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EXISTENCE OF A POSITIVE SOLUTION TO INFINITE SEMIPOSITONE PROBLEMS

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Abstract. We establish an existence result for a positive solution to the Schrödinger-type singular semipositone problem: $-\Delta u + V(x)u = \lambda \frac{f(u)}{u^\alpha}$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N, N > 2, \lambda \in \mathbb{R}$ is a positive parameter, $V \in L^\infty(\Omega)$, $0 < \alpha < 1$, $f \in C([0,\infty), \mathbb{R})$ with $f(0) < 0$. In particular, when $\frac{f(s)}{s^\alpha}$ is sublinear at infinity, we establish the existence of a positive solutions for $\lambda \gg 1$. The proofs are mainly based on the sub and supersolution method. Further, we extend our existence result to infinite semipositone problems with mixed boundary conditions.

1. Introduction and a main Result

We are concerned with the existence of a positive solution of the following infinite semipositone Schrödinger-type problem with Dirichlet boundary condition

$$\begin{cases} -\Delta u + V(x)u = \lambda \frac{f(u)}{u^\alpha}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a nonempty bounded domain in $\mathbb{R}^N, N \geq 2$, with a smooth boundary $\partial\Omega$, $V \in L^\infty(\Omega)$, $\alpha \in (0,1)$ and $\lambda > 0$ is a real parameter. We assume that $f \in C([0,\infty), \mathbb{R})$ with $f(0) < 0$ satisfies the following hypotheses:

- $(H1)$ There exist $\gamma > 0$ and $A > 0$ such that $\alpha \leq \gamma < \alpha + 1$ and $f(s) \leq As^\gamma$ for $s \geq 0$.
- $(H2)$ There exist $0 < \beta < \frac{1}{2}(1-\alpha^2)$ and $B > 0$ such that $f(s) \geq Bs^\beta$ for $s \gg 1$.

We further assume that $V \in L^\infty(\Omega)$ satisfies the following condition:

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($H3$) There exists $c_v > 0$ such that $V(x) \geq -c_v > -\frac{1}{\|e\|_\infty}$ for $x \in \Omega$, where e is the positive solution of

$$\begin{cases} -\Delta e = 1, \text{ in } \Omega, \\ e = 0, \text{ on } \partial\Omega. \end{cases}$$

The equation (1) is derived based on the nonlinear Schrödinger equation, which is detailed in [23]. Nonlinear Schrödinger equations have been widely studied to investigate the existence of solutions that act according to V on the whole space $\mathbb{R}^N$ (see [4, 16, 21] or references therein) or on bounded domains with linear boundary conditions in [14].

In the case when $V \equiv 0$, the existence of positive solutions of (1) has dramatically changed according to the sign of $f(0)$. Studying existence of positive solutions of positone problems ($f(0) > 0$) has a rich history for a long time (see [7] and [22] for $\alpha = 0$ or references therein), and it is well known that there exists a positive solution for each $\lambda > 0$ under the assumption $\lim_{s\to\infty} \frac{f(s)}{s} = 0$. The condition $f(0) < 0$ (semipositone problem) causes mathematical challenge as pointed out by P. L. Lions [20]. However, in the past 30 years, there has been considerable progress on the research of semipositone problems (see [1, 3, 6, 8, 9] or references therein). One of main tools to study is the sub and supersolution method, which is introduced in Section 2. Main difficulty employing this tool for semipositone problems is the construction of a positive subsolution ψ since this subsolution must satisfy that $-\Delta\psi < 0$ near $\partial\Omega$ while $-\Delta\psi > 0$ in a large part of the interior of Ω. Moreover, since our nonlinearity satiesfies $\lim_{s\to 0^+} \frac{f(s)}{s^\alpha} = -\infty$ (infinite semipositone problem), a subsolution ψ should be constructed in such a way that $-\Delta\psi$ is sufficiently small near $\partial\Omega$.

