 2.2(1))} \\
&= \bigwedge_{y\in Y}\left[\bigvee_{t\in Y}[e^r_X(y,t)\odot A(t)] \Rightarrow e^r_X(y,z)\right] \\
&= \bigwedge_{y\in Y}\bigwedge_{t\in Y}[[e^r_X(y,t)\odot A(t)] \Rightarrow e^r_X(y,z)] \text{ (by Lemma 2.2(2))} \\
&= \bigwedge_{y\in Y}\bigwedge_{t\in Y}[A(t) \Rightarrow [e^r_X(y,t) \Rightarrow e^r_X(y,z)]] \text{ (by Lemma 2.2(3))} \\
&= \bigwedge_{t\in Y}\left[A(t) \Rightarrow \bigwedge_{y\in Y}[e^r_X(y,t) \Rightarrow e^r_X(y,z)]\right] \text{ (by Lemma 2.2(2))} \\
&= \bigwedge_{t\in Y}[A(t) \Rightarrow e^r_X(t,z)] \text{ (by Lemma 3.3(1))} \\
&\leq e^r_X(x,z) \text{ (by Eq. (2)).}
\end{aligned}$$

Moreover,

$$\begin{aligned}
i^{\rightarrow}_{r*}((e^r_X)^x|_Y)(w) &= \bigvee_{y\in Y}[e^r_X(w,y)\odot (e^r_X)^x|_Y(y)] \\
&= \bigvee_{y\in Y}[e^r_X(w,y)\odot e^r_X(y,x)] \\
&\leq e^r_X(w,x).
\end{aligned}$$

Therefore $x = \bigsqcup_r i^{\rightarrow}_{r*}((e^r_X)^x|_Y)$.

The converse is trivial.

(2) Assume that Y is an r-meet-dense in X. Let $x \in X$. Then there exists $A \in L^Y$ such that $x = \sqcap_r i^{*\rightarrow}_r(A)$. By (RM1), we have

$$i^{*\rightarrow}_r(A)(t) \leq e^r_X(x,t) \text{ for all } t \in X. \tag{3}$$

By (RM2), $\bigwedge_{w\in X}[i^{*\rightarrow}_r(A)(w) \to e^r_X(t,w)] \leq e^r_X(t,x)$ for all $t \in X$. Since

$$\bigwedge_{w\in X}[i^{*\rightarrow}_r(A)(w) \to e^r_X(t,w)] = \bigwedge_{y\in Y}[A(y) \to e^r_X(t,y)] \text{ by Theorem 3.4(2),}$$

we have

$$\bigwedge_{y\in Y}[A(y) \to e^r_X(t,y)] \leq e^r_X(t,x) \text{ for all } t \in X. \tag{4}$$

Note that

$$\begin{aligned}
&\bigwedge_{w\in X}\left[i_r^{*\to}\left((e_X^r)_x|_Y\right)(w)\to e_X^r(z,w)\right]\\
&=\bigwedge_{y\in Y}\left[\left((e_X^r)_x|_Y\right)(y)\to e_X^r(z,y)\right] \text{ (by Theorem 3.4(2))}\\
&=\bigwedge_{y\in Y}\left[e_X^r(x,y)\to e_X^r(z,y)\right] \text{ (by Lemma 3.3(2))}\\
&\le\bigwedge_{y\in Y}\left[i_r^{*\to}(A)(y)\to e_X^r(z,y)\right] \text{ (by Eq. (3) and Lemma 2.2(1))}\\
&=\bigwedge_{y\in Y}\left[\bigvee_{t\in Y}\left[A(t)\odot e_X^r(t,y)\right]\to e_X^r(z,y)\right]\\
&=\bigwedge_{y\in Y}\bigwedge_{t\in Y}\left[\left[A(t)\odot e_X^r(t,y)\right]\to e_X^r(z,y)\right] \text{ (by Lemma 2.2(2))}\\
&=\bigwedge_{y\in Y}\bigwedge_{t\in Y}\left[A(t)\to\left[e_X^r(t,y)\to e_X^r(z,y)\right]\right] \text{ (by Lemma 2.2(3))}\\
&=\bigwedge_{t\in Y}\left[A(t)\to\bigwedge_{y\in Y}\left[e_X^r(t,y)\to e_X^r(z,y)\right]\right] \text{ (by Lemma 2.2(2))}\\
&=\bigwedge_{t\in Y}\left[A(t)\to e_X^r(z,t)\right] \text{ (by Lemma 3.3(2))}\\
&\le e_X^r(z,x) \text{ (by Eq. (4))}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
i_r^{*\to}\left((e_X^r)_x|_Y\right)(w) &= \bigvee_{y\in Y}\left[(e_X^r)_x|_Y(y)\odot e_X^r(y,w)\right]\\
&= \bigvee_{y\in Y}\left[e_X^r(x,y)\odot e_X^r(y,w)\right]\\
&\le e_X^r(x,w).
\end{aligned}$$

