East Asian Math. J.
Vol. 40 (2024), No. 3, pp. 329–339
http://dx.doi.org/10.7858/eamj.2024.022

OBTUSE MATRIX OF ARITHMETIC TABLE

Eunmi Choi

Abstract. In the work we generate arithmetic matrix $P^{(c,b,a)}$ of $(cx^2+bx+a)^n$ from a Pascal matrix $P^{(1,1)}$. We extend an identity $P^{(1,1)}O^{(1,1)} = P^{(1,1,1)}$ with an obtuse matrix $O^{(1,1)}$ to k degree polynomials. For the purpose we study ${P^{(1,1)}}^k O^{(1,1)}$ and find generating polynomials of ${O^{(1,1)}}^k$.

1. Introduction

Let $P^{(1,1)}$ be an arithmetic matrix (abbr, AM) of $(x+1)^n$. Let $O^{(1,1)}$, called an obtuse matrix of $P^{(1,1)}$, be that each i^{th}row of $O^{(1,1)}$ is the i^{th}row of $P^{(1,1)}$ shifted i places rightward for all $i \geq 0$. Then

$$P^{(1,1)}O^{(1,1)} = \begin{bmatrix} 1 \\ 11 \\ 121 \\ 1331 \\ \cdots \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 11 & & \\ & & 121 & \\ & & & 1331 \\ & & & \cdots \end{bmatrix} = \begin{bmatrix} 1 \\ 111 \\ 12321 \\ 1367631 \\ \cdots \end{bmatrix} = P^{(1,1,1)}$$

where $P^{(1,1,1)}$ is an AM of $(x^2+x+1)^n$ ([3]). Generally, we call O an obtuse matrix of M if each i^{th}row of O is formed by shifting the i^{th}row of M i places rightward. For instance, the obtuse of $P^{(1,1,1)}$ is $O^{(1,1,1)} = \begin{bmatrix} 1 & & & \\ & 111 & & \\ & & 12321 & \\ & & & 13676 \\ & & & \cdots \end{bmatrix}$.

In this work we study the obtuse matrix $O^{(1,1)}$ of $P^{(1,1)}$ and explore $O^{(1,1,1)}$ together with $P^{(1,1,1)}$ yields an AM of $\sum_{i=0}^{k} x^i$. Then we extend these study to AM of $(cx^2+bx+a)^n$ for any $a, b, c \in \mathbb{Z}$. For the purpose we investigate matrices ${P^{(1,1)}}^k O^{(1,1)}$ in Theorem 2.2. And we find generating polynomials of ${O^{(1,1)}}^k$ and $P^{(1,1)}{O^{(1,1)}}^k$ in Theorem 4.2 and Theorem 4.5.

Some articles on arithmetic tables, including [2], [4], [6], have been presented, in which binomial and multinomial coefficients were calculated for expanding polynomials. A distinctive feature of this work is to study Pascal matrix [1] and the obtuse matrix to explores arithmetic tables related to polynomials.

Throughout the work we denote by $P^{(c,b,a)}$ an AM of $(cx^2+bx+a)^n$. Since the expansion $(cx^2+bx+a)^n$ results in different shapes depending on ascending

Received February 7, 2024; Accepted May 21, 2024.
2020 *Mathematics Subject Classification.* 11C08, 15A16.
Key words and phrases. arithmetic matrix, Pascal matrix, generating polynomial.
* This work was supported by 2023 Hannam University Research Fund.

(pISSN 1226-6973, eISSN 2287-2833)

or descending order in x, we write $P^{(c,b,a)} = P^{(c,b,a)\downarrow}$ to indicate descending order expansion. Otherwise we write by $P^{(c,b,a)\uparrow}$. Let $r_i(M)$ and $c_j(M)$ be the i^{th}row and j^{th}column of a matrix M, respectively. Let $\bar{s}_k = (s, \dots, s)$ be a k-tuple of s, and $(\bar{0}_k; r_i(M); \bar{0}_t)$ be a row matrix consisting of k zeros followed by $r_i(M)$ and then t zeros. Let $(a_1, \dots, a_k)(b_1, \dots, b_k) = (a_1, \dots, a_k)\begin{bmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_k \end{bmatrix}$ $= (a_1 b_1, \dots, a_k b_k)$, and write a diagonal matrix $\begin{bmatrix} 1 & & & \\ & a & & \\ & & a^2 & \\ & & & \ddots \end{bmatrix}$ by diag$[a^i]$.

2. Expressions of $P^{(a,b,c)}$ by P

In analog to the obtuse matrix $O^{(c,b,a)}$ of $P^{(c,b,a)}$, let $O^{(c,b,a)\uparrow}$ be the corresponding object of $P^{(c,b,a)\uparrow}$. Clearly $P^{(1,1,1)\uparrow} = P^{(1,1,1)\downarrow} = P^{(1,1,1)}$ and $P^{(c,b,a)\uparrow} = P^{(a,b,c)\downarrow}$. When $c = 0$, write $P^{(0,b,a)} = P^{(b,a)}$ and $O^{(b,a)}$.

Theorem 2.1. *$r_n(P^{(b,a)}) = r_n(P^{(1,1)})(b^n, \dots, b, 1)(1, a, \dots, a^n)$ and $P^{(b,a)} = {P^{(1,1)}}^b diag[a^n] = {P^{(1,1)}}^{b-1} P^{(1,a)}$ for any nonzero a, b.*

Proof. We mainly refer to [3] for the proof. In fact, we simply write $P^{(1,1)} = P$. Firstly we notice $P^{(3,1)} = \begin{bmatrix} (1)(1) \\ (1,1)(3,1) \\ (1,2,1)(3^2,3,1) \\ \cdots \end{bmatrix} = \begin{bmatrix} r_0(P)(1) \\ r_1(P)(3,1) \\ r_2(P)(3^2,3,1) \\ \cdots \end{bmatrix} = P^3$, so $r_n(P^{(3,1)}) = r_n(P)(3^n, \dots, 3, 1)$. Assume an inductive hypothesis $r_{n-1}(P^{(b,1)}) = r_{n-1}(P)(b^{n-1}, \dots, b, 1)$ for some n. Then we have

$$\begin{aligned} r_n(P^{(b,1)}) \begin{bmatrix} x^n \\ \vdots \\ x \\ 1 \end{bmatrix} &= (bx+1)^n = (bx+1)\ r_{n-1}(P^{(b,1)}) \begin{bmatrix} x^{n-1} \\ \vdots \\ x \\ 1 \end{bmatrix} \\ &= r_{n-1}(P)(b^n, \dots, b^2, b) \begin{bmatrix} x^n \\ \vdots \\ x^2 \\ x \end{bmatrix} + r_{n-1}(P)(b^{n-1}, \dots, b, 1) \begin{bmatrix} x^{n-1} \\ \vdots \\ x \\ 1 \end{bmatrix} \\ &= ((r_{n-1}(P); 0) + (0; r_{n-1}(P)))\,(b^n, \dots, 1) \begin{bmatrix} x^n \\ \vdots \\ x \\ 1 \end{bmatrix} = r_n(P)(b^n, \dots, 1) \begin{bmatrix} x^n \\ \vdots \\ x \\ 1 \end{bmatrix}, \end{aligned}$$

so $r_n(P^{(b,1)}) = r_n(P)(b^n, \dots, b, 1)$ and $P^{(b,1)} = \begin{bmatrix} r_0(P)(1) \\ r_1(P)(b,1) \\ r_2(P)(b^2,b,1) \\ \cdots \end{bmatrix} = P^b$.