Recently, Schrödinger-type equations with $V \not\equiv 0$ have been studied by many authors. The existence and multiplicity results of the problem satisfying $f(0) > 0$ with Dirichlet boundary conditions has been studied in [17]. The case when $\alpha = 0$, the existence result of the equation satisfying $f(0) < 0$ with Dirichlet boundary conditions and $f(0) < 0$ and mixed boundary conditions were established in [8] and [18], respectively.

Using sub and supersolution method to an infinite semipositone problem (1) we must construct a subsolution $\psi > 0$ such that $-\Delta\psi + V(x)\psi \leq -\infty$ near $\partial\Omega$. Since V changes sign in Ω, trivial test functions like the first eigenfunction of Laplacian do not serve as a subsolution of (1). By use of the solutions of (4) and (5) we construct a positive subsolution of (1).

In this paper, we establish the existence of positive solutions of (1) for a large value of $\lambda > 0$ by the method of sub and supersolution when $V \not\equiv 0$ is bounded in Ω. The existence result of a positive solution of (1) can be easily extended to the problem with mixed boundary conditions. We state our main result.

Theorem 1.1. *Assume* ($H1$), ($H2$) *and* ($H3$). *Then the problem* (1) *has a positive solution* $u_\lambda \in C^2(\Omega) \cap C(\bar{\Omega})$ *for* $\lambda \gg 1$.

This paper is organized as follows: In the next Section 2, we introduce the method of sub and supersolution for singular problems and some Lemmas. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we introduce the mixed boundary value problem to which we extend the main result, and we provide a brief proof of Theorem 4.1.

2. Preliminary

In this section, we define a subsolution and a supersoluton of (1) and introduce the theorem of sub and supersolution for (1). Further, we provide some lemmas needed for constructing a subsolution and a supersolution of (1).

A subsolution of (1) is defined as a function $\psi \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying

$$\begin{cases} -\Delta\psi + V(x)\psi \leq \lambda \frac{f(\psi)}{\psi^\alpha}, & x \in \Omega, \\ \psi > 0, & x \in \Omega, \\ \psi \leq 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

while a supersolution of (1) is defined as a function $Z \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying

$$\begin{cases} -\Delta Z + V(x)Z \geq \lambda \frac{f(Z)}{Z^\alpha}, & x \in \Omega, \\ Z > 0, & x \in \Omega, \\ Z \geq 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

Now we introduce the theorem of sub and supersolution for (1).

Lemma 2.1. *(see [10]). If there exist a subsolution ψ and a supersolution Z of (1) such that $\psi \leq Z$ on Ω, then (1) has at least one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying $\psi \leq u \leq Z$ on Ω.*

Lemma 2.2. *Assume $(H3)$. Then the problem*

$$\begin{cases} -\Delta\phi + V(x)\phi = 1, \text{ in } \Omega, \\ \phi = 0, \text{ on } \partial\Omega \end{cases} \tag{4}$$

has a solution $\phi \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $\phi(x) > 0$ for $x \in \Omega$ and $\frac{\partial\phi}{\partial\eta} < 0$ on $\partial\Omega$ where η is the outward unit normal to $\partial\Omega$.

Proof. It can be proven by similar way as in [18]. For the reader's convenience, we provide the details of the proof. From $(H3)$ it holds that $c_v\|e\|_\infty < 1$, which implies that there exists $K > 0$ such that $1 - c_v\|e\|_\infty > \frac{1}{K}$. Let $Z = Ke$. Then we have

$$\begin{aligned} -\Delta Z + V(x)Z = K(-\Delta e + V(x)e) &= K(1 + V(x)e) \\ &\geq K(1 - c_v\|e\|_\infty) = 1 \end{aligned}$$

in Ω and $Z = 0$ on $\partial\Omega$. Hence, Z is a positive supersolution of (4). Note that $\psi \equiv 0$ is a subsolution of (4) but not a solution of (4). By Lemma 2.1, we can see that there exists a solution $\phi \in C^2(\Omega) \cap C(\bar{\Omega})$ of (4) such that $0 \leq \phi \leq Z$ in Ω. Now we claim that $\phi > 0$ in Ω and $\frac{\partial\phi}{\partial\eta} < 0$ on $\partial\Omega$. Suppose that there exists $x_0 \in \Omega$ such that $\phi(x_0) = 0$. From the first equation of (4) we see that