Therefore $x=\sqcap_r i_r^{*\to}\left((e_X^r)_x|_Y\right)$.

The converse is trivial.

(3) Assume that Y is an l-join-dense in X. Let $x\in X$. Then there exists $A\in L^Y$ such that $x=\bigsqcup_l i_{l*}^{\to}(A)$. By (LJ1), we have

$$i_{l*}^{\to}(t)=e_X^l(t,x) \text{ for all } t\in X. \tag{5}$$

By (LJ2), $\bigwedge_{w\in X}\left[i_{l*}^{\to}(A)(w)\to e_X^l(w,t)\right]\le e_X^l(x,t)$ for all $t\in X$. Since

$$\bigwedge_{w\in X}\left[i_{l*}^{\to}(A)(w)\to e_X^l(w,t)\right]=\bigwedge_{y\in Y}\left[A(y)\to e_X^l(y,t)\right] \text{ by Theorem 3.4(3),}$$

we have

$$\bigwedge_{y\in Y}\left[A(y)\to e_X^l(y,t)\right]\le e_X^l(x,t) \text{ for all } t\in X. \tag{6}$$

Note that

$$\begin{aligned}
&\bigwedge_{w\in X}\left[i_{l*}^{\to}\left((e_X^l)^x|_Y\right)(w)\to e_X^l(w,z)\right]\\
&=\bigwedge_{y\in Y}\left[\left((e_X^l)^x|_Y\right)(y)\to e_X^l(y,z)\right] \text{ (by Theorem 3.4(3))}\\
&=\bigwedge_{y\in Y}\left[e_X^l(y,x)\to e_X^l(y,z)\right]\\
&\le\bigwedge_{y\in Y}\left[i_{i*}^{\to}(A)(y)\to e_X^l(y,z)\right] \text{ (by Eq. (5) and Lemma 2.2(1))}\\
&=\bigwedge_{y\in Y}\left[\bigvee_{t\in Y}\left[A(t)\odot e_X^l(y,t)\right]\to e_X^l(y,z)\right]\\
&=\bigwedge_{y\in Y}\bigwedge_{t\in Y}\left[\left[A(t)\odot e_X^l(y,t)\right]\to e_X^l(y,z)\right] \text{ (by Lemma 2.2(2))}\\
&=\bigwedge_{y\in Y}\bigwedge_{t\in Y}\left[A(t)\to\left[e_X^l(y,t)\to e_X^l(y,z)\right]\right] \text{ (by Lemma 2.2(3))}\\
&=\bigwedge_{t\in Y}\left[A(t)\to\bigwedge_{y\in Y}\left[e_X^l(y,t)\to e_X^l(y,z)\right]\right] \text{ (by Lemma 2.2(2))}\\
&=\bigwedge_{t\in Y}\left[A(t)\to e_X^l(t,z)\right] \text{ (by Lemma 3.3(3))}\\
&\le e_X^l(x,z) \text{ (by Eq. (6))}.
\end{aligned}$$

Moreover,

$$\begin{aligned} i_{l*}^{\rightarrow}\left(\left(e_X^l\right)^x|_Y\right)(w) &= \bigvee_{y\in Y}\left[\left(e_X^l\right)^x|_Y(y)\odot e_X^l(w,y)\right] \\ &= \bigvee_{y\in Y}\left[e_X^l(y,x)\odot e_X^l(w,y)\right] \\ &\le e_X^l(w,x). \end{aligned}$$

Therefore $x=\bigsqcup_l i_{l*}^{\rightarrow}\left(\left(e_X^l\right)^x|_Y\right)$.