Thus with any nonzero a and b, we have

$$\begin{aligned} r_n(P^{(b,a)}) \begin{bmatrix} x^n \\ \vdots \\ 1 \end{bmatrix} &= a^n r_n(P^{(a^{-1}b,1)}) \begin{bmatrix} x^n \\ \vdots \\ 1 \end{bmatrix} = a^n r_n(P)((a^{-1}b)^n, \dots, a^{-1}b, 1) \begin{bmatrix} x^n \\ \vdots \\ 1 \end{bmatrix} \\ &= r_n(P)(b^n, ab^{n-1}, \dots, a^n) \begin{bmatrix} x^n \\ \vdots \\ 1 \end{bmatrix} = r_n(P)(b^n, \dots, b, 1)(1, a, \dots, a^n) \begin{bmatrix} x^n \\ \vdots \\ 1 \end{bmatrix}, \end{aligned}$$

so $r_n(P^{(b,a)}) = r_n(P^{(b,1)})(1, a, \dots, a^n)$. Hence it follows that

$$P^{(b,a)} = \begin{bmatrix} r_0(P^{(b,1)})(1) \\ r_1(P^{(b,1)})(1,a) \\ r_2(P^{(b,1)})(1,a,a^2) \\ \cdots \end{bmatrix} = P^{(b,1)}\text{diag}[a^i] = P^b \text{diag}[a^i] = P^{b-1} P^{(1,a)}. \qquad \square$$

Similarly for $P^{(b,1)\uparrow}$ of $(bx+1)^n$, we have

$$r_n(P^{(b,1)\uparrow})\begin{bmatrix}1\\ \cdots\\ x^n\end{bmatrix} = b^n r_n(P^{(1,b^{-1})\uparrow})\begin{bmatrix}1\\ \cdots\\ x^n\end{bmatrix} = b^n r_n(P^{(b^{-1},1)})\begin{bmatrix}1\\ \cdots\\ x^n\end{bmatrix}$$
$$= b^n r_n(P)(b^{-n},\dots,b^{-1},1)\begin{bmatrix}1\\ \cdots\\ x^n\end{bmatrix} = r_n(P)(1,\dots,b^n)\begin{bmatrix}1\\ \cdots\\ x^n\end{bmatrix},$$

so $r_n(P^{(b,1)\uparrow}) = r_n(P)(1,\dots,b^n)$. Theorem 2.1 is generalized as follows.

Theorem 2.2. *$r_n(P^{(1,1,a)\uparrow}) = r_n(P^{(1,a)})O^{(1,1)}$ and $P^{(a,1,1)} = P^{(1,1,a)\uparrow} = {P^{(1,1)}}^a O^{(1,1)}$.*

Proof. We write $P^{(1,1)} = P$ and $O^{(1,1)} = O$ for simplicity. When $a = 1$, $PO = P^{(1,1,1)}$ is clear ([3]). If $a = 2$ then $P^2O = \begin{bmatrix}1\\ 2\,1\,1\\ 4\,4\,5\,2\\ 8\,12\,18\,13\\ \cdots\end{bmatrix} = P^{(2,1,1)} = P^{(1,1,2)\uparrow}$. Consider an AM $P^{(1,1,a)\uparrow}$ of $f^n(x) = (x^2+x+a)^n$. If we let $X = x^2+x$ then $X^i = x^i r_i(P)(1,\dots,x^i)^T = r_i(P)(x^i,\dots,x^{2i})^T = (\bar{0}_i; r_i(P); \bar{0}_{2(n-i)})(1,\dots,x^{2n})^T$, so we have

$$(1,X,\dots,X^n)^T = \begin{bmatrix}(r_0(P);\bar{0}_{2n})\\ (0;r_1(P);\bar{0}_{2(n-1)})\\ \cdots\\ (\bar{0}_n;r_n(P))\end{bmatrix}(1,\dots,x^{2n})^T = O^{(1,1)}(1,\dots,x^{2n})^T.$$

Thus with the n^{th}row $r_n(P^{(1,a)\uparrow})$ of $P^{(1,a)\uparrow}$, $f^n(x) = (X+a)^n$ gives

$$f^n(x) = r_n(P^{(1,a)\uparrow})(1,X,\dots,X^n)^T = r_n(P^{(1,a)\uparrow})O^{(1,1)}(1,x,\dots,x^{2n})^T.$$

But since $f^n(x) = (x^2+x+a)^n = r_n(P^{(1,1,a)\uparrow})(1,x,\dots,x^{2n})^T$ with the n^{th}row $r_n(P^{(1,1,a)\uparrow})$, we have $r_n(P^{(1,1,a)\uparrow}) = r_n(P^{(1,a)\uparrow})O^{(1,1)}$. Therefore

$$P^{(1,1,a)\uparrow} = \begin{bmatrix}r_0(P^{(1,1,a)\uparrow})\\ r_1(P^{(1,1,a)\uparrow})\\ \cdots\\ r_n(P^{(1,1,a)\uparrow})\end{bmatrix} = \begin{bmatrix}r_0(P^{(1,a)\uparrow})\\ r_1(P^{(1,a)\uparrow})\\ \cdots\\ r_n(P^{(1,a)\uparrow})\end{bmatrix}O^{(1,1)} = P^{(1,a)\uparrow}O^{(1,1)}.$$

Since $P^{(1,a)\uparrow} = P^a$ by Theorem 1, we conclude $P^aO^{(1,1)} = P^{(1,1,a)\uparrow}$. □

Corollary 2.3. *$P^{(1,1,a)\uparrow} = P^{(1,a)\uparrow}O^{(1,1)} = {P^{(1,1)}}^a O^{(1,1)} = P^{(1,1)}P^{(1,1,a-1)\uparrow}$ and $P^{(a,1,1)\uparrow} = P^{(1,1)}O^{(a,1)\uparrow}$, where $O^{(a,1)\uparrow}$ is the obtuse matrix of $P^{(a,1)\uparrow}$.*

The proof is easy from Theorem 2.2. Indeed, $P^2O = \begin{bmatrix}1\\ 211\\ 44521\\ \cdots\end{bmatrix} = P^{(1,1,2)\uparrow}$ and $P^3O = \begin{bmatrix}1\\ 3\;1\;1\\ 9\;6\;7\;2\;1\\ 27\,27\,36\,19\,2\\ \cdots\end{bmatrix} = P^{(1,1,3)\uparrow}$. Now consider a polynomial $(ax^k + x^{k-1} + \cdots + 1)^n$, an AM $P^{(a,1,\dots,1)} = P^{(a,\bar{1}_k)}$ and obtuse matrix $O^{(a,\bar{1}_k)}$ for $k \ge 1$.

Theorem 2.4. *$P^{(\bar{1}_{k+1})} = P^{(1,1)}O^{(\bar{1}_k)}$, moreover $P^{(\bar{1}_k,a)\uparrow} = P^{(1,a)\uparrow}O^{(\bar{1}_k)}$.*

Proof. Theorem 2.2 shows $PO = P^{(1,1,1)}$ with $P = P^{(1,1)}$, $O = O^{(1,1)}$. Consider an AM $P^{(\bar{1}_4)}$ of $f^n(x) = (x^3+x^2+x+1)^n$. Then with $X = x(x^2+x+1)$,

$$r_n(P^{(\bar{1}_4)})(1,x,\dots,x^{3n})^T = f^n(x) = (X+1)^n = r_n(P)(1,X,\dots,X^n)^T.$$

But since

$$X^i = r_i(P^{(\bar{1}_3)})(x^i,\dots,x^{3i})^T = (\bar{0}_i; r_i(P^{(\bar{1}_3)}); \bar{0}_{3(n-i)})(1,\dots,x^{3i},\dots,x^{3n})^T,$$

we have

$$(1, X, \dots, X^n)^T = \begin{bmatrix} (r_0(P^{(\bar{1}_3)}); \bar{0}_{3n}) \\ (0; r_1(P^{(\bar{1}_3)}); \bar{0}_{3(n-1)}) \\ \cdots \\ (\bar{0}_n; r_n(P^{(\bar{1}_3)})) \end{bmatrix} (1, \dots, x^{3n})^T = O^{(\bar{1}_3)}(1, \dots, x^{3n})^T.$$

Thus it follows

$$r_n(P^{(\bar{1}_4)})(1, \dots, x^{3n})^T = r_n(P)O^{(\bar{1}_3)}(1, \dots, x^{3n})^T$$

and $r_n(P^{(\bar{1}_4)}) = r_n(P)O^{(\bar{1}_3)}$ for all n. Hence $PO^{(\bar{1}_3)} = P^{(\bar{1}_4)}$. This process can be extended to a k degree polynomial $f(x) = x^k + \cdots + x + 1$ so that $f^n(x) = (X+1)^n$ with $X = x(x^{k-1} + \cdots + 1)$ and $PO^{(\bar{1}_k)} = P^{(\bar{1}_{k+1})}$.

Consider $f(x) = x^k + \cdots + x + a$. Theorem 2.2 is for $k = 2$. When $k = 3$,

$$f^n(x) = (x^3 + x^2 + x + a)^n = r_n(P^{(\bar{1}_3, a)\uparrow})(1, x, \dots, x^{3n})^T.$$

Let $X = x^3 + x^2 + x$. Then $X^i = (\bar{0}_i; r_i(P^{(\bar{1}_3)}); \bar{0}_{3(n-i)})(1, \dots, x^i, \dots, x^{3n})^T$ and $(1, X, \dots, X^n)^T = O^{(\bar{1}_3)}(1, \dots, x^{3n})^T$, so $f^n(x) = r_n(P^{(1,a)\uparrow})O^{(\bar{1}_3)}(1, \dots, x^{3n})^T$. Hence we have $r_n(P^{(\bar{1}_3, a)\uparrow}) = r_n(P^{(1,a)\uparrow})O^{(\bar{1}_3)}$ and $P^{(\bar{1}_3, a)\uparrow} = P^{(1,a)\uparrow}O^{(\bar{1}_3)}$.