$1 = -\Delta\phi(x_0) + V(x_0)\phi(x_0) \leq 0$ since V is bounded, which is a contradiction. Therefore, $\phi > 0$ in Ω, and hence by Hopf's maximum principle we find that $\frac{\partial\phi}{\partial\eta} < 0$ on $\partial\Omega$. $\square$

Lemma 2.3. *(see [17]). Assume (H3). Then the problem*

$$\begin{cases} -\Delta\zeta + V(x)\zeta = \frac{1}{\zeta^\alpha}, \ \textit{in } \Omega, \\ \zeta = 0, \ \textit{on } \partial\Omega \end{cases} \tag{5}$$

has a solution $\zeta \in C^2(\Omega) \cap C(\bar{\Omega})$ *such that* $\zeta(x) > 0$ *for* $x \in \Omega$ *and* $\frac{\partial\zeta}{\partial\eta} < 0$ *on* $\partial\Omega$.

3. Proof of Theorem 1.1

Proof. We first construct a positive subsolution of (1) for large value of λ. Let $w = \frac{\zeta}{\|\zeta\|_\infty}$, where ζ which is the solution of (5). Then w satisfies

$$\begin{cases} -\Delta w + V(x)w = \frac{1}{\|\zeta\|_\infty^{1+\alpha}} \frac{1}{w^\alpha}, \ \text{in } \Omega \\ w = 0, \ \text{on } \partial\Omega \end{cases} \tag{6}$$

From Lemma 2.3 we can see that $w > 0$ in Ω and $\frac{\partial w}{\partial\eta} < 0$ on $\partial\Omega$. Since $w > 0$ in Ω and $w = 0$ and $\frac{\partial w}{\partial\eta} < 0$ on $\partial\Omega$, there exist $\delta > 0$ and $m > 0$ such that

$$(\frac{2}{1+\alpha})\left[(\frac{1-\alpha}{1+\alpha})|\nabla w|^2 - \left(\frac{c_v(1-\alpha)}{2} + \frac{1}{\|\zeta\|_\infty^{1+\alpha}}\right)w^{1-\alpha}\right] \geq m \text{ in } \bar{\Omega}_\delta, \tag{7}$$

where $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$. Further, there exists $\mu \in (0,1)$ such that

$$\mu \leq w(x) \leq 1 \text{ in } \Omega \setminus \bar{\Omega}_\delta. \tag{8}$$

Define $\psi = \lambda^r w^{\frac{2}{1+\alpha}}$, where $r \in (\frac{1}{1+\alpha}, \frac{1}{1+\alpha-\beta})$. From the following simple calculations

$$\nabla\psi = \lambda^r(\frac{2}{1+\alpha})w^{\frac{1-\alpha}{1+\alpha}}\nabla w$$

we have

$$\begin{aligned} -\Delta\psi &= -\lambda^r(\frac{2}{1+\alpha})\left[w^{\frac{1-\alpha}{1+\alpha}}\Delta w + (\frac{1-\alpha}{1+\alpha})w^{-\frac{2\alpha}{1+\alpha}}|\nabla w|^2\right] \\ &= -\lambda^r(\frac{2}{1+\alpha})\left[w^{\frac{1-\alpha}{1+\alpha}}(V(x)w - \frac{1}{\|\zeta\|_\infty^{1+\alpha}}\frac{1}{w^\alpha}) + (\frac{1-\alpha}{1+\alpha})w^{-\frac{2\alpha}{1+\alpha}}|\nabla w|^2\right] \\ &= -\lambda^r(\frac{2}{1+\alpha})w^{-\frac{2\alpha}{1+\alpha}}\left[V(x)w^2 - \frac{1}{\|\zeta\|_\infty^{1+\alpha}}w^{1-\alpha} + (\frac{1-\alpha}{1+\alpha})|\nabla w|^2\right]. \end{aligned}$$