The converse is trivial.

(4) Assume that Y is an l-meet-dense in X. Let $x\in X$. Then there exists $A\in L^Y$ such that $x=\sqcap_l i_l^{*\rightarrow}(A)$. By (LM1), we have

$$i_l^{*\rightarrow}(A)(t)\le e_X^l(x,t) \text{ for all } t\in X. \tag{7}$$

By (LM2), $\bigwedge_{w\in X}\left[i_l^{*\rightarrow}(A)(w)\Rightarrow e_X^l(t,w)\right]\le e_X^l(t,x)$ for all $t\in X$. Since

$$\bigwedge_{w\in X}\left[i_l^{*\rightarrow}(A)(w)\Rightarrow e_X^l(t,w)\right]=\bigwedge_{y\in Y}\left[A(y)\Rightarrow e_X^l(t,y)\right] \text{ by Theorem 3.4(4)},$$

we have

$$\bigwedge_{y\in Y}\left[A(y)\Rightarrow e_X^l(t,y)\right]\le e_X^l(t,x) \text{ for all } t\in X. \tag{8}$$

Note that

$$\begin{aligned} &\bigwedge_{w\in X}\left[i_l^{*\rightarrow}\left(\left(e_X^l\right)_x|_Y\right)(w)\Rightarrow e_X^l(z,w)\right] \\ &=\bigwedge_{y\in Y}\left[\left(\left(e_X^l\right)_x|_Y\right)(y)\Rightarrow e_X^l(z,y)\right] \text{ (by Theorem 3.4(4))} \\ &=\bigwedge_{y\in Y}\left[e_X^l(x,y)\Rightarrow e_X^l(z,y)\right] \\ &\le\bigwedge_{y\in Y}\left[i_l^{*\rightarrow}(A)(y)\Rightarrow e_X^l(z,y)\right] \text{ (by Eq. (7) and Lemma 2.2(1))} \\ &=\bigwedge_{y\in Y}\left[\bigvee_{t\in Y}\left[e_X^l(t,y)\odot A(t)\right]\Rightarrow e_X^l(z,y)\right] \\ &=\bigwedge_{y\in Y}\bigwedge_{t\in Y}\left[\left[e_X^l(t,y)\odot A(t)\right]\Rightarrow e_X^l(z,y)\right] \text{ (by Lemma 2.2(2))} \\ &=\bigwedge_{y\in Y}\bigwedge_{t\in Y}\left[A(t)\Rightarrow\left[e_X^l(t,y)\Rightarrow e_X^l(z,y)\right]\right] \text{ (by Lemma 2.2(3))} \\ &=\bigwedge_{t\in Y}\left[A(t)\Rightarrow\bigwedge_{y\in Y}\left[e_X^l(t,y)\Rightarrow e_X^l(z,y)\right]\right] \text{ (by Lemma 2.2(2))} \\ &=\bigwedge_{t\in Y}\left[A(t)\Rightarrow e_X^l(z,t)\right] \text{ (by Lemma 3.3(4))} \\ &\le e_X^l(z,x) \text{ (by Eq. (8))}. \end{aligned}$$

Moreover,

$$\begin{aligned} i_l^{*\rightarrow}\left(\left(e_X^l\right)_x|_Y\right)(w) &= \bigvee_{y\in Y}\left[e_X^l(y,w)\odot (e_X^l)_x|_Y(y)\right] \\ &= \bigvee_{y\in Y}\left[e_X^l(y,w)\odot e_X^r(x,y)\right] \\ &\le e_X^l(x,w). \end{aligned}$$

Therefore $x=\sqcap_l i_l^{*\rightarrow}\left(\left(e_X^l\right)_x|_Y\right)$. □

By Lemma 2.5 and Theorems 3.4-3.5, we have the following.