It is not hard to generalize to any polynomial $f(x) = x^k + \cdots + x + a$ for $k \geq 0$, hence we have $P^{(\bar{1}_k, a)\uparrow} = P^{(1,a)\uparrow}O^{(\bar{1}_k)}$. □

3. Recurrences on Obtuse matrix

The obtuse matrix $O^{(\bar{1}_k)}$ was shown to play an important role in creation $P^{(\bar{1}_{k+1})}$ from the Pascal matrix $P = P^{(1,1)}$. We study $O^{(\bar{1}_k)}$ explicitly.

Theorem 3.1. *$O^{(1,1)} = [o_{i,j}^{(1,1)}]$ with $o_{i,j}^{(1,1)} = \begin{cases} e_{i,j-i}^{(1,1)} & \text{if } 0 \leq i \leq j \leq 2i \\ 0 & \text{otherwise} \end{cases}$. Each column is $c_j(O^{(1,1)}) = (\bar{0}_{\lfloor \frac{j-1}{2} \rfloor + 1}; e^{(1,1)}_{\lfloor \frac{j-1}{2} \rfloor + 1, j - \lfloor \frac{j-1}{2} \rfloor - 1}, e^{(1,1)}_{\lfloor \frac{j-1}{2} \rfloor + 2, j - \lfloor \frac{j-1}{2} \rfloor - 2}, \dots, e_{j,0}^{(1,1)})^T$ satisfying $\big(0; (c_{j-1}(O^{(1,1)}) + c_j(O^{(1,1)}))\big) = c_{j+1}(O^{(1,1)})$ and $o_{i,j-1}^{(1,1)} + o_{i,j}^{(1,1)} = o_{i+1,j+1}^{(1,1)}$.*

Proof. Let $P^{(1,1)} = P = [e_{i,j}]$ and $O^{(1,1)} = O = [o_{i,j}]$. Then $r_i(O) = (\bar{0}_i; r_i(P))$ and $O = \begin{bmatrix} e_{0,0} & & & & & \\ & e_{1,0} & e_{1,1} & & & \\ & & e_{2,0} & e_{2,1} & e_{2,2} & \\ & & & e_{3,0} & e_{3,1} & e_{3,2} \\ & & & & e_{4,0} & e_{4,1} \cdots \\ & & & & & \cdots \end{bmatrix}$. Clearly $c_2(O) = \begin{bmatrix} 0 \\ e_{1,1} \\ e_{2,0} \\ \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ e_{0,0} \\ \cdots \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e_{1,0} \\ \cdots \end{bmatrix} = (0; c_0(O)) + (0; c_1(O))$. Moreover

$$c_3(O) = \begin{bmatrix} 0 \\ 0 \\ e_{2,1} \\ e_{3,0} \\ \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e_{1,0} + e_{1,1} \\ e_{2,0} \\ \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e_{1,0} \\ \cdots \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e_{1,1} \\ e_{2,0} \\ \cdots \end{bmatrix} = (0; c_1(O)) + (0; c_2(O))$$

$$c_4(O) = \begin{bmatrix} 0 \\ 0 \\ e_{2,2} \\ e_{3,1} \\ e_{4,0} \\ \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e_{1,1} \\ e_{2,0} + e_{2,1} \\ e_{3,0} \\ \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e_{1,1} \\ e_{2,0} \\ \cdots \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ e_{2,1} \\ e_{3,0} \\ \cdots \end{bmatrix} = (0; c_2(O)) + (0; c_3(O))$$

$$c_5(O) = \begin{bmatrix} \left.\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\}3 \\ e_{3,2} \\ e_{4,1} \\ e_{5,0} \\ \cdots \end{bmatrix} = \begin{bmatrix} \left.\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\}3 \\ e_{2,1} \\ e_{3,0} \\ \cdots \end{bmatrix} + \begin{bmatrix} \left.\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\}3 \\ e_{2,2} \\ e_{3,1} \\ e_{4,0} \\ \cdots \end{bmatrix} = (0; c_3(O)) + (0; c_4(O))$$

Thus any j^{th}column is $c_j(O) = \begin{bmatrix} \left.\begin{matrix} 0 \\ \cdots \\ 0 \end{matrix}\right\} \lfloor \frac{j-1}{2} \rfloor + 1 \\ \left.\begin{matrix} e_{\lfloor \frac{j-1}{2} \rfloor+1, j-\lfloor \frac{j-1}{2} \rfloor-1} \\ e_{\lfloor \frac{j-1}{2} \rfloor+2, j-\lfloor \frac{j-1}{2} \rfloor-2} \\ \cdots \\ e_{j,0} \end{matrix}\right\} j - \lfloor \frac{j-1}{2} \rfloor \end{bmatrix}$ where $e_{j,0} = e_{\lfloor \frac{j-1}{2} \rfloor+1+(j-\lfloor \frac{j-1}{2} \rfloor-1), j-\lfloor \frac{j-1}{2} \rfloor-1-(j-\lfloor \frac{j-1}{2} \rfloor-1)}$, and satisfies a recurrence

$$(0; c_{j-1}(O)) + (0; c_j(O))$$
$$= \begin{bmatrix} 0 \\ \left.\begin{matrix} 0 \\ \cdots \\ 0 \end{matrix}\right\} \lfloor \frac{j-2}{2} \rfloor + 1 \\ e_{\lfloor \frac{j-2}{2} \rfloor+1, j-1-\lfloor \frac{j-2}{2} \rfloor-1} \\ e_{\lfloor \frac{j-2}{2} \rfloor+2, j-1-\lfloor \frac{j-2}{2} \rfloor-2} \\ \cdots \\ e_{j-1,0} \end{bmatrix} + \begin{bmatrix} 0 \\ \left.\begin{matrix} 0 \\ \cdots \\ 0 \end{matrix}\right\} \lfloor \frac{j-1}{2} \rfloor + 1 \\ e_{\lfloor \frac{j-1}{2} \rfloor+1, j-\lfloor \frac{j-1}{2} \rfloor-1} \\ e_{\lfloor \frac{j-1}{2} \rfloor+2, j-\lfloor \frac{j-1}{2} \rfloor-2} \\ \cdots \\ e_{j,0} \end{bmatrix} = \begin{bmatrix} \left.\begin{matrix} 0 \\ \cdots \\ 0 \end{matrix}\right\} \lfloor \frac{j}{2} \rfloor + 1 \\ e_{\lfloor \frac{j}{2} \rfloor+1, j+1-\lfloor \frac{j}{2} \rfloor-1} \\ e_{\lfloor \frac{j}{2} \rfloor+2, j+1-\lfloor \frac{j}{2} \rfloor-2} \\ \cdots \\ e_{j+1,0} \end{bmatrix}$$
$$= c_{j+1}(O).$$

Indeed when $j = 2t$, $e_{\lfloor \frac{j-2}{2} \rfloor+1, j-\lfloor \frac{j-2}{2} \rfloor-2} + e_{\lfloor \frac{j-1}{2} \rfloor+1, j-\lfloor \frac{j-1}{2} \rfloor-1} = e_{t,t-1} + e_{t,t} = e_{t+1,t} = e_{\lfloor \frac{j}{2} \rfloor+1, j-\lfloor \frac{j}{2} \rfloor}$, and so on. On the other hand when $j = 2t+1$, $e_{\lfloor \frac{j-2}{2} \rfloor+1, j-\lfloor \frac{j-2}{2} \rfloor-2} = e_{\lfloor \frac{j}{2} \rfloor+1, j-\lfloor \frac{j}{2} \rfloor}$ and $e_{\lfloor \frac{j-2}{2} \rfloor+2, j-\lfloor \frac{j-2}{2} \rfloor-3} + e_{\lfloor \frac{j-1}{2} \rfloor+1, j-\lfloor \frac{j-1}{2} \rfloor-1} = e_{\lfloor \frac{j}{2} \rfloor+1, j-\lfloor \frac{j}{2} \rfloor}$, etc.