By $(H3)$ we obtain

$$
\begin{aligned}
&-\Delta\psi + V(x)\psi \\
&= -\lambda^r(\frac{2}{1+\alpha})w^{-\frac{2\alpha}{1+\alpha}}\left[V(x)w^2 - \frac{1}{\|\zeta\|_\infty^{1+\alpha}}w^{1-\alpha} + (\frac{1-\alpha}{1+\alpha})|\nabla w|^2 - (\frac{1+\alpha}{2})V(x)w^2\right] \\
&= -\lambda^r(\frac{2}{1+\alpha})w^{-\frac{2\alpha}{1+\alpha}}\left[(\frac{1-\alpha}{1+\alpha})|\nabla w|^2 + \frac{1-\alpha}{2}V(x)w^2 - \frac{1}{\|\zeta\|_\infty^{1+\alpha}}w^{1-\alpha}\right] \\
&\leq -\lambda^r(\frac{2}{1+\alpha})w^{-\frac{2\alpha}{1+\alpha}}\left[(\frac{1-\alpha}{1+\alpha})|\nabla w|^2 - \frac{c_v(1-\alpha)}{2}w^2 - \frac{1}{\|\zeta\|_\infty^{1+\alpha}}w^{1-\alpha}\right].
\end{aligned}
$$

Now using the fact $\|w\|_\infty \leq 1$, we find

$$
\begin{aligned}
- \quad & \Delta\psi + V(x)\psi \qquad (9)\\
\leq \quad & -\lambda^r(\frac{2}{1+\alpha})w^{-\frac{2\alpha}{1+\alpha}}\left[(\frac{1-\alpha}{1+\alpha})|\nabla w|^2 - \left(\frac{c_v(1-\alpha)}{2} + \frac{1}{\|\zeta\|_\infty^{1+\alpha}}\right)w^{1-\alpha}\right].
\end{aligned}
$$

First, we evaluate $-\Delta\psi + V(x)\psi$ in $\bar{\Omega}_\delta$. Let

$$
f_m := \min_{s\in[0,\infty)} f(s).
$$

Noting that $f_m < 0$ and $1 - r - r\alpha < 0$, it follows from (7) for $\lambda \gg 1$

$$
-(\frac{2}{1+\alpha})\left[(\frac{1-\alpha}{1+\alpha})|\nabla w|^2 - \left(\frac{c_v(1-\alpha)}{2} + \frac{1}{\|\zeta\|_\infty^{1+\alpha}}\right)w^{1-\alpha}\right] \leq -m \leq \lambda^{1-r-r\alpha}f_m.
$$

Hence, we have in $\bar{\Omega}_\delta$,

$$
\begin{aligned}
-\Delta\psi + V(x)\psi &\leq \lambda^r w^{-\frac{2\alpha}{1+\alpha}}\lambda^{1-r-r\alpha}f_m \\
&= \lambda\frac{f_m}{(\lambda^r w^{\frac{2}{1+\alpha}})^\alpha} \\
&\leq \lambda\frac{f(\lambda^r w^{\frac{2}{1+\alpha}})}{(\lambda^r w^{\frac{2}{1+\alpha}})^\alpha} = \lambda\frac{f(\psi)}{\psi^\alpha}. \qquad (10)
\end{aligned}
$$

Next, we estimate $-\Delta\psi + V(x)\psi$ in $\Omega \setminus \bar{\Omega}_\delta$. Since $w \geq \mu > 0$ in $\Omega \setminus \bar{\Omega}_\delta$, it holds that by $(H2)$

$$
f(\lambda^r w^{\frac{2}{1+\alpha}}) \geq B(\lambda^r w^{\frac{2}{1+\alpha}})^\beta \qquad (11)
$$

for $\lambda \gg 1$. Further, noting that $1 + r(\beta-\alpha) - r > 0$, we can see that for $\lambda \gg 1$

$$
(\frac{2}{1+\alpha})\left(\frac{c_v(1-\alpha)}{2} + \frac{1}{\|\zeta\|_\infty^{1+\alpha}}\right) \leq B\lambda^{1+r(\beta-\alpha)-r}. \qquad (12)
$$