Theorem 3.6. *Let $\left(X,e_X^r,e_X^l\right)$ be a bi-partially ordered set. Let $Y\subseteq X$. Then the following hold.*

(1) Y is an r-join-dense in X if and only if $\bigwedge_{y\in Y}\left[e_X^r(y,x)\Rightarrow e_X^r(y,z)\right]=e_X^r(x,z)$ for all $x,z\in X$.

(2) Y is an r-meet-dense in X if and only if $\bigwedge_{y\in Y}\left[e_X^r(x,y)\rightarrow e_X^r(z,y)\right]=e_X^r(z,x)$ for all $x,z\in X$.

(3) *Y is an l-join-dense in X if and only if* $\bigwedge_{y\in Y}\left[e^l_X(y,x)\to e^l_X(y,z)\right] = e^l_X(x,z)$ *for all* $x,z\in X$.
(4) *Y is an l-meet-dense in X if and only if* $\bigwedge_{y\in Y}\left[e^l_X(x,y)\Rightarrow e^l_X(z,y)\right] = e^l_X(z,x)$ *for all* $x,z\in X$.

Proof. (1) By Theorem 3.5(1), Y is an r-join-dense in X if and only if

$$x=\bigsqcup_r i^{\to}_{r^*}\left((e^r_X)^x|_Y\right)\ \text{ for all } x\in X,$$

which is equivalent by Lemma 2.5(1) that

$$\bigwedge_{w\in X}\left[i^{\to}_{r^*}\left((e^r_X)^x|_Y\right)(w)\Rightarrow e^r_X(w,z)\right]=e^r_X(x,z)\,\text{for all } x,z\in X,$$

which is equivalent by Theorem 3.4(1) that

$$\bigwedge_{y\in Y}\left[e^r_X(y,x)\Rightarrow e^r_X(y,z)\right]=e^r_X(x,z)\,\text{for all } x,z\in X.$$

(2) By Theorem 3.5(2), Y is an r-meet-dense in X if and only if

$$x=\sqcap_r i^{*\to}_r\left((e^r_X)_x|_Y\right)\ \text{ for all } x\in X,$$

which is equivalent by Lemma 2.5(2) that

$$\bigwedge_{w\in X}\left[i^{*\to}_r\left((e^r_X)_x|_Y\right)(w)\to e^r_X(z,w)\right]=e^r_X(z,x)\,\text{for all } x,z\in X,$$

which is equivalent by Theorem 3.4(2) that

$$\bigwedge_{y\in Y}\left[e^r_X(x,y)\to e^r_X(z,y)\right]=e^r_X(z,x)\,\text{for all } x,z\in X.$$

(3) By Theorem 3.5(3), Y is an l-join-dense in X if and only if

$$x=\bigsqcup_l i^{\to}_{l*}\left(\left(e^l_X\right)^x|_Y\right)\ \text{ for all } x\in X,$$

which is equivalent by Lemma 2.5(3) that

$$\bigwedge_{w\in X}\left[i^{\to}_{l^*}\left(\left(e^l_X\right)^x|_Y\right)(w)\to e^l_X(w,z)\right]=e^l_X(x,z)\,\text{for all } x,z\in X,$$

which is equivalent by Theorem 3.4(3) that

$$\bigwedge_{y\in Y}\left[e^l_X(y,x)\to e^l_X(y,z)\right]=e^l_X(x,z)\,\text{for all } x,z\in X.$$

(4) By Theorem 3.5(4), Y is an l-meet-dense in X if and only if

$$x=\sqcap_l i^{*\to}_l\left(\left(e^l_X\right)_x|_Y\right)\ \text{ for all } x\in X,$$

which is equivalent by Lemma 2.5(4) that

$$\bigwedge_{w\in X}\left[i^{*\to}_l\left(\left(e^l_X\right)_x|_Y\right)(w)\Rightarrow e^l_X(z,w)\right]=e^l_X(z,x)\,\text{for all } x,z\in X,$$

which is equivalent by Theorem 3.4(4) that

$$\bigwedge_{y\in Y}\left[e^l_X(x,y)\Rightarrow e^l_X(z,y)\right]=e^l_X(z,x)\,\text{for all } x,z\in X.$$