Clearly we also have $o_{i,j-1} + o_{i,j} = e_{i,j-i-1} + e_{i,j-i} = e_{i+1,j-i} = o_{i+1,j+1}$. □

Theorem 3.2. *Let* $P^{(\bar{1}_k)} = [e^{(\bar{1}_k)}_{i,j}]$, $O^{(\bar{1}_k)} = [o^{(\bar{1}_k)}_{i,j}]$. *Then* $c_j(O^{(\bar{1}_k)}) = (\bar{0}_{\lfloor \frac{j-1}{k} \rfloor+1}; e^{(\bar{1}_k)}_{\lfloor \frac{j-1}{k} \rfloor+1, j-\lfloor \frac{j-1}{k} \rfloor-1}, \dots, e^{(\bar{1}_k)}_{j,0})^T$ *satisfying* $\big(0; (c_{j-k+1}(O^{(\bar{1}_k)}) + \cdots + c_{j-1}(O^{(\bar{1}_k)}) + c_j(O^{(\bar{1}_k)}))\big) = c_{j+1}(O^{(\bar{1}_k)})$ *and* $o^{(\bar{1}_k)}_{i,j-k+1} + \cdots + o^{(\bar{1}_k)}_{i,j-1} + o^{(\bar{1}_k)}_{i,j} = o^{(\bar{1}_k)}_{i+1,j+1}$.

Proof. When $k = 3$, $O^{(\bar{1}_3)} = \begin{bmatrix} e^{(\bar{1}_3)}_{0,0} & & & & \\ & e^{(\bar{1}_3)}_{1,0} e^{(\bar{1}_3)}_{1,1} e^{(\bar{1}_3)}_{1,2} & & & \\ & & e^{(\bar{1}_3)}_{2,0} e^{(\bar{1}_3)}_{2,1} e^{(\bar{1}_3)}_{2,2} & \cdots \\ & & & \cdots \end{bmatrix} = \begin{bmatrix} r_0(P^{(\bar{1}_3)}) \\ (0; r_1(P^{(\bar{1}_3)})) \\ (\bar{0}_2; r_2(P^{(\bar{1}_3)})) \\ \cdots \end{bmatrix}$, so

$$c_4(O^{(\bar{1}_3)}) = \begin{bmatrix} 0 \\ 0 \\ e^{(\bar{1}_3)}_{2,2} \\ e^{(\bar{1}_3)}_{3,1} \\ e^{(\bar{1}_3)}_{4,0} \\ \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e^{(\bar{1}_3)}_{1,0} + e^{(\bar{1}_3)}_{1,1} + e^{(\bar{1}_3)}_{1,2} \\ e^{(\bar{1}_3)}_{2,0} + e^{(\bar{1}_3)}_{2,1} \\ e^{(\bar{1}_3)}_{3,0} \\ \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e^{(\bar{1}_3)}_{1,0} \\ \cdots \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^{(\bar{1}_3)}_{1,1} \\ e^{(\bar{1}_3)}_{2,0} \\ \cdots \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^{(\bar{1}_3)}_{1,2} \\ e^{(\bar{1}_3)}_{2,1} \\ e^{(\bar{1}_3)}_{3,0} \\ \cdots \end{bmatrix}$$
$$= (0; c_1(O^{(\bar{1}_3)})) + (0; c_2(O^{(\bar{1}_3)})) + (0; c_3(O^{(\bar{1}_3)}))$$

$$c_5(O^{(\bar{1}_3)}) = \begin{bmatrix} 0 \\ 0 \\ e^{(\bar{1}_3)}_{2,3} \\ e^{(\bar{1}_3)}_{3,2} \\ e^{(\bar{1}_3)}_{4,1} \\ e^{(\bar{1}_3)}_{5,0} \\ \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e^{(\bar{1}_3)}_{1,1} + e^{(\bar{1}_3)}_{1,2} \\ e^{(\bar{1}_3)}_{2,0} + e^{(\bar{1}_3)}_{2,1} + e^{(\bar{1}_3)}_{2,2} \\ e^{(\bar{1}_3)}_{3,1} + e^{(\bar{1}_3)}_{2,1} \\ e^{(\bar{1}_3)}_{4,0} \ \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e^{(\bar{1}_3)}_{1,1} \\ e^{(\bar{1}_3)}_{2,0} \\ \cdots \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^{(\bar{1}_3)}_{1,2} \\ e^{(\bar{1}_3)}_{2,1} \\ e^{(\bar{1}_3)}_{3,0} \\ \cdots \end{bmatrix} + \begin{bmatrix} \left.\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\} 3 \\ e^{(\bar{1}_3)}_{2,2} \\ e^{(\bar{1}_3)}_{3,1} \\ e^{(\bar{1}_3)}_{4,0} \\ \cdots \end{bmatrix}$$
$$= (0; c_2(O^{(\bar{1}_3)})) + (0; c_3(O^{(\bar{1}_3)})) + (0; c_4(O^{(\bar{1}_3)})).$$

And

$$c_6(O^{(\bar{1}_3)}) = \begin{bmatrix} 0 \\ 0 \\ e^{(\bar{1}_3)}_{1,2} \\ e^{(\bar{1}_3)}_{2,1} \\ e^{(\bar{1}_3)}_{3,0} \\ \vdots \end{bmatrix} + \begin{bmatrix} \left.\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\}3 \\ e^{(\bar{1}_3)}_{2,2} \\ e^{(\bar{1}_3)}_{3,1} \\ e^{(\bar{1}_3)}_{4,0} \\ \vdots \end{bmatrix} + \begin{bmatrix} \left.\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\}3 \\ e^{(\bar{1}_3)}_{2,3} \\ e^{(\bar{1}_3)}_{3,2} \\ e^{(\bar{1}_3)}_{4,1} \\ e^{(\bar{1}_3)}_{5,0} \\ \vdots \end{bmatrix} = \big(0; (c_3(O^{(\bar{1}_3)}) + c_4(O^{(\bar{1}_3)}) + c_5(O^{(\bar{1}_3)}))\big).$$

Thus any j^{th}column is $c_j(O^{(\bar{1}_3)}) = \begin{bmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\}\lfloor\frac{j-1}{3}\rfloor + 1 \\ \left.\begin{matrix} e^{(\bar{1}_3)}_{\lfloor\frac{j-1}{3}\rfloor+1,\,j-\lfloor\frac{j-1}{3}\rfloor-1} \\ e^{(\bar{1}_3)}_{\lfloor\frac{j-1}{3}\rfloor+2,\,j-\lfloor\frac{j-1}{2}\rfloor-2} \\ \vdots \\ e^{(\bar{1}_3)}_{j,0} \end{matrix}\right\}j - \lfloor\frac{j-1}{3}\rfloor \end{bmatrix}$ where $e^{(\bar{1}_3)}_{j,0} = e^{(\bar{1}_3)}_{\lfloor\frac{j-1}{3}\rfloor+(j-\lfloor\frac{j-1}{3}\rfloor),\,j-\lfloor\frac{j-1}{3}\rfloor-(j-\lfloor\frac{j-1}{3}\rfloor)}$. And

$$\begin{aligned}
&(0; c_{j-2}(O^{(\bar{1}_3)})) + (0; c_{j-1}(O^{(\bar{1}_3)})) + (0; c_j(O^{(\bar{1}_3)})) \\
&= \begin{bmatrix} 0 \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\}\lfloor\frac{j-3}{3}\rfloor + 1 \\ e^{(\bar{1}_3)}_{\lfloor\frac{j-3}{3}\rfloor+1,\,j-\lfloor\frac{j-3}{3}\rfloor-3} \\ e^{(\bar{1}_3)}_{\lfloor\frac{j-3}{3}\rfloor+2,\,j-\lfloor\frac{j-3}{3}\rfloor-4} \\ \vdots \\ e^{(\bar{1}_3)}_{j-2,0} \end{bmatrix} + \begin{bmatrix} 0 \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\}\lfloor\frac{j-2}{3}\rfloor + 1 \\ e^{(\bar{1}_3)}_{\lfloor\frac{j-2}{3}\rfloor+1,\,j-\lfloor\frac{j-2}{3}\rfloor-2} \\ e^{(\bar{1}_3)}_{\lfloor\frac{j-2}{3}\rfloor+2,\,j-\lfloor\frac{j-2}{3}\rfloor-3} \\ \vdots \\ e^{(\bar{1}_3)}_{j-1,0} \end{bmatrix} + \begin{bmatrix} 0 \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\}\lfloor\frac{j-1}{3}\rfloor + 1 \\ e^{(\bar{1}_3)}_{\lfloor\frac{j-1}{3}\rfloor+1,\,j-\lfloor\frac{j-1}{3}\rfloor-1} \\ e^{(\bar{1}_3)}_{\lfloor\frac{j-1}{3}\rfloor+2,\,j-\lfloor\frac{j-1}{3}\rfloor-2} \\ \vdots \\ e^{(\bar{1}_3)}_{j,0} \end{bmatrix} \\
&= \begin{bmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\}\lfloor\frac{j}{3}\rfloor + 1 \\ e^{(\bar{1}_3)}_{\lfloor\frac{j}{3}\rfloor+1,\,j-\lfloor\frac{j}{3}\rfloor} \\ e^{(\bar{1}_3)}_{\lfloor\frac{j}{3}\rfloor+2,\,j-\lfloor\frac{j}{3}\rfloor-1} \\ \vdots \\ e^{(\bar{1}_3)}_{j+1,0} \end{bmatrix} = c_{j+1}(O^{(\bar{1}_3)}), \text{ because } e^{(\bar{1}_3)}_{i,j-2} + e^{(\bar{1}_3)}_{i,j-1} + e^{(\bar{1}_3)}_{i,j} = e^{(\bar{1}_3)}_{i+1,j}.
\end{aligned}$$