As $0 < \beta < \frac{1}{2}(1-\alpha^2)$, it follows that $2\beta - 2 + \alpha^2 < 0$. Observing that $\mu \leq w \leq 1$ in $\Omega \setminus \bar{\Omega}_\delta$, we find

$$
(w^{\frac{2}{1+\alpha}})^{\beta-\alpha-1}w^{\alpha+1} = w^{\frac{2\beta-2+\alpha^2}{1+\alpha}} \geq 1 \text{ in } \Omega \setminus \bar{\Omega}_\delta,
$$

which implies that

$$
B\lambda^{1+r(\beta-\alpha)-r} \leq B\lambda^{1+r(\beta-\alpha)-r}(w^{\frac{2}{1+\alpha}})^{\beta-\alpha-1}w^{\alpha+1}. \qquad (13)
$$

From (12) and (14), we can see

$$(\frac{2}{1+\alpha})\left(\frac{c_v(1-\alpha)}{2}+\frac{1}{\|\zeta\|_\infty^{1+\alpha}}\right)\leq B\lambda^{1+r(\beta-\alpha)-r}(w^{\frac{2}{1+\alpha}})^{\beta-\alpha-1}w^{\alpha+1},$$

which means

$$\lambda^r(\frac{2}{1+\alpha})\left(\frac{c_v(1-\alpha)}{2}+\frac{1}{\|\zeta\|_\infty^{1+\alpha}}\right)w^{\frac{2}{1+\alpha}-(\alpha+1)}\leq B\lambda^{1+r(\beta-\alpha)}(w^{\frac{2}{1+\alpha}})^{\beta-\alpha}. \tag{14}$$

Now from (9) we estimate that in $\Omega\setminus\bar{\Omega}_\delta$

$$\begin{aligned}
&-\Delta\psi+V(x)\psi\\
&\leq-\lambda^r(\frac{2}{1+\alpha})w^{-\frac{2\alpha}{1+\alpha}}\left[(\frac{1-\alpha}{1+\alpha})|\nabla w|^2-\left(\frac{c_v(1-\alpha)}{2}+\frac{1}{\|\zeta\|_\infty^{1+\alpha}}\right)w^{1-\alpha}\right]\\
&\leq\lambda^r(\frac{2}{1+\alpha})w^{-\frac{2\alpha}{1+\alpha}}\left(\frac{c_v(1-\alpha)}{2}+\frac{1}{\|\zeta\|_\infty^{1+\alpha}}\right)w^{1-\alpha}\\
&=\lambda^r(\frac{2}{1+\alpha})\left(\frac{c_v(1-\alpha)}{2}+\frac{1}{\|\zeta\|_\infty^{1+\alpha}}\right)w^{\frac{2}{1+\alpha}-(\alpha+1)}.
\end{aligned}$$

By (11) and (14), we obtain that in $\Omega\setminus\bar{\Omega}_\delta$

$$\begin{aligned}
-\Delta\psi+V(x)\psi &\leq B\lambda^{1+r(\beta-\alpha)}(w^{\frac{2}{1+\alpha}})^{\beta-\alpha}\\
&=\lambda\frac{B(\lambda^r w^{\frac{2}{1+\alpha}})^\beta}{(\lambda^r w^{\frac{2}{1+\alpha}})^\alpha}\\
&\leq\lambda\frac{f(\lambda^r w^{\frac{2}{1+\alpha}})}{(\lambda^r w^{\frac{2}{1+\alpha}})^\alpha}=\lambda\frac{f(\psi)}{\psi^\alpha}.
\end{aligned}\tag{15}$$

Therefore, from (10) and (15) it concludes that ψ is a positive subsolution of (1) in Ω for $\lambda\gg 1$.