□

Example 3.7. Let $K=\{(x,y)\in\mathbb{R}^2\mid x>0\}$ be a set where $\mathbb{R}$ is the set of all real numbers. Define a binary operation $\otimes: K\times K\to K$ by $(x_1,y_1)\otimes(x_2,y_2)=(x_1x_2,x_1y_2+y_1)$. Then one can see that $(K,\otimes)$ is a non-commutative group where $e=(1,0)$ is the identity and $(x,y)^{-1}=(\frac{1}{x},-\frac{y}{x})$ for all $(x,y)\in K$.

Let $P=\{(a,b)\in\mathbb{R}^2\mid \ (1<a)\text{ or }(a=1\text{ and }0\le b)\ \}$. One can see that $P\cap P^{-1}=\{(1,0)\}$, $P\otimes P\subseteq P$, $(a,b)^{-1}\otimes P\otimes(a,b)=P$ for all $(a,b)\in K$ and $P\cup P^{-1}=K$. Then P is a positive cone of K.

For all $(x_1,y_1),(x_2,y_2)\in K$, define

$$(x_1,y_1)\le(x_2,y_2)\qquad\text{if }\ (x_1,y_1)^{-1}\otimes(x_2,y_2)\in P.$$

Then $(K,\le,\otimes)$ is a lattice-group (see [3, 4]). Note that $(x_1,y_1)\le(x_2,y_2)$ if and only if either $(x_1<x_2)$ or $(x_1=x_2$ and $y_1\le y_2)$.

Let $L=\left\{(x,y)\in K\mid \left(\frac{1}{2},1\right)\le(x,y)\le(1,0)\right\}$. Define three binary operations $\odot$, $\Rightarrow$, $\to$: $L\times L\to L$ by

$$\begin{aligned}(x_1,y_1)\odot(x_2,y_2)&=[(x_1,y_1)\otimes(x_2,y_2)]\vee\left(\tfrac{1}{2},1\right)&&=(x_1x_2,x_1y_2+y_1)\vee\left(\tfrac{1}{2},1\right),\\(x_1,y_1)\Rightarrow(x_2,y_2)&=\left[(x_1,y_1)^{-1}\otimes(x_2,y_2)\right]\wedge(1,0)&&=\left(\tfrac{x_2}{x_1},\tfrac{y_2-y_1}{x_1}\right)\wedge(1,0),\\(x_1,y_1)\to(x_2,y_2)&=\left[(x_2,y_2)\otimes(x_1,y_1)^{-1}\right]\wedge(1,0)&&=\left(\tfrac{x_2}{x_1},-\tfrac{x_2y_1}{x_1}+y_2\right)\wedge(1,0).\end{aligned}$$

One can see that the structure $\left(L,\odot,\Rightarrow,\to,\left(\frac{1}{2},1\right),(1,0)\right)$ is a generalized residuated lattice where $\perp=\left(\frac{1}{2},1\right)$ is the least element and $\top=(1,0)$ is the greatest element.

Let $X=\{a,b,c\}$ be a set. Define $e^r_X,e^l_X: X\times X\to L$ by

$$e^r_X=\begin{pmatrix}(1,0)&\left(\frac{5}{8},-5\right)&\left(\frac{5}{6},1\right)\\\left(\frac{5}{7},2\right)&(1,0)&\left(\frac{5}{6},-1\right)\\\left(\frac{6}{7},\frac{18}{5}\right)&\left(\frac{3}{4},-\frac{36}{5}\right)&(1,0)\end{pmatrix},\ e^l_X=\begin{pmatrix}(1,0)&\left(\frac{2}{3},-1\right)&\left(\frac{5}{6},-1\right)\\\left(\frac{4}{7},-1\right)&(1,0)&\left(\frac{6}{7},-1\right)\\\left(\frac{2}{3},-\frac{1}{3}\right)&\left(\frac{4}{5},-\frac{9}{5}\right)&(1,0)\end{pmatrix}.$$

One can check that e^r_X is an r-partial order and e^l_X is an l-partial order. Hence $\left(X,e^r_X,e^l_X\right)$ is a bi-partially ordered set. But e^r_X is not an l-partial order because $e^r_X(c,a)\odot e^r_X(b,c)\not\le e^r_X(b,a)$.