In fact, for instance, if $j = 3t+2$ then $\lfloor\frac{j-3}{3}\rfloor = t-1$ and $\lfloor\frac{j-2}{3}\rfloor = \lfloor\frac{j-1}{3}\rfloor = t$, so

$$\begin{aligned}
&(0; c_{j-2}(O^{(\bar{1}_3)})) + (0; c_{j-1}(O^{(\bar{1}_3)})) + (0; c_j(O^{(\bar{1}_3)})) \\
&= \begin{bmatrix} 0 \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\}t \\ e^{(\bar{1}_3)}_{t,2t} \\ e^{(\bar{1}_3)}_{t+1,2t-1} \\ \vdots \\ e^{(\bar{1}_3)}_{3t,0} \end{bmatrix} + \begin{bmatrix} 0 \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\}t+1 \\ e^{(\bar{1}_3)}_{t+1,2t} \\ e^{(\bar{1}_3)}_{t+2,2t-1} \\ \vdots \\ e^{(\bar{1}_3)}_{3t+1,0} \end{bmatrix} + \begin{bmatrix} 0 \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\}t+1 \\ e^{(\bar{1}_3)}_{t+1,2t+1} \\ e^{(\bar{1}_3)}_{t+2,2t} \\ \vdots \\ e^{(\bar{1}_3)}_{3t+2,0} \end{bmatrix} = \begin{bmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\}t+1 \\ e^{(\bar{1}_3)}_{t+1,2t+2} \\ e^{(\bar{1}_3)}_{t+2,2t+1} \\ \vdots \\ e^{(\bar{1}_3)}_{3t+3,0} \end{bmatrix} = c_{j+1}(O^{(\bar{1}_3)}).
\end{aligned}$$

Moreover we clearly have

$o^{(\bar{1}_3)}_{i,j-2} + o^{(\bar{1}_3)}_{i,j-1} + o^{(\bar{1}_3)}_{i,j} = e^{(\bar{1}_3)}_{i,j-2-i} + e^{(\bar{1}_3)}_{i,j-1-i} + e^{(\bar{1}_3)}_{i,j-i} = e^{(\bar{1}_3)}_{i+1,j-i} = o^{(\bar{1}_3)}_{i+1,j+1}$.

These can be extended to $O^{(\bar{1}_k)}$ that $\big(0; (c_{j-k+1}(O^{(\bar{1}_k)}) + \cdots + c_j(O^{(\bar{1}_k)}))\big) = c_{j+1}(O^{(\bar{1}_k)})$ and $c_j(O^{(\bar{1}_k)}) = (\bar{0}_{\lfloor\frac{j-1}{k}\rfloor+1}; e^{(\bar{1}_k)}_{\lfloor\frac{j-1}{k}\rfloor+1,\,j-\lfloor\frac{j-1}{k}\rfloor-1}, \ldots, e^{(\bar{1}_k)}_{j,0})^T$. □

Theorem 3,2 and Theorem 2.1 give a way to get $P^{(\bar{1}_k)}$ explicitly. Let $r^{[t]}_i(P^{(\bar{1}_3)})$ be the i^{th}row $r_i(P^{(\bar{1}_3)})$ deleted the first t entries, i.e., $r^{[t]}_i(P^{(\bar{1}_3)}) = (e^{(\bar{1}_3)}_{i,t}, e^{(\bar{1}_3)}_{i,t+1}, \ldots)$.

For example, $e^{(\bar{1}_3)}_{6,j} = r_6(P)c_j(O) = r^{[1]}_6(P)(c_{j-2}(O) + c_{j-1}(O))$ for $j > 1$. So
$e^{(\bar{1}_3)}_{6,3} = r^{[1]}_6(P)(c_1(O) + c_2(O)) = r^{[2]}_6(P)((e_{1,0}) + (e_{1,1}, e_{2,0}))^T$
$\quad = r^{[2]}_6(P)(e_{1,0} + e_{1,1}, e_{2,0})^T = (e_{6,2}, e_{6,3})(e_{2,1}, e_{2,0})^T = (15, 20)(2, 1)^T = 50,$
$e^{(\bar{1}_3)}_{6,4} = r^{[1]}_6(P)(c_2(O) + c_3(O)) = r^{[2]}_6(P)((e_{1,1}, e_{2,0}) + (0, e_{2,1}, e_{3,0}))^T$
$\quad = r^{[2]}_6(P)(e_{1,1}, e_{2,0} + e_{2,1}, e_{3,0})^T = (e_{6,2}, e_{6,3}, e_{6,4})(e_{1,1}, e_{3,1}, e_{3,0})^T = 90,$
$e^{(\bar{1}_3)}_{6,5} = r^{[3]}_6(P)(e_{2,1} + e_{2,2}, e_{3,0} + e_{3,1}, e_{4,0})^T = 126,$
etc, so it shows $r_6(P^{(\bar{1}_3)}) = (1, 6, 21, 50, 90, 126, 141, 126, 90, 50, 21, 6, 1)$.

4. Generating polynomial of $O^{(1,1)^k}$

Let $P(g)$ be an AM of a polynomial $g^n(x)$ and $O(g)$ be the obtuse matrix of $P(g)$. If $f_1(x) = x + 1$ then $P(f_1) = P^{(1,1)}$ and $O(f_1) = O^{(1,1)}$. And $P(f_1)^k = P(kx+1)^{\downarrow}$ and $P(f_1)^k O(f_1) = P(kx^2+x+1)^{\downarrow}$ for any k by Theorems 2.1 and 2.2.

Theorem 4.1. *$O(f_1)^i = O(f_i)$ with $f_1(x) = x+1$, $f_2(x) = (1, 2, 2, 1)(x^3, x^2, x, 1)^T$, $f_3(x) = (1, 3, 6, 9, 10, 8, 4, 1)(x^7, \ldots, x, 1)^T$ and $f_4(x) = (1, 4, 12, 30, 64, 118, 188, 258, 310, 298, 244, 162, 84, 32, 8, 1)(x^{15}, \ldots, x, 1)^T$.*

Proof. Simple matrix multiplications show $O(f_1)^2 = \begin{bmatrix} 1 \\ & 1 & 2 & 2 & 1 \\ & & 1 & 4 & 8 & 10 & 8 & 4 & 1 \\ & & & 1 & 6 & 18 & 35 & 48 & 48 \\ & & & & 1 & 8 & 32 & 84 & \ldots \end{bmatrix}$ for $P(f_1) = P$ and $O(f_1) = O$. Let $f_2(x) = x^3 + 2x^2 + 2x + 1$. By expanding $f_2^i(x)$ for $i \geq 0$, we have an AM $P(f_2) = \begin{bmatrix} 1 \\ 1 & 2 & 2 & 1 \\ 1 & 4 & 8 & 10 & 8 & 4 & 1 \\ 1 & 6 & 18 & 35 & 48 & 48 & \ldots \end{bmatrix} = [u_{i,j}]$ satisfying

$u_{i,j-2} + 2u_{i,j-1} + 2u_{i,j} + u_{i,j+1} = u_{i+1,j+1}$.

