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ON THE EQUATIONS DEFINING SOME RATIONAL CURVES OF MAXIMAL GENUS IN $\mathbb{P}^3$

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Abstract. For a nondegenerate irreducible projective variety, it is a classical problem to describe its defining equations and the syzygies among them. In this paper, we precisely determine a minimal generating set and the minimal free resolution of defining ideals of some rational curves of maximal genus in $\mathbb{P}^3$.

1. Introduction

Throughout this paper, we work over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. Let $\mathbb{P}^r$ and $R = \mathbb{K}[X_0, X_1, \ldots, X_r]$ be respectively the projective r-space and the homogeneous coordinate ring of $\mathbb{P}^r$. Let $X \subset \mathbb{P}^r$ be a nondegenerate projective variety and I_X be the defining ideal of X. To understand a given variety X, it is a natural problem to study a minimal generating set of I_X and the syzygies among them. Although there have been many results on this problem (cf. [1], [2], [3], [4], [6], [8], [10] and so on), to the authors best knowledge, it is still a very difficult problem.

Along this line, the main purpose of this paper is to provide a complete description of a minimal generating set and the minimal free resolution for some rational curves in $\mathbb{P}^3$ which are attained the possibly maximal arithmetic genus. For a reduced, irreducible and nondegenerate curve $C \subset \mathbb{P}^r$ of degree d, in the classical paper [1], Castelnuovo showed that the arithmetic genus g of C can not exceed the value $\pi_0(d, r)$ which is explicitly determined by d and r. And he also classified the extremal case. These curves are arithmetically Cohen-Macaulay and contained in a surface of minimal degree.

Let $T := \mathbb{K}[s, t]$ be the homogeneous coordinate ring of $\mathbb{P}^1$. And let T_k be the k-th graded component of T for each $k \geq 1$. Then we call the curve $\widetilde{C}$

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parameterized as

$$\widetilde{C} = \{[s^d(P) : s^{d-1}t(P) : \cdots : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\}$$

a rational normal curve of degree d in $\mathbb{P}^d$. Indeed, $\widetilde{C}$ is defined to be the image of the d-th Veronese embedding $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$ of $\mathbb{P}^1$. Note that it is well known that the defining ideal $I_{\widetilde{C}}$ is minimally generated by the set $\{X_iX_j - X_{i-1}X_{j+1} \mid 1 \le i \le j \le d-1\}$ in the sense of Notation and Remarks 2.1.(A). Let $C_d \subset \mathbb{P}^r$ be a nondegenerate rational curve of degree $d \ge r$. Since the normalization of C_d is the rational normal curve $\widetilde{C}$, C_d can be described as the image of the linear projection of $\widetilde{C} \subset \mathbb{P}^d$ from a linear subspace $\Lambda \cong \mathbb{P}^{d-r-1}$ of $\mathbb{P}^d$. That is, C_d is obtained by the parametrization

$$C_d = \{[f_0(P) : f_1(P) : \cdots : f_r(P)] \mid P \in \mathbb{P}^1\}$$

where $f_0, f_1, \ldots, f_r$ are $\mathbb{K}$-linearly independent forms of degree d in T_d. In particular, we focus our interest to determine a minimal generating set and the minimal free resolution of the defining ideals of rational curves $C_d \subset \mathbb{P}^3$ which are parametrized as

$$C_d = \{[s^d(P) : s^2t^{d-2}(P) : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\} \quad \text{for} \quad d \ge 4. \tag{1}$$

In [9], the authors studied the possible arithmetic genus of curves which are contained in a surface of minimal degree. For the curve C_d in (1), the result is

Theorem 1.1 (Theorem 3.3, [9]). *Let $C_d \subset \mathbb{P}^3$ be a rational curve as stated in (1). Then*

(1) *C_d is contained in the rational normal surface scroll $S(0,2)$ as a divisor linear equivalent to dF where F is a ruling line of $S(0,2)$.*

(2) *C_d has the arithmetic genus*

$$g = \begin{cases} (k-1)^2 & \text{if } d = 2k \\ k(k-1) & \text{if } d = 2k+1 \end{cases}$$

In particular, the genus g of C_d is possibly maximal.