(1) Let $Y=\{a,b\}$. Let $i: Y\to X$ be the inclusion map.

By Theorem 3.6(1), Y is an r-join-dense in X if and only if

$$\bigwedge_{y\in Y}\left[e^r_X(y,x)\Rightarrow e^r_X(y,z)\right]=e^r_X(x,z)\,\text{for all } x,z\in X.\tag{9}$$

By a direct computation, one can see that Eq. (9) holds. Hence Y is an r-join-dense in X.

By Theorem 3.6(3), Y is an l-join-dense in X if and only if

$$\bigwedge_{y\in Y}\left[e^l_X(y,x)\to e^l_X(y,z)\right]=e^l_X(x,z)\ \text{ for all } x,z\in X. \tag{10}$$

Since

$$\begin{aligned}\textstyle\bigwedge_{y\in Y}\left[e^l_X(y,c)\to e^l_X(y,b)\right] &= \left[e^l_X(a,c)\to e^l_X(a,b)\right]\wedge\left[e^l_X(b,c)\to e^l_X(b,b)\right]\\ &= [(\tfrac{5}{6},-1)\to(\tfrac{2}{3},-1)]\wedge[(\tfrac{6}{7},-1)\to(1,0)]\\ &= (\tfrac{4}{5},-\tfrac{1}{5})\end{aligned}$$

and $e^l_X(c,b)=(\frac{4}{5},-\frac{9}{5})$, Eq. (10) does not hold. Hence Y is not an l-join-dense in X.
(2) Let $U=\{a,c\}$. Let $i:U\to X$ be the inclusion map.
By Theorem 3.6(1), U is an r-join-dense in X if and only if

$$\bigwedge_{y\in U}[e^r_X(y,x)\Rightarrow e^r_X(y,z)]=e^r_X(x,z)\ \text{ for all } x,z\in X. \tag{11}$$

Since

$$\begin{aligned}\textstyle\bigwedge_{y\in U}[e^r_X(y,b)\Rightarrow e^r_X(y,a)] &= [e^r_X(a,b)\Rightarrow e^r_X(a,a)]\wedge[e^r_X(c,b)\Rightarrow e^r_X(c,a)]\\ &= [(\tfrac{5}{8},-5)\Rightarrow(1,0)]\wedge[(\tfrac{3}{4},-\tfrac{36}{5})\Rightarrow(\tfrac{6}{7},\tfrac{18}{5})]\\ &= (1,0)\end{aligned}$$

and $e^r_X(b,a)=(\frac{5}{7},2)$, Eq. (11) does not hold. Hence U is not an r-join-dense in X.
By Theorem 3.6(3), U is an l-join-dense in X if and only if

$$\bigwedge_{y\in U}\left[e^l_X(y,x)\to e^l_X(y,z)\right]=e^l_X(x,z)\,\text{for all } x,z\in X. \tag{12}$$

Since

$$\begin{aligned}\textstyle\bigwedge_{y\in U}\left[e^l_X(y,b)\to e^l_X(y,a)\right] &= \left[e^l_X(a,b)\to e^l_X(a,a)\right]\wedge\left[e^l_X(c,b)\to e^l_X(c,a)\right]\\ &= [(\tfrac{2}{3},-1)\to(1,0)]\wedge[(\tfrac{4}{5},-\tfrac{9}{5})\to(\tfrac{2}{3},-\tfrac{1}{3})]\\ &= (\tfrac{5}{6},\tfrac{7}{6})\end{aligned}$$

and $e^l_X(b,a)=(\frac{4}{7},-1)$, Eq. (12) does not hold. Hence U is not an l-join dense in X.

4. Conclusion

Throughout these concepts introduced in this paper, we have investigated the characteristics of bi-partially ordered sets on complete generalized residuated lattices. In the future, we might to investigate various completions on these spaces.

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