So the i^{th}row $r_i(P(f_2))$ are $r_4(P(f_2)) = (1, 8, 32, 84, 160, 232, 262, 232, 160, 84, 32, 8, 1)$, and $r_5(P(f_2)) = (1, 10, 50, 165, 400, 752, 1130, 1380, 1380, 1130, 752, 400, 165, 50, 10, 1)$, and so on, which are sequences of coefficients of expansions of $f_2^i(x)$. Hence the obtuse matrix is $O(f_2) = \begin{bmatrix} 1 \\ & 1 & 2 & 2 & 1 \\ & & 1 & 4 & 8 & 10 & 8 & 4 & 1 \\ & & & 1 & 6 & 18 & 35 & 48 & 48 & 35 & 18 & 6 & 1 \\ & & & & 1 & 8 & 32 & 84 & 160 & 232 & 262 & 232 & \ldots \end{bmatrix}$ which equals $O(f_1)^2$. We also notice $O(f_1)^3 = O^3 = \begin{bmatrix} 1 \\ & 1 & 3 & 6 & 9 & 10 & 8 & 4 & 1 \\ & & 1 & 6 & 21 & 54 & 110 & 184 & 257 \\ & & & 1 & 9 & 45 & 162 & 462 & 1095 \\ & & & & 1 & 12 & 78 & 360 & \ldots \end{bmatrix}$. By letting $f_3(x) = x^7 + 3x^6 + 6x^5 + 9x^4 + 10x^3 + 8x^2 + 4x + 1$ and expanding $f_3^i(x)$ for $i \geq 0$, we have AM $P(f_3)$. In fact, $r_i(P(f_3))$ are the coefficients of $f_3^i(x)$:

$r_2(P(f_3)) = (1, 6, 21, 54, 110, 184, 257, 302, 298, 244, 162, 84, 32, 8, 1)$

$r_3(P(f_3)) = (1, 9, 45, 162, 462, 1095, 2217, 3900, 6024, 8220, 9936, 10638, \ldots)$.

So $P(f_3) = \begin{bmatrix} 1 \\ 1 & 3 & 6 & 9 & 10 & 8 & 4 & 1 \\ 1 & 6 & 21 & 54 & 110 & 184 & 257 & 298 \\ 1 & 9 & 45 & 162 & 462 & 1095 & 2217 & \ldots \end{bmatrix} = [u_{i,j}]$ and it satisfies a recurrence

$u_{i,j-6} + 4u_{i,j-5} + 8u_{i,j-4} + 10u_{i,j-3} + 9u_{i,j-2} + 6u_{i,j-1} + 3u_{i,j} + u_{i,j+1} = u_{i+1,j+1}$,

from which i^{th}rows $r_i(P(f_3))$ are obtained that

$r_4(P(f_3)) = (1, 12, 78, 360, 1309, 3956, 10258, 23296, 46986, 84984, \dots),$

etc. Hence the obtuse matrix is $O(f_3) = \begin{bmatrix} 1 & & & & & & & & & & & \\ & 1 & 3 & 6 & 9 & 10 & 8 & 4 & 1 & & & \\ & & 1 & 6 & 21 & 54 & 110 & 184 & 257 & 302 & & \\ & & & 1 & 9 & 45 & 162 & 462 & 1095 & 2217 & & \\ & & & & 1 & 12 & 78 & 360 & 1309 & \dots & & \end{bmatrix} = O(f_1)^3.$

Note $O(f_1)^4 = \begin{bmatrix} 1 & & & & & & & & & & & & & & & & \\ & 1 & 4 & 12 & 30 & 64 & 118 & 188 & 258 & 302 & 298 & 244 & 162 & 84 & 32 & 8 & 1 \\ & & 1 & 8 & 40 & 156 & 512 & 1468 & 3756 & 8692 & \dots & & & & & & \\ & & & 1 & 12 & 84 & 442 & 1920 & 7218 & 24144 & \dots & & & & & & \end{bmatrix}$. Considering $f_4(x) = (1, 4, 12, 30, 64, 118, 188, 258, 310, 298, 244, 162, 84, 32, 8, 1)(x^{15}, \dots, 1)^T$, we get $P(f_4) = \begin{bmatrix} 1 & & & & & & & & & & & \\ 1 & 4 & 12 & 30 & 64 & 118 & 188 & 258 & 302 & 298 & 244, & 162, 84, 32, 8, 1 \\ 1 & 8 & 40 & 156 & 512 & 1468 & 3756 & 8692 & 18356 & 35588 & 63588 & \dots \\ 1 & 12 & 84 & 442 & 1920 & 7218 & 24144 & \dots & & & & \end{bmatrix} =$ $[u_{i,j}]$ satisfying $u_{i,j-14} + 8u_{i,j-13} + 32u_{i,j-12} + 84u_{i,j-11} + 162u_{i,j-10} + 244u_{i,j-9} + 298u_{i,j-8} + 302u_{i,j-7} + 258u_{i,j-6} + 188u_{i,j-5} + 118u_{i,j-4} + 64u_{i,j-3} + 30u_{i,j-2} + 12u_{i,j-1} + 4u_{i,j} + u_{i,j+1} = u_{i+1,j+1}$. So it is observed $O(f_4) = O(f_1)^4$. □

The polynomials $f_i(x)$ such that $O(f_i) = O^i$ $(1 \le i \le 4)$ in Theorem 4.1 are $f_2(x) = (x+1)(x^2+x+1) = f_1(x)(x^2 + f_1(x))$, $f_3(x) = (x+1)(x^2+x+1)(x^4 + x^3 + 2x^2 + 2x + 1) = f_2(x)(x^4 + f_2(x))$ and

$$f_4(x) = (x+1)(x^2+x+1)(x^4+x^3+2x^2+2x+1)$$
$$(x^8 + x^7 + 3x^6 + 6x^5 + 9x^4 + 10x^3 + 8x^2 + 4x + 1) = f_3(x)(x^8 + f_3(x)).$$

Theorem 4.2. *$O(f_i) = O^i$ with $f_{i+1}(x) = f_i(x)(x^{2^i} + f_i(x))$ and $f_1(x) = x+1$.*

Proof. For $f_2(x) = x^2 f_1(x) + f_1^2(x)$, since $x^2 f_1(x) = x^2 r_1(P)(x, 1)^T = (1, 1)(x^3, x^2)^T$ and $f_1^2(x) = r_2(P)(x^2, x, 1)^T = (1, 2, 1)(x^2, x, 1)^T$, we have

$$f_2(x) = (1, 1)(x^3, x^2)^T + (0, 1, 2, 1)(x^3, x^2, x, 1)^T = (1, 2, 2, 1)(x^3, x^2, x, 1)^T,$$

so $r_1(O(f_2)) = (0, 1, 2, 2, 1)$. But since $r_1(O)$ is of size 1×3, we have

$$r_1(O^2) = r_1(O)[c_0(O)| \cdots |c_4(O)] = (0, 1, 1) \begin{bmatrix} e_{0,0} & & & & \\ & e_{1,0} & e_{1,1} & & \\ & & e_{2,0} & e_{2,1} & e_{2,2} \end{bmatrix} = (0, 1, 2, 2, 1),$$

so $r_1(O^2) = r_1(O(f_2))$ and $O^2 = O(f_2)$.

From $x^4 f_2(x) = x^4 r_1(P(f_2))(x^3, \dots, 1)^T = (1, 2, 2, 1)\ (x^7, \dots, x^4)^T$ and $f_2^2(x) = r_2(P(f_2))(x^6, \dots, 1)^T = (1, 4, 8, 10, 8, 4, 1)(x^6, \dots, 1)^T$, we have $f_3(x) = x^4 f_2(x) + f_2^2(x) = (1, 3, 6, 9, 10, 8, 4, 1)(x^7, \dots, 1)^T$ so $r_1(O(f_3)) = (0, 1, 3, 6, 9, 10, 8, 4, 1)$. Note $r_1(O^2)$ is of size 1×5, and $\lfloor \frac{j-1}{2} \rfloor + 1 \ge 5$ if $j \ge 9$. Thus for $j \ge 9$, $c_j(O)$ begins with more than 5 zeros by Theorem 3.1, so $r_1(O^2)c_j(O) = 0$. Hence

$$r_1(O^3) = r_1(O^2)[c_0(O)| \cdots |c_8(O)] = (0, 1, 2, 2, 1) \begin{bmatrix} 1 & & & & & & & & \\ & 1 & 1 & & & & & & \\ & & 1 & 2 & 1 & & & & \\ & & & 1 & 3 & 3 & 1 & & \\ & & & & 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$
$$= (0, 1, 3, 6, 9, 10, 8, 4, 1) = r_1(O(f_3)).$$

Similarly $x^8 f_3(x) = x^8 r_1(P(f_3))(x^7, \dots, 1)^T = (1, 3, 6, 9, 10, 8, 4, 1)(x^{15}, \dots, x^8)^T$ and $f_3^2(x) = r_2(P(f_3))(x^{14}, \dots, 1)^T = (1, 6, 21, 54, 110, 184, 257, 302, 298, 244, 162, 84, 32, 8, 1)(x^{14}, \dots, x, 1)^T$ yield

$$f_4(x) = x^8 f_3(x) + f_3^2(x)$$
$$= (1, 4, 12, 30, 64, 118, 188, 258, 302, 298, 244, 162, 84, 32, 8, 1)(x^{15}, \dots, x, 1)^T.$$

Thus $r_1(O(f_4)) = (0, 1, 4, 12, 30, 64, 118, 188, 258, 302, 298, 244, 162, 84, 32, 8, 1)$.