Now we construct a positive supersolution Z of (1) with $Z\geq\psi$ in Ω. Since $1+\alpha-\gamma>0$ and $\gamma-\alpha>0$, we can choose $M_\lambda\gg 1$ such that

$$M_\lambda^{1+\alpha-\gamma}\geq\lambda A\phi^{\gamma-\alpha}, \tag{16}$$

where ϕ is the solution of (4) in Lemma 2.2. Let $Z=M_\lambda\phi$. Then by (21) it follows that

$$-\Delta Z+V(x)Z=M_\lambda\geq\lambda\frac{A(M_\lambda\phi)^\gamma}{(M_\lambda\phi)^\alpha}\geq\lambda\frac{f(M_\lambda\phi)}{(M_\lambda\phi)^\alpha}$$

in Ω, where the last inequality was obtained by the estimate from $(H1)$. Also, $Z=0$ on $\partial\Omega$. Hence, Z is a positive supersolution of (1) with $Z\geq\psi$ if we choose M_λ sufficiently large so that $M_\lambda\phi\geq\lambda^r w^{\frac{2}{1+\alpha}}$ in Ω for each $\lambda\gg 1$.

Therefore, Lemma 2.1 concludes that there exists a positive solution u_λ of (1) such that $\psi\leq u_\lambda\leq Z$ in Ω for $\lambda\gg 1$. $\square$

4. Extension

In this section, we introduce a singular Schrödinger-type problem with mixed boundary conditions to which the result of Theorem 1.1 can be easily extended.

4.1. Mixed boundary problem

We consider an infinite semipositone Schrödinger-type problem with mixed boundary condition

$$\begin{cases} -\Delta u + V(x)u = \lambda \frac{f(u)}{u^\alpha}, & x \in \Omega = \Omega_1 \setminus \bar{\Omega}_2, \\ \frac{\partial u}{\partial \eta} + g(u)u = 0, & x \in \partial\Omega_1, \\ u = 0, & x \in \partial\Omega_2, \end{cases} \tag{17}$$

where Ω_1 and Ω_2 are subsets of $\Omega \subset \mathbb{R}^N$ with $\bar{\Omega}_2 \subset \Omega_1$, which are nonempty bounded domains in $\mathbb{R}^N, N > 2$, $\partial\Omega_1$ is a smooth boundary of Ω_1 with outward normal η, $\partial\Omega_2$ is a smooth boundary of Ω_2 λ is a positive parameter, $f \in C^1([0,\infty),\mathbb{R})$ with $f(0) < 0$ and $g \in C^\beta([0,\infty),(0,\infty))$ for some $0 < \beta < 1$ satisfies the following hypothesis:

(H4) There exists $m > 0$ such that $g(s) \geq m$ for $s \geq 0$.

We further assume that $V \in L^\infty(\Omega)$ satisfies the following condition:

$(H5)$ There exists $c_V > 0$ such that $V(x) \geq -c_V > -\frac{1}{\|\tilde{e}\|_\infty}$ for $x \in \Omega$, when e is the positive solution of

$$\begin{cases} -\Delta \tilde{e} = 1, \text{ in } \Omega, \\ \frac{\partial \tilde{e}}{\partial \eta} + m\tilde{e} = 0, \text{ on } \partial\Omega. \end{cases}$$

The nonlinear boundary condition (17) naturally arises in several applications, for example, in thermal explosion models [15, 19], convection-diffusion systems, corrosion/oxidation models, and metal-insulator or metal-oxide semiconductor systems in [2, 5, 11, 13]. Recently, the existence of a positive solution of non-singular problem (17) when $\alpha = 0$ has been investigated in [18].

Now we establish the existence result for the infinite semipositone problem when $\alpha \neq 0$.

Theorem 4.1. *Assume* $(H1), (H2), (H4)$ *and* $(H5)$. *Then the problem* (17) *has a positive solution* $u_\lambda \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *for* $\lambda \gg 1$.