The following is our main theorem in this paper:

Theorem 1.2. *Let $C_d \subset \mathbb{P}^3$ be a rational curve as stated in (1). Then,*

(1) *The minimal free resolution of C_d is of the form*

$$\begin{array}{ll} 0 \longrightarrow R(-k-2) \longrightarrow R(-2) \oplus R(-k) \longrightarrow I_{C_d} \longrightarrow 0 & \text{if } d = 2k, \\ 0 \longrightarrow R(-k-2)^2 \longrightarrow R(-2) \oplus R(-k-1)^2 \longrightarrow I_{C_d} \longrightarrow 0 & \text{if } d = 2k+1. \end{array}$$

(2) *The defining ideal I_{C_d} of C_d is minimally generated as follows:*

$$\begin{array}{ll} I_{C_d} = \langle X_1X_3 - X_2^2,\ X_1^k - X_0X_3^{k-1} \rangle & \text{if } d = 2k, \\ I_{C_d} = \langle X_1X_3 - X_2^2,\ X_1^{k+1} - X_0X_2X_3^{k-1},\ X_1^kX_2 - X_0X_3^k \rangle & \text{if } d = 2k+1. \end{array}$$

Proof. See Theorem 2.3 and Theorem 2.5. □

2. Main Theorem

Keeping the notation in the previous section, we provide a complete description of equations which generate the defining ideals of rational curves in Theorem 1.2 and the syzygies among them.

Notation and Remarks 2.1. (A) For a nondegenerate projective variety $X \subset \mathbb{P}^r$, let

$$M = \{F_{i,j} \in \mathbb{K}[X_0, X_1, \ldots, X_r] \quad | \quad F_{i,j} \in I_X \quad \text{for } 2 \leq i \leq m \text{ and } 1 \leq j \leq \ell_i\}$$

be the set of homogeneous polynomials $F_{i,j}$ of degree i in the homogeneous coordinate ring $R = \mathbb{K}[X_0, X_1, \ldots, X_r]$. Then we call M a minimal set of generators of I_X if the following three conditions hold:

(*i*) I_X is generated by the polynomials in M (i.e., $I_X = \langle M \rangle$).

(*ii*) $F_{i,1}, F_{i,2}, \ldots, F_{i,\ell_i}$ are $\mathbb{K}$-linearly independent forms of degree i for each $2 \leq i \leq m$.

(*iii*) $F_{i,j} \notin \langle (I_X)_{\leq i-1}, F_{i,1}, \ldots, F_{i,j-1} \rangle$ for each $2 \leq i \leq m$ and $1 \leq j \leq \ell_i$ where $(I_X)_{\leq t}$ is the set of all homogeneous polynomials in I_X which degree do not exceed t.

(B) For the vector bundle

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$$

on $\mathbb{P}^1$, the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E})$ defines the birational morphism $\phi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^3$ and its image is the rational normal surface scroll $S := S(0,2) \subset \mathbb{P}^3$ of degree 2. Then it is well known that $\mathrm{Pic}\big(\mathbb{P}(\mathcal{E})\big)$ is freely generated by the hyperplane class $[\widetilde{H}] := [\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)]$ and the class of fibre $[\widetilde{F}] := [\pi^*\mathcal{O}_{\mathbb{P}^1}(1)]$ of the projection $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$. Note that the divisor class group of $S(0,2)$ is freely generated by F which is the image of $\widetilde{F}$ via ϕ. The rational normal surface scroll $S := S(0,2)$ can be described as

$$S = \{[0 : s^2 : st : t^2] \mid (s,t) \in \mathbb{K}^2 \setminus (0,0)\} \subset \mathbb{P}^3$$

and the defining ideal I_S of S is generated by $X_1X_3 - X_2^2$.

Let $C_d \subset \mathbb{P}^3$ be a nondegenerate projective curve parameterized as (1). Then C_d is contained in the singular rational normal surface scroll $S(0,2)$ by Theorem 1.1.(1) and hence it is always arithmetically Cohen-Macaulay (see [5, Example 5.2]). This follows that the minimal free resolution of C_d is same with its general hyperplane section. The following lemma will play a crucial role to understand the graded Betti numbers of C_d.