Note $r_1(O^3)$ is of size 1×9, and $\lfloor \frac{j-1}{2} \rfloor + 1 \geq 9$ if $j \geq 17$. Thus for $j \geq 17$, $c_j(O)$ begins with more than 9 zeros by Theorem 3.1 so $r_1(O^3)c_j(O) = 0$. Hence

$$r_1(O^4)= r_1(O^3)[c_0(O)|\cdots|c_{16}(O)] = (0,1,3,6,9,10,8,4,1)\begin{bmatrix}1 \\ 1\,1 \\ 1\,2\,1 \\ \cdots \\ 1,8,28,\ldots,1\end{bmatrix}$$

$$= (0,1,4,12,30,64,118,188,258,302,298,244,162,84,32,8,1) = r_1(O(f_4)).$$

We now assume $O^k = O(f_k)$ for some k. Since $\deg f_k(x) = 2^k - 1$, write $f_k(x) = (1, a_{2^k-2}, \ldots, a_1, 1)(x^{2^k-1}, \ldots, 1)^T$. Then $r_1(P(f_k)) = (1, a_{2^k-2}, \ldots, 1)$ and $P(f_k) = [u_{i,j}]$ hold a recurrence

$(1, a_1, \ldots, a_{2^k-3}, a_{2^k-2}, 1) \circ (u_{i,j-2^k+1}, \ldots, u_{i,j-1}, u_{i,j}, u_{i,j+1}) = u_{i+1,j+1}$.

Hence the 2^{nd}row $r_2(P(f_k))$ is obtained from $r_1(P(f_k))$ that

$$r_2(P(f_k)) = (1, (a_{2^k-2}, 1) \circ (1, a_{2^k-2}), (a_{2^k-3}, a_{2^k-2}, 1) \circ (1, a_{2^k-2}, a_{2^k-3}), (a_{2^k-4}, a_{2^k-3}, a_{2^k-2}, 1) \circ (1, a_{2^k-2}, a_{2^k-3}, a_{2^k-4}), \ldots, 1),$$

thus it gives

$$f_k^2(x) = (1, (a_{2^k-2}, 1) \circ (1, a_{2^k-2}), (a_{2^k-3}, a_{2^k-2}, 1) \circ (1, a_{2^k-2}, a_{2^k-3}), (a_{2^k-4}, a_{2^k-3}, a_{2^k-2}, 1) \circ (1, a_{2^k-2}, a_{2^k-3}, a_{2^k-4}), \ldots, 1)\,(x^{2^{k+1}-2}, \ldots, 1)^T.$$

Therefore

$$f_{k+1}(x) = x^{2^k} f_k(x) + f_k^2(x)$$
$$= (1, a_{2^k-2}, a_{2^k-3}, \ldots, a_1, 1)(x^{2^{k+1}-1}, \ldots, x^{2^k+1}, x^{2^k})^T$$
$$+(0, 1, (a_{2^k-2}, 1) \circ (1, a_{2^k-2}), (a_{2^k-3}, a_{2^k-2}, 1) \circ (1, a_{2^k-2}, a_{2^k-3}),$$
$$(a_{2^k-4}, a_{2^k-3}, a_{2^k-2}, 1)\circ(1, a_{2^k-2}, a_{2^k-3}, a_{2^k-4}), \ldots, 1)(x^{2^{k+1}-1}, x^{2^{k+1}-2}, \ldots, 1)^T$$
$$= (1, a_{2^k-2}+1, a_{2^k-3}+2a_{2^k-2}, a_{2^k-4}+2a_{2^k-3}+a_{2^k-2}^2, \ldots)(x^{2^{k+1}-1}, \ldots, 1)^T.$$

Hence

$$r_1(O(f_{k+1})) = (0, 1, a_{2^k-2}+1, a_{2^k-3}+2a_{2^k-2}, a_{2^k-4}+2a_{2^k-3}+a_{2^k-2}^2, \ldots, 1)$$
$$= (0, 1, a_{2^k-2}, \ldots, a_1, 1)\begin{bmatrix}1 \\ 1\,1 \\ 1\,2\,1 \\ 1\,3\,3\,1 \\ \cdots\end{bmatrix} = r_1(O(f_k))O = r_1(O^k)O = r_1(O^{k+1})$$

by the inductive hypothesis. Therefore $O(f_{k+1}) = O^{k+1} = O(f_1)^{k+1}$. □

Corollary 4.3. *(1)* $f_i(x) = f_1(x)\prod_{k=1}^{i-1}(x^{2^k} + f_k(x))$ *of degree* $2^i - 1$ *for* $i > 1$.

(2) Let $F_0(x) = x$ *and* $F_{i+1}(x) = F_i(x + x^2)$ *for* $i \geq 0$. *Then* $f_i(x) = x^{-1}F_i(x)$.

These formulae of $f_i(x)$ are easily followed. The coefficients of x^i in $F_i(x)$ make the below matrix M which shows $r_i(M) = r_1(P(f_i))$ for all $i \geq 1$.

$$M = \begin{bmatrix}
1 \\
1\,1 \\
1\,2\,2\;\;1 \\
1\,3\,6\;\;9\;\;10\;\;8\;\;4\;\;1 \\
1\,4\,12\;30\;64\;118\;188\;258\;302\;298\;244\;162\;84,32,8,1 \\
1\,5\,20\;70\;220\;630\;1656\;4014\;8994\;18654\;35832\;63750\;\ldots \\
1\,6\,30\;135\;560\;2170\;7916\;27326\;89582\;279622\;832680\;\ldots \\
\cdots
\end{bmatrix}$$

See [5] for M. The next theorem on $P(f_i)$ is an analog of Theorem 2.2.

Theorem 4.4. $P(f_2) = P(f_1)\ O^{(1,2,2)\uparrow}$.

Proof. Note $P(f_1) = P^{(1,1)} = P$. Then $f_2^n(x) = (x^3 + 2x^2 + 2x + 1)^n = (X+1)^n = r_n(P)(1, X, \cdots, X^n)^T$ with $X = x(x^2 + 2x + 2)$. But since

$X^i = x^i r_i(P^{(1,2,2)\uparrow})(1, x, \dots, x^{2i})^T = (\bar{0}_i; r_i(P^{(1,2,2)\uparrow}); \bar{0}_{3(n-i)})(1, \dots, x^{3n})^T$,
we have

$$(1, \dots, X^n)^T = \begin{bmatrix} (r_0(P^{(1,2,2)\uparrow}); \bar{0}_{3n}) \\ (0; r_1(P^{(1,2,2)\uparrow}); \bar{0}_{3(n-1)}) \\ \dots \\ (\bar{0}_n; r_i(P^{(1,2,2)\uparrow})) \end{bmatrix} (1, \dots, x^{3n})^T = O^{(1,2,2)\uparrow}(1, \dots, x^{3n})^T.$$

It therefore follows that

$$r_n(P(f_2))(1, \cdots, x^{3n})^T = f_2^n(x) = r_n(P(f_1))O^{(1,2,2)\uparrow}(1, \cdots, x^{3n})^T,$$

hence $r_n(P(f_2)) = r_n(P(f_1))O^{(1,2,2)\uparrow}$ and $P(f_2) = P(f_1)O^{(1,2,2)\uparrow}$. □

Theorem 2.2 and Theorem 4.2 show $P^nO = P^{(n,1,1)}$ and $O^n = O(f_n)$. We ask about a generating polynomial $\theta_{PO^n}(x)$ of PO^n. Clearly $\theta_P(x) = x + 1 = f_1(x)$ and $\theta_{PO}(x) = x^2 + x + 1 = x^2 + \theta_P(x) = x^2 + f_1(x)$ because $PO = P^{(\bar{1}_3)}$. And $PO^2 = \begin{bmatrix} 1 \\ 1\,1\,2\,2\ \,1 \\ 1\,2\,5\,8\ \,10 \\ 1\,3\,9\,19 \dots \end{bmatrix}$, $PO^3 = \begin{bmatrix} 1 \\ 1\,1\,3\ \,6\ \,9\ \,10\ \,8\ \ \ 4, 1 \\ 1\,2\,7\ \,18\,39\,74\ \,126 \\ 1\,3\,12\,37\,99\,237 \dots \end{bmatrix}$ yield generating polynomials $\theta_{PO^2}(x) = x^4 + x^3 + 2x^2 + 2x + 1 = x^4 + f_2(x)$ and $\theta_{PO^3}(x) = x^8 + x^7 + 3x^6 + 6x^5 + 9x^4 + 10x^3 + 8x^2 + 4x + 1 = x^8 + f_3(x)$.