4.2. The method of sub and supersolution and lemmas

A subsolution of (17) is defined as a function $\psi \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfying

$$\begin{cases} -\Delta\psi + V(x)\psi \leq \lambda f(\psi), & x \in \Omega, \\ \psi > 0, & x \in \Omega, \\ \frac{\partial \psi}{\partial \eta} + g(\psi)\psi \leq 0, & x \in \partial\Omega_1, \\ \psi \leq 0, & x \in \partial\Omega_2, \end{cases} \tag{18}$$

while a supersolution of (17) is defined as a function $Z \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfying

$$\begin{cases} -\Delta Z + V(x)Z \geq \lambda f(Z), & x \in \Omega, \\ Z > 0, & x \in \Omega, \\ \frac{\partial Z}{\partial \eta} + g(Z)Z \geq 0, & x \in \partial\Omega_1, \\ Z \geq 0, & x \in \partial\Omega_2. \end{cases} \tag{19}$$

Lemma 4.2. *(Theorem for sub and supersolution in* [10] *and* [17]*). If there exist a subsolution ψ and a supersolution Z of* (17) *such that $\psi \leq Z$ on Ω, then* (17) *has at least one solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfying $\psi \leq u \leq Z$ on Ω.*

Lemma 4.3. *(see* [17]*). Assume* $(H3)$*. Then the problem*

$$\begin{cases} -\Delta \xi + V(x)\xi = 1, \ \text{in } \Omega, \\ \frac{\partial \xi}{\partial \eta} + m\xi = 0, \ \text{on } \partial\Omega \end{cases} \tag{20}$$

has a solution $\zeta \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that $\zeta(x) > 0$ for $x \in \bar{\Omega}$ and $\frac{\partial \zeta}{\partial \eta} < 0$ on $\partial\Omega$.

4.3. Proof of Theorem 4.1

Proof. We consider the problem (5) in $\Omega = \Omega_1 \setminus \bar{\Omega}_2$ in which (17) is defined. Then the function $\psi = \lambda^r w^{\frac{2}{1+\alpha}}$ constructed in Theorem 1.1 is positive in Ω and satisfies the boundary condition $\psi = 0$ and $\frac{\partial \psi}{\partial \eta} < 0$ on $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$. Hence, ψ is also a positive subsoltion of (17) since it holds $\frac{\partial \psi}{\partial \eta} + g(\psi)\psi \leq 0$ on $\partial\Omega_1$ and $\psi \leq 0$ on $\partial\Omega_2$.

Now we construct a positive supersolution Z of (17) with $Z \geq \psi$ in Ω. Note that the function $Z = M_\lambda \phi$ constructed in theorem 1.1 is not a supersolution of (17) since $Z = M_\lambda \phi$ does not satisfy the boundary condition on $\partial\Omega_1$ in (19). Hence, we construct a positive supersolution Z using the function ξ in Lemma 4.3. Since $1 + \alpha - \gamma > 0$ and $\gamma - \alpha > 0$, we can choose $M_\lambda \gg 1$ such that

$$M_\lambda^{1+\alpha-\gamma} \geq \lambda A \xi^{\gamma-\alpha}. \tag{21}$$

Let $Z = M_\lambda \xi$. Then by (21) it follows that

$$-\Delta Z + V(x)Z = M_\lambda \geq \lambda \frac{A(M_\lambda \xi)^\gamma}{(M_\lambda \xi)^\alpha} \geq \lambda \frac{f(M_\lambda \xi)}{(M_\lambda \xi)^\alpha}, \text{ in } \Omega,$$

where the last inequality was obtained by the estimate $f(M_\lambda \xi) \leq A(M_\lambda \xi)^\gamma$ from $(H1)$. We also have

$$-\frac{\partial Z}{\partial \eta} + g(Z)Z = M_\lambda \xi(-m + g(M_\lambda \xi)) \geq 0 \text{ on } \partial\Omega_1 \text{ and } Z > 0 \text{ on } \partial\Omega_2$$

as $g(s) \geq m$ for all $s \geq 0$ and $\xi > 0$ on $\partial\Omega$. Hence, Z is a positive supersolution of (17) with $Z \geq \xi$ if we choose M_λ sufficiently large so that $M_\lambda \xi \geq \lambda^r w^{\frac{2}{1+\alpha}}$ in Ω for each $\lambda \gg 1$. Therefore, by Lemma 4.2 there exists a positive solution u_λ of (17) such that $\psi \leq u_\lambda \leq Z$ in Ω for $\lambda \gg 1$. □

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