Lemma 2.2. *Let $\Gamma \subset \mathbb{P}^n$ be a finite subscheme of length $|\Gamma| \geq n+1$. Suppose that Γ lies on a rational normal curve $D \subset \mathbb{P}^n$. Then it can be written as*

$|\Gamma| = tn + 1 - p$ for some integer t and $0 \le p \le n-1$, and the minimal free resolution of Γ is as follows:

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow I_\Gamma \longrightarrow 0$$

where $F_i = R(-1-i)^{\alpha_i} \oplus R(-t+1-i)^{\beta_i} \oplus R(-t-i)^{\gamma_i}$ for $i = 1, \ldots, n$ and

$$\alpha_i = i\binom{n}{i+1} \qquad \text{for } 1 \le i \le n$$
$$\beta_i = \begin{cases} (p+1-i)\binom{n}{i-1} & \text{for } 1 \le i \le p \\ 0 & \text{for } p+1 \le i \le n \end{cases}$$
$$\gamma_i = \begin{cases} 0 & \text{for } 1 \le i \le p \\ (i-p)\binom{n}{i} & \text{for } p+1 \le i \le n \end{cases}$$

Proof. We refer the reader to see [11, Proposition 2.2] and [12, Proposition 2.3]. □

Theorem 2.3. *Let $C_d \subset \mathbb{P}^3$, $d \ge 4$ be a curve stated as in (1). Then the minimal free resolution of C_d is of the form*

$$0 \longrightarrow R(-k-2) \longrightarrow R(-2) \oplus R(-k) \longrightarrow I_{C_d} \longrightarrow 0 \qquad \text{if } d = 2k \text{ and}$$
$$0 \longrightarrow R(-k-2)^2 \longrightarrow R(-2) \oplus R(-k-1)^2 \longrightarrow I_{C_d} \longrightarrow 0 \quad \text{if } d = 2k+1.$$

Proof. Let Γ and D be respectively general hyperplane sections of C_d and $S(0,2)$. Since C_d is arithmetically Cohen-Macaulay, the minimal free resolution of C_d is same with that of Γ. On the other hand, the curve D is a rational normal curve of degree 2 in $\mathbb{P}^2$ and hence the minimal free resolution of Γ on D is obtained from Lemma 2.2. Indeed, we consider the two cases for $d = |\Gamma| = 2k$ and $d = |\Gamma| = 2k+1$ for $k \ge 2$. Then it holds that $t = k$ and $p = 1$ if $d = 2k$, and $t = k$ and $p = 0$ if $d = 2k+1$ by applying Lemma 2.2. Thus we obtain the following graded Betti numbers:

$$\begin{cases} \alpha_1 = 1, \beta_1 = 1, \gamma_1 = 0 \\ \alpha_2 = 0, \beta_2 = 0, \gamma_2 = 1 \end{cases} \qquad \text{if } d = 2k \text{ and}$$
$$\begin{cases} \alpha_1 = 1, \beta_1 = 0, \gamma_1 = 2 \\ \alpha_2 = 0, \beta_2 = 0, \gamma_2 = 2 \end{cases} \qquad \text{if } d = 2k+1$$

This completes the proof. □

Now we describe a set of minimal generators of the defining ideal I_{C_d} of C_d. First we begin with some simple examples.

Example 2.4. For $d = 4, 5, 6, 7, 8, 9, 10$, let $C_d \subset \mathbb{P}^3$ be curves defined as the parametrization (1). Then by means of the Computer Algebra System Singular [7], I_{C_d} are respectively minimally generated as follows:

(*i*) $I_{C_4} = \langle X_2^2 - X_1X_3, X_1^2 - X_0X_3 \rangle$,

(*i*) $I_{C_5} = \langle X_2^2 - X_1X_3, X_1^3 - X_0X_2X_3, X_1^2X_2 - X_0X_3^2 \rangle$, (*i*) $I_{C_6} = \langle X_2^2 - X_1X_3, X_1^3 - X_0X_3^2\rangle$,

(*i*) $I_{C_7} = \langle X_2^2 - X_1X_3, X_1^4 - X_0X_2X_3^2, X_1^3X_2 - X_0X_3^3\rangle$,

(*i*) $I_{C_8} = \langle X_2^2 - X_1X_3, X_1^4 - X_0X_3^3\rangle$,

(*ii*) $I_{C_9} = \langle X_2^2 - X_1X_3, X_1^5 - X_0X_2X_3^3, X_1^4X_2 - X_0X_3^4\rangle$,

(*iii*) $I_{C_{10}} = \langle X_2^2 - X_1X_3, X_1^5 - X_0X_3^4\rangle$,

This example enables us to pose the theorem:

Theorem 2.5. *Let $C_d \subset \mathbb{P}^3$, $d \geq 4$ be a curve stated as in (1). Then I_{C_d} is minimally generated as follows: For $k \geq 2$,*

$$I_{C_d} = \langle X_1X_3 - X_2^2,\ X_1^k - X_0X_3^{k-1}\rangle \qquad \textit{if } d = 2k \textit{ and}$$
$$I_{C_d} = \langle X_1X_3 - X_2^2,\ X_1^{k+1} - X_0X_2X_3^{k-1},\ X_1^kX_2 - X_0X_3^k\rangle \quad \textit{if } d = 2k+1.$$

Proof. First we denote by M_{2k} and M_{2k+1} respectively the sets

$$M_{2k} = \{X_1X_3 - X_2^2,\ X_1^k - X_0X_3^{k-1}\} \quad \text{and}$$
$$M_{2k+1} = \{X_1X_3 - X_2^2,\ X_1^{k+1} - X_0X_2X_3^{k-1},\ X_1^kX_2 - X_0X_3^k\}.$$

And we also denote by $I_{M_d} := \langle M_d\rangle$ the ideals generated by the set M_d for $d = 2k$ and $d = 2k+1$. Then it is easy to see that the equations in M_d vanish on C_d in (1) for each case. That is, $I_{M_d} \subseteq I_{C_d}$. Now we will show that the sets M_{2k} and M_{2k+1} are minimal generating sets of ideals I_{M_d} for $d = 2k$ and $d = 2k+1$, respectively. Then we conclude that the equality $I_{M_d} = I_{C_d}$ holds by Theorem 2.3. In particular, this follows that I_{C_d} is minimally generated by the set M_d for each case. To this aim, it suffices to show that M_d and I_{M_d} for $d = 2k$ and $d = 2k+1$ satisfy two conditions (ii) and (iii) in Notations and Remarks 2.1. Thus we show the following statements:

(*a*) When $d = 2k$, $X_1^k - X_0X_3^{k-1} \notin \langle X_1X_3 - X_2^2\rangle$.

(*b*) When $d = 2k+1$,

(*b*.1) $X_1^{k+1} - X_0X_2X_3^{k-1}$ and $X_1^kX_2 - X_0X_3^k$ are $\mathbb{K}$-linearly independent,

(*b*.2) $X_1^{k+1} - X_0X_2X_3^{k-1} \notin \langle X_1X_3 - X_2^2\rangle$, and

(*b*.3) $X_1^kX_2 - X_0X_3^k \notin \langle X_1X_3 - X_2^2, X_1^{k+1} - X_0X_2X_3^{k-1}\rangle$.

To verify (*a*), suppose that

$$X_1^k - X_0X_3^{k-1} = F_{k-2}(X_1X_3 - X_2^2) \tag{2}$$

where $F_{k-2} \in \mathbb{K}[X_0, X_1, X_2, X_3]$ is a homogeneous polynomial of degree $k-2$. Then the equality in (2) fails to satisfy at the point $p = [0, 1, 0, 0] \in \mathbb{P}^3$. Suppose that $d = 2k+1$. Then it is obviously that two polynomial $X_1^{k+1} - X_0X_2X_3^{k-1}$ and $X_1^kX_2 - X_0X_3^k$ are $\mathbb{K}$-linearly independent as the exclusive monomials of each polynomial. This proves $(b.1)$. To verify $(b.2)$, suppose that

$$X_1^{k+1} - X_0X_2X_3^{k-1} = G_{k-1}(X_1X_3 - X_2^2) \tag{3}$$

where $G_{k-1} \in \mathbb{K}[X_0, X_1, X_2, X_3]$ is a homogeneous polynomial of degree $k-1$. Then it is easy to see that the point $p = [0,1,0,0] \in \mathbb{P}^3$ gives a failure for the equality in (3). This proves $(b.2)$. Finally, suppose that

$$X_1^kX_2 - X_0X_3^k = H_{k-1}(X_1X_3 - X_2^2) + b(X_1^{k+1} - X_0X_2X_3^{k-1}) \tag{4}$$

where $H_{k-1} \in \mathbb{K}[X_0, X_1, X_2, X_3]$ is a homogeneous polynomial of degree $k-1$ and b is a constant. In particular, we may assume that b is nonzero. Otherwise, the equality fails to satisfy at the point $p = [0,1,1,1] \in \mathbb{P}^3$. We also observe that the equality (4) is represented as $0 = b$ on the point $[0,1,0,0] \in \mathbb{P}^3$ which can not happen. This completes the proof of statements in (b).

□

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