Theorem 4.5. *$\theta_{PO^n}(x) = x^{2^n} + f_n(x)$ and $f_{n+1}(x) = f_n(x)\theta_{PO^n}(x)$ for $n > 0$.*

Proof. Clearly $\theta_{PO^n}(x) = x^{2^n} + f_n(x)$ for $1 \le n \le 3$. Since $\deg f_n(x) = 2^n - 1$ with leading coefficient 1, write $f_n(x) = x^{2^n-1} + a_{2^n-2}x^{2^n-2} + \cdots + a_1x + 1$. Then

$$PO^n = PO(f_n) = \begin{bmatrix} 1 \\ 1\,1 \\ 1\,2\,1 \\ 1\,3\,3\,1 \\ 1\,4\,6\,4\,1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & 1\; a_{2^n-2} & \dots & a_1 & 1 \\ & & 1 & & \dots \\ & & & 1 & \dots \end{bmatrix} = \begin{bmatrix} 1\,0 \\ 1\,1\; a_{2^n-2} & \dots\; a_1\; 1 \\ 1\,2\; a_{2^n-2} + 1 \dots \end{bmatrix}$$

by Theorem 4.2, so the generating polynomial of PO^n is

$$\theta_{PO^n}(x) = x^{2^n} + x^{2^n-1} + a_{2^n-2}x^{2^n-2} + \cdots + a_1x + 1 = x^{2^n} + f_n(x). \qquad \square$$

Finally we prove that $P(f_n)$ is obtained from $P(f_{n-1})$ for all $n \ge 1$.

Theorem 4.6. *$r_i(P(f_n)) = \sum_{k=0}^{i} \binom{i}{k}(\bar{0}_k; r_{i+k}(P(f_{n-1})); \bar{0}_{2^{n-1}(i-k)})$ for all i.*

Proof. Note $P(f_1) = P^{(1,1)} = P$ and $f_{n+1}(x) = f_n(x)(x^{2^n} + f_n(x))$. Then from $f_2^2(x) = x^4 f_1^2(x) + 2x^2 f_1^3(x) + f_1^4(x)$ we have

$$r_2(P(f_2))(x^6, \cdots, x, 1)^T = f_2^2(x)$$
$$= r_2(P)(x^6, x^5, x^4)^T + 2r_3(P)(x^5, x^4, x^3, x^2)^T + r_4(P)(x^4, \dots, x, 1)^T$$
$$= ((r_2(P); \bar{0}_4) + 2(0; r_3(P); \bar{0}_2) + (\bar{0}_2; r_4(P)))\,(x^6, \dots, x, 1)^T.$$

Also $f_2^3(x) = x^6 f_1^3(x) + 3x^4 f_1^4(x) + 3x^2 f_1^5(x) + f_1^6(x)$ implies

$$r_3(P(f_2))(x^9, \dots, x, 1)^T = f_2^3(x)$$
$$= ((r_3(P); \bar{0}_6) + 3(0; r_4(P); \bar{0}_4) + 3(\bar{0}_2; r_5(P); \bar{0}_2) + (\bar{0}_3; r_6(P)))\,(x^9, \dots, 1)^T.$$

Thus any i^{th}row $r_i(P(f_2))$ of $P(f_2)$ satisfies

$$r_i(P(f_2))$$
$$= (r_i(P); \bar{0}_{2i}) + \binom{i}{1}(0; r_{i+1}(P); \bar{0}_{2i-2}) + \cdots + \binom{i}{i}(\bar{0}_i; r_{2i}(P))$$
$$= \sum_{k=0}^{i} \binom{i}{k}(\bar{0}_k; r_{i+k}(P); \bar{0}_{2(i-k)}).$$

Now for $f_3(x) = x^4 f_2(x) + f_2^2(x)$ of degree 7 and $P(f_3)$, we have
$r_1(P(f_3))(x^7, \dots, 1)^T = f_3(x) = r_1(P(f_2))(x^7, \dots, x^4)^T + r_2(P(f_2))(x^6, \dots, 1)^T$
$= ((r_1(P(f_2)); \bar{0}_4) + (0; r_2(P(f_2)))\,(x^7, \dots, x, 1)^T$.
And $f_3^2(x) = x^8 f_2^2(x) + 2x^4 f_2^3(x) + f_2^4(x)$ implies
$r_2(P(f_3))(x^{14}, \dots, x, 1)^T = f_3^2(x)$
$= r_2(P(f_2))(x^{14}, \dots, x^8)^T + 2r_3(P(f_2))(x^{13}, \dots, x^4)^T + r_4(P(f_2))(x^{12}, \dots, 1)^T$
$= ((r_2(P(f_2)); \bar{0}_8) + 2(0; r_3(P(f_2)); \bar{0}_4) + (\bar{0}_2; r_4(P(f_2))))\,(x^{14}, \dots, x, 1)^T$.
Therefore for any $i \geq 0$, it follows immediately

$$r_i(P(f_3)) = \binom{i}{0}(r_i(P(f_2)); \bar{0}_{4i}) + \cdots + \binom{i}{i}(\bar{0}_i; r_{i+i}(P(f_2)))$$
$$= \sum_{k=0}^{i} \binom{i}{k}(\bar{0}_k; r_{i+k}(P(f_2)); \bar{0}_{4(i-k)}).$$

Now for any n, since $\deg f_n(x) = 2^n - 1$ we have
$r_1(P(f_{n+1}))(x^{2^{n+1}-1}, \dots, x, 1)^T = f_{n+1}(x) = x^{2^n} f_n(x) + f_n^2(x)$
$= x^{2^n} r_1(P(f_n))(x^{2^n-1}, \dots, x, 1)^T + r_2(P(f_n))(x^{2(2^n-1)}, \dots, x, 1)^T$
$= ((r_1(P(f_n)); \bar{0}_{2^n}) + (0; r_2(P(f_n))))\,(x^{2^{n+1}-1}, \dots, x, 1)^T$,
so $r_1(P(f_{n+1})) = (r_1(P(f_n)); \bar{0}_{2^n}) + (0; r_2(P(f_n)))$.
Similarly from $f_{n+1}^2(x) = x^{2^{n+1}} f_n^2(x) + 2x^{2^n} f_n^3(x) + f_n^4(x)$, we have
$r_2(P(f_{n+1}))(x^{2(2^{n+1}-1)}, \dots, x, 1)^T = f_{n+1}^2(x)$
$= x^{2^{n+1}} r_2(P(f_n))(x^{2(2^n-1)}, \dots, 1)^T + 2x^{2^n} r_3(P(f_n))(x^{3(2^n-1)}, \dots, 1)^T$
$+ r_4(P(f_n))(x^{4(2^n-1)}, \dots, 1)^T$
$= ((r_2(P(f_n)); \bar{0}_{2^{n+1}}) + 2(0; r_3(P(f_n)); \bar{0}_{2^n}) + (\bar{0}_2; r_4(P(f_n))))\,(x^{2(2^{n+1}-1)}, \dots, 1)^T$,
so $r_2(P(f_{n+1})) = (r_2(P(f_n)); \bar{0}_{2^{n+1}}) + 2(0; r_3(P(f_n)); \bar{0}_{2^n}) + (\bar{0}_2; r_4(P(f_n)))$.
Continuing this process to i^{th}row of $P(f_{n+1})$, it follows that
$r_i(P(f_{n+1}))\ (x^{(2^{n+1}-1)i}, \dots, x, 1)^T = f_{n+1}^i(x)$

$$= \Big((r_i(P(f_n)); \bar{0}_{2^{n+1}}) + \binom{i}{1}(0; r_{i+1}(P(f_n)); \bar{0}_{2^n}) + \binom{i}{2}(\bar{0}_2; r_{i+2}(P(f_n)))$$
$$+ \cdots + (\bar{0}_i; r_{2i}(P(f_n)))\Big)\,(x^{(2^{n+1}-1)i}, \dots, x, 1)^T$$
$$= \sum_{k=0}^{i} \binom{i}{k}(\bar{0}_k; r_{i+k}(P(f_{n-1})); \bar{0}_{2^{n-1}(i-k)})(x^{(2^{n+1}-1)i}, \dots, x, 1)^T. \qquad \square$$

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Choi, Eunmi
Department of Mathematics, HanNam University, Daejeon, Korea
Email address: emc@hnu